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ON THE DISTANCE BETWEEN GENERALIZED FIBONACCI NUMBERS

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Abstract. For an integer $k \geq 2$, let $(F_n^{(k)})_n$ be the k-Fibonacci sequence which starts with $0, \dots, 0, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. This paper completes a previous work of Marques (2014) which investigated the spacing between terms of distinct k-Fibonacci sequences.

1. Introduction and preliminary results. For $k \geq 2$, we consider the *k-generalized Fibonacci sequence* or, for simplicity, the *k-Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$ given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2, \tag{1.1}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We shall refer to $F_n^{(k)}$ as the nth *k-Fibonacci number*. We note that in fact each choice of k produces a distinct sequence which is a generalization of the usual Fibonacci sequence $(F_n)_{n \geq 0}$, obtained for $k = 2$.

The first direct observation is that the first $k + 1$ nonzero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2} \quad \text{for all } 2 \leq n \leq k+1, \tag{1.2}$$

while the next term is $F_{k+2}^{(k)} = 2^k - 1$. In fact, $F_n^{(k)} < 2^{n-2}$ for all $n \geq k+2$ (see [1]). In general, Cooper and Howard [4] proved the following nice formula:

LEMMA 1.1. *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor (n+k)/(k+1) \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

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where

$$C_{n,j} = (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or one of a or b is negative, and denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x. For example, if $k+2 \le n \le 2k+2$, then Cooper and Howard's formula becomes the identity

$$F_n^{(k)} = 2^{n-2} - (n-k) \cdot 2^{n-k-3} \quad \text{for all } k+2 \le n \le 2k+2. \tag{1.3}$$

In the present paper, we investigate the differences between generalized Fibonacci numbers, extending and completing the work of D. Marques [8]. Our goal here is to remove some restrictions considered by Marques in his work. To be more precise, we study the Diophantine equation

$$F_n^{(k)} - F_m^{(\ell)} = c \tag{1.4}$$

in integers m, n, ℓ, k and c with $\ell \ge k \ge 2$ and $n, m \ge 2$.

Marques [8] obtained the following partial result concerning the solutions of (1.4).

Theorem 1.2. *If* (m, n, ℓ, k) *is a solution of* (1.4) *with* $\ell \ge k \ge 2$, $n > k+2$, $m > \ell + 2$ *and* $m \ne n$, *then* $\max\{m, n, \ell, k\} < M$ *for some effectively computable constant* M *which can be taken as*

$$M = \max\{c_1, 1.9 \times 10^{146} c_2^{24} \log^{27} c_2, 8 \times 10^{246}\}$$

where $c_1 := 5\log(|c|+1) + 2$ *and* $c_2 := 4\log(|c|+5)/\log 2$.

For $m = n$, Marques showed the following result.

Theorem 1.3. *If* $c = r2^{r-3} - s2^{s-3}$ *where* r *and* s *are integers such that* $0 \le r < s$, *then for all* $k \ge 2$,

$$(n, m, \ell) = (k+s, k+s, k+s-r)$$

is a solution of (1.4) *with* $k \ge s-1$. *Conversely, if* (1.4) *has a solution with* $m = n \le 2k+1$, *then* $c = r2^{r-3} - s2^{s-3}$ *for some integers* $r < s$.

We note that the case $n = k+2$ and $m = \ell+2$ can be included in Theorem 1.2, whereas Theorem 1.3 only considers the case when $\max\{m, n, \ell\} \le 2k+1$. Our main aim here is to complete the analysis of the case $n = m$ in Theorem 1.2. Furthermore, we treat the other cases involving n, k and m, ℓ. Our principal results are given in Section 3, in particular in Theorems 3.1 and 3.4.

To prove our main results we use lower bounds for linear forms in logarithms (Baker's theory) and a method developed by Bravo and Luca in [1, 2], based on the fact that when k is large then the dominant root of the characteristic polynomial of $F^{(k)}$ is exponentially close to 2. In addition, the formula of Cooper and Howard [4] is needed for some important estimates. We follow the approach and the presentation in [8].

Before proceeding further it may be mentioned that the characteristic polynomial of $F^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible in $\mathbb{Q}[x]$ and has just one zero real outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single zero. The other roots are strictly inside the unit circle, so $\alpha(k)$ is a Pisot number of degree k. Moreover, it is also known that $\alpha(k)$ is between $2(1-2^{-k})$ and 2 (see [6, Lemma 2.3] or [11, Lemma 3.6]). To simplify notation, we shall omit the dependence of α on k.

We now consider the function $f_k(x) = (x-1)/(2+(k+1)(x-2))$ for an integer $k \geq 2$ and $x > 2(1-2^{-k})$. It is easy to see that the inequalities

$$1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad |f_k(\alpha^{(i)})| < 1, \quad 2 \leq i \leq k, \tag{1.5}$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, by computing norms from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$, for example, we see that the number $f_k(\alpha)$ is not an algebraic integer.

With the above notation, Dresden and Du [5] showed that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \quad \text{and} \quad |F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| < 1/2 \tag{1.6}$$

for all $n \geq 1$ and $k \geq 2$.

In addition, Bravo and Luca [2] proved that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{for all } n \geq 1 \text{ and } k \geq 2. \tag{1.7}$$

The observations in (1.6) and (1.7) lead us to call α the *dominant zero* of $F^{(k)}$.

It was also proved in [2] that if we write

$$\alpha^{r-1} = 2^{r-1} + \delta \quad \text{and} \quad f_k(\alpha) = f_k(2) + \eta, \qquad \text{where } 1 \leq r < 2^{k/2},$$

then $|\delta| < 2^r/2^{k/2}$, $|\eta| < 2k/2^k$ and

$$|f_k(\alpha)\alpha^{r-1} - 2^{r-2}| < \frac{2^{r-1}}{2^{k/2}} + \frac{2^r k}{2^k} + \frac{2^{r+1}k}{2^{3k/2}}.$$

Furthermore, if $k \geq 10$, then $2k/2^k + 4k/2^{3k/2} < 1/2^{k/2}$, thus

$$|f_k(\alpha)\alpha^{r-1} - 2^{r-2}| < \frac{2^r}{2^{k/2}}. \tag{1.8}$$

To conclude this section, we briefly present the concept of Sidon sets which will be used later. The history of Sidon sets began in 1932, when Sidon [10], motivated by considerations of Fourier analysis, asked how large a set $\mathcal{A}$ of integers from $\{1,\dots,N\}$ can be if it has the property that all sums $a+b$ with $a,b\in\mathcal{A}$, $a\le b$, are distinct. Sets of integers with this property are now called *Sidon sets*, B_2 *sets*, or $B_2[1]$ *sets*. Since an equivalent condition is that the differences are all distinct, we see that $\mathcal{A}$ is a Sidon set if all the nonzero differences $a-a'$ $(a,a'\in\mathcal{A})$ are distinct.

Similarly, a sequence of positive integers is called a *Sidon sequence* if the pairwise sums of its members are all different. We also say that the elements form a difference-set. As an example, it is a straightforward exercise to check that all powers of two form an infinite Sidon sequence.

2. Linear forms in logarithms. In order to prove our main results, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers. Such a bound was given by Matveev [9]. We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d+a_1x^{d-1}+\cdots+a_d=a_0\prod_{i=1}^{d}(x-\eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta)=\frac{1}{d}\Big(\log a_0+\sum_{i=1}^{d}\log\max\{|\eta^{(i)}|,1\}\Big)$$

is called the *logarithmic height* of η. In particular, if $\eta=p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta)=\log\max\{|p|,q\}$.

We let $\mathbb{K}=\mathbb{Q}(\alpha)$. Knowing that $\mathbb{Q}(\alpha)=\mathbb{Q}(f_k(\alpha))$ and $|f_k(\alpha^{(i)})|\le 1$ for all $i=1,\dots,k$ and $k\ge 2$, we obtain $h(\alpha)=(\log\alpha)/k$ and $h(f_k(\alpha))=(\log a_0)/k$, where a_0 is the leading coefficient of the minimal primitive polynomial of $f_k(\alpha)$ over the integers. Define

$$g_k(x)=\prod_{i=1}^{k}(x-f_k(\alpha^{(i)}))\in\mathbb{Q}[x]\quad\text{ and }\quad\mathcal{N}=\mathrm{N}_{\mathbb{K}/\mathbb{Q}}\big(2+(k+1)(\alpha-2)\big)\in\mathbb{Z}.$$

We conclude that $\mathcal{N}g_k(x)\in\mathbb{Z}[x]$ vanishes at $f_k(\alpha)$. Thus, a_0 divides $|\mathcal{N}|$. But, for $k\ge 2$,

$$\begin{aligned}|\mathcal{N}|&=\Big|\prod_{i=1}^{k}(2+(k+1)(\alpha^{(i)}-2))\Big|=(k+1)^k\Big|\prod_{i=1}^{k}\Big(2-\frac{2}{k+1}-\alpha^{(i)}\Big)\Big|\\&=(k+1)^k\Big|\varPsi_k\Big(2-\frac{2}{k+1}\Big)\Big|=\frac{2^{k+1}k^k-(k+1)^{k+1}}{k-1}<2^kk^k.\end{aligned}$$

Hence, we will use the inequalities

(2.1) $$h(\alpha) < 0.7/k \quad \text{and} \quad h(f_k(\alpha)) < 2\log k \quad \text{for all } k \geq 2.$$

Matveev [9] proved the following deep theorem.

THEOREM 2.1 (Matveev's theorem). *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \dots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \dots, b_t$ be rational integers. Set*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \quad \textit{and} \quad B \geq \max\{|b_1|, \dots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ be real numbers for $i = 1, \dots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\bigl(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t\bigr).$$

3. Main results. First of all, we point out that the case $c = 0$ was studied by Bravo–Luca in [3] and independently by Marques [7]. From now on, we assume that (n, m, ℓ, k) is a solution of (1.4) with $\ell \geq k \geq 2$, $n, m, \geq 2$ and $c \neq 0$. We begin by considering the case when $n \leq k+1$ and $m \leq \ell+1$, where by (1.2), $F_n^{(k)}$ and $F_m^{(\ell)}$ are powers of two. Then it follows from (1.4) that

$$1/2 \leq |1 - 2^{-|n-m|}| = 4|c|/2^{\max\{n,m\}}.$$

Hence, $\max\{n, m\} < 3 + 2\log|c|$. Even more, for fixed ℓ and k, the equation $2^{n-2} - 2^{m-2} = c$ has a unique solution since the powers of two form an infinite Sidon sequence, as mentioned in Section 1.

3.1. The case $n \geq k + 2$ and $m \geq \ell + 2$. Here, we set $\beta = \alpha(\ell)$ and use equation (1.4) and the results of Dresden–Du (1.6), to get

(3.1) $$|f_k(\alpha)\alpha^{n-1} - f_\ell(\beta)\beta^{m-1}| \leq |c| + 1.$$

The left-hand side above is nonzero (see [3, p. 2125]). Dividing the above expression by the term involving $\delta := \max\{n, m\}$, we get

(3.2) $$\left|\left(\frac{f_k(\alpha)}{f_\ell(\beta)}\right)^{\varepsilon} \alpha^{\varepsilon(n-1)}\beta^{-\varepsilon(m-1)} - 1\right| \leq \frac{2(|c|+1)}{\phi^{\delta-1}},$$

with some $\varepsilon \in \{\pm 1\}$ and $\phi := \alpha(2) = (\sqrt{5}+1)/2$. In fact, $\varepsilon = 1$ if $\delta = n$ and $\varepsilon = -1$ if $\delta = m$. In the right-hand side of (3.2) we shall use a linear form in $t := 3$ logarithms:

$$\gamma_1 := f_k(\alpha)/f_\ell(\beta), \qquad \gamma_2 := \alpha, \qquad \gamma_3 := \beta.$$

We take $b_1 := \varepsilon$, $b_2 := \varepsilon(n-1)$ and $b_3 := -\varepsilon(m-1)$. The field $\mathbb{Q}(\alpha, \beta)$ containing all these numbers has degree $D \leq k\ell$. Further, in the notation of Theorem 2.1, we can take the following parameters: $A_1 := 4\ell^2 \log \ell$, $A_2 = A_3 := 0.7\ell$ and $B = \delta - 1$. Applying Theorem 2.1, we find that $|\gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1|$

exceeds

$$\exp\bigl(-1.4 \times 30^6 \times 3^{4.5} \times \ell^4(1 + 2\log \ell)(1 + \log(\delta - 1)) \times (4\ell^2 \log \ell)(0.7\ell)^2\bigr).$$

The absolute value of the number under the exponential is

$$< 2.25 \times 10^{12}\ell^8(\log \ell)^2 \log(\delta - 1),$$

where we have used the fact that $1 + 2\log \ell \leq 4\log \ell$ and $1 + \log(\delta - 1) < 2\log(\delta - 1)$ for $\ell \geq 2$ and $\delta - 1 \geq 3$. Comparing this with (3.2), we get

$$\frac{\delta - 1}{\log(\delta - 1)} < \frac{2.25}{\log \phi} \times 10^{12}\ell^8(\log \ell)^2 + \frac{\log(2(|c| + 1))}{\log(\delta - 1)\log \phi}. \tag{3.3}$$

However, if $\delta > \ell_0 := (|c| + 5)^3 > 1 + (2(|c| + 1))^{1/\log \phi}$, then

$$\frac{\log(2(|c| + 1))}{\log(\delta - 1)\log \phi} < 1.$$

Thus,

$$\frac{\delta - 1}{\log(\delta - 1)} < 4.7 \times 10^{12}\ell^8(\log \ell)^2.$$

From this, and using the fact that the inequality $x/\log x < A$ implies $x < 2A\log A$ whenever $A \geq 3$ (see [1, p. 74]), we have

$$\delta < 4.8 \times 10^{14}\ell^8(\log \ell)^3. \tag{3.4}$$

We now need to upper bound ℓ polynomially in terms of k.

CASE 1: $\ell < \max\{240, \ell_0\}$. Then

$$\delta \leq M := \max\{m, n, \ell, k\} < \max\{8.7 \times 10^{35}, 4.8 \times 10^{14}\ell_0^8(\log \ell_0)^3\} := H_0.$$

From now on, we work under the assumption $\delta \geq H_0$, and so we must be in the following case:

CASE 2: $\ell \geq \max\{240, \ell_0\}$. Then $m \leq \delta < 4.8 \times 10^{14}\ell^8(\log \ell)^3 < 2^{\ell/2}$. Using Bravo–Luca's argument (1.8) and (3.1), we conclude that

$$|f_k(\alpha)\alpha^{n-1}2^{-(m-2)} - 1| < \frac{|c| + 5}{2^{\ell/2}}. \tag{3.5}$$

Since $f_k(\alpha)$ is not an algebraic integer, $\Lambda := f_k(\alpha)\alpha^{n-1}2^{-(m-2)} - 1$ is nonzero. We apply again Matveev's Theorem 2.1 to bound the left-hand side of (3.5) from below. Here, we take $t := 3$, $\gamma_1 := f_k(\alpha)$, $\gamma_2 := \alpha$, $\gamma_3 := 2$; hence $\mathbb{K} := \mathbb{Q}(\alpha)$ and so $D := k$. Also, we take $b_1 := 1$, $b_2 := n - 1$ and $b_3 := -(m - 2)$.

Here one can take $A_1 := 2k\log k$, $A_2 = A_3 := 0.7$ and $B = \delta$. Applying Theorem 2.1, we deduce from (3.5) that

$$\exp(-8.5 \times 10^{11}k^3(\log k)^2 \log \delta) < |\Lambda| < \frac{|c| + 5}{2^{\ell/2}}.$$

By inequality (3.4), we conclude that $\log\delta < 56\log\ell$. Thus,

$$\frac{\ell}{\log\ell} < \frac{2}{\log 2}\big(4.77\times 10^{13}k^3(\log k)^2\big) + \frac{2\log(|c|+5)}{\log 2\log\ell}.$$

Keeping in mind $\ell \ge \ell_0 > (|c|+5)^{2/\log 2}$, we deduce that

$$2\log(|c|+5)/(\log 2\log\ell) < 1.$$

So, from the above we find that

$$\ell < 1.4\times 10^{16}k^3(\log k)^3. \tag{3.6}$$

In addition, combining inequalities (3.4) and (3.6), we finally arrive at

$$\delta < 1.3\times 10^{149}k^{24}(\log k)^{27}. \tag{3.7}$$

Case 3: $k < \max\{1670, k_0\}$ with $k_0 := 3\log(2|c|+18)$. Then

$$\delta \le M < \max\{9.2\times 10^{249}, 1.3\times 10^{149}k_0^{24}(\log k_0)^{27}\} := H_1.$$

We now assume that $\delta > H_1$, and therefore we are in the following case:

Case 4: $k \ge \max\{1670, k_0\}$. Here,

$$\delta < 1.3\times 10^{149}k^{24}(\log k)^{27} < 2^{0.499k} < 2^{k/2} \le 2^{\ell/2}.$$

Using the Bravo–Luca argument (1.8) once more, we infer that

$$|2^{n-2} - 2^{m-2}| \le \frac{2^n}{2^{k/2}} + \frac{2^m}{2^{\ell/2}} + |c| + 1.$$

Dividing both sides of the above inequality by 2^δ, we obtain

$$|1 - 2^{-|n-m|}| \le \frac{|c|+9}{2^{k/2}}.$$

Case 5: $n \ne m$. In this case, the absolute value of the left-hand side of the above expression is $\ge 1/2$, so

$$k < 2\log(2|c|+18)/\log 2 < k_0,$$

which is impossible.

We record what we have just proved.

Theorem 3.1. *Let $c \ne 0$ be an integer. If (m,n,ℓ,k) is a solution of the Diophantine equation $F_n^{(k)} - F_m^{(\ell)} = c$ with $n \ge k+2$, $m \ge \ell+2$, $\ell \ge k$ and $n \ne m$, then*

$$M := \max\{m,n,\ell,k\} < H_1 := \max\{9.2\times 10^{249}, 1.3\times 10^{149}k_0^{24}(\log k_0)^{27}\}.$$

To deal with the case $n = m$, we will use the following results:

Lemma 3.2. *If $r < 2^k$, then*

$$F_r^{(k)} = 2^{r-2}\left(1 + \frac{k-r}{2^{k+1}} + \zeta(k,r)\right),$$

where $\zeta = \zeta(k,r)$ is a real number such that $|\zeta| < 4r^2/2^{2k+2}$.

Proof. From Cooper and Howard's formula of Lemma 1.1, we get

$$|\zeta| \leq \sum_{j=2}^{\lfloor (r+k)/(k+1)\rfloor -1} \frac{|C_{r,j}|}{2^{(k+1)j}} < \sum_{j\geq 2} \frac{2r^j}{2^{(k+1)j}(j-2)!}$$
$$< \frac{2r^2}{2^{2k+2}} \sum_{j\geq 2} \frac{(r/2^{k+1})^{j-2}}{(j-2)!} < \frac{2r^2}{2^{2k+2}} e^{r/2^{k+1}}.$$

Further, since $r < 2^k$ we have $e^{r/2^{k+1}} < e^{1/2} < 2$. Thus, $|\zeta| < 4r^2/2^{2k+2}$. ∎

LEMMA 3.3. *The sequence $T = (t2^t)_{t\geq 1}$ is an infinite Sidon sequence.*

Proof. We can assume that

$$x2^x - y2^y = a2^a - b2^b \tag{3.8}$$

for some positive integers x, y, a, b with $x > y$, $a > b$ and $x > a$. Then $b < a < x$. Note that, if $y < a$, then it is easy to see that $x - y \geq 2$ and $x - a \geq 1$. We now observe that expression (3.8) can be written as $x2^x - a2^a = y2^y - b2^b$. Dividing the above equality by $x2^x$ and taking absolute value, we obtain

$$\left|1 - \frac{a/x}{2^{x-a}}\right| < \frac{y/x}{2^{x-y}} + \frac{b/x}{2^{x-b}} < \frac{2}{2^{x-y}}. \tag{3.9}$$

But this is a contradiction because the left-hand side is $> 1/2$ while the right-hand side is $\leq 1/2$. If, on the contrary, $a < y$, then $x-y \geq 1$ and $x-a \geq 2$. Here, a similar argument applied to the expression $x2^x - y2^y = a2^a - b2^b$ also gives an absurdity. Thus, it remains to deal with the case when $y = a$. Here the equality

$$x2^x + b2^b = a2^a + y2^y = a2^{a+1}$$

obtained from (3.8) is impossible for $x \geq a+1$. ∎

CASE 6: $n = m$. Since $\ell > k$, we have $c < 0$. Here, we distinguish the cases $k+2 \leq n \leq 2k+2$ and $n > 2k+2$.

Turning back to our problem, we recall that Marques proved (Theorem 1.3) that if (1.4) has a solution with $k+2 \leq m = n \leq 2k+2$, then $c = r2^{r-3} - s2^{s-3}$ for some positive integers $r < s$. Even more, $m \leq 2\ell + 2$ because $k \leq \ell$. So, from Lemma 1.1 (see also (1.3)), we get

$$F_n^{(k)} = 2^{n-2} - (n-k)2^{n-k-3} \quad \text{and} \quad F_m^{(\ell)} = 2^{n-2} - (n-\ell)2^{n-\ell-3}.$$

Hence, the Diophantine equation (1.4) becomes

$$(n-\ell)2^{n-\ell-3} - (n-k)2^{n-k-3} = r2^{r-3} - s2^{s-3},$$

and, in view of Lemma 3.3, we obtain $n - \ell = r$ and $n - k = s$. Thus, in this case ($k+2 \leq n \leq 2k+2$), equation (1.4) has no solutions when

$c \neq r2^{r-3} - s2^{s-3}$, while all the solutions are given by

$$(n, m, \ell, k) = (k+s, k+s, k+s-r, k) \quad \text{for } k \geq s-2,$$

when $c = r2^{r-3} - s2^{s-3}$.

Suppose now that $n > 2k+2$. We first consider $\ell = k+1$ without any restriction on $c < 0$. By Lemma 3.3, we can write

$$F_n^{(k)} = 2^{n-2}\left(1 + \frac{k-n}{2^{k+1}} + \zeta_1\right), \qquad F_n^{(k+1)} = 2^{n-2}\left(1 + \frac{k+1-n}{2^{k+2}} + \zeta_2\right),$$

with $\zeta_1 \neq 0$, $|\zeta_1| < 4n^2/2^{2k+2}$ and $|\zeta_2| < n^2/2^{2k+2}$. Substituting these values in (1.4) and rearranging some terms, we get

$$2^{n-k-4}(n-k+1) - |c| = 2^{n-2}(\zeta_2 - \zeta_1).$$

Dividing by $2^{n-k-4}(n-k+1) > 0$ (because $n > 2k+2$), and taking into account $n < 2^{k/2}$ and $n-k+1 \geq k+4$, we obtain, after some elementary algebra,

$$\left|1 - \frac{|c|}{2^{n-k-4}(n-k+1)}\right| < \frac{8}{k+4}. \tag{3.10}$$

On the other hand, by using the facts that $n-k+1 \geq k+4 > 3\log(2|c|+18)$ and $2^{n-k} > (2|c|+18)^{3\log 2}$, which hold because

$$n-k > k+2 > 3\log(2|c|+18),$$

we get

$$\begin{aligned}\frac{|c|}{2^{n-k-4}(n-k+1)} &< \left(\frac{|c|}{2|c|+18}\right)\left(\frac{16}{3(2|c|+18)^{3\log 2-1}\log(2|c|+18)}\right)\\ &< \frac{16}{3\cdot 20^{3\log 2-1}\log 20} < 0.0701633.\end{aligned}$$

With this data and (3.10), we arrive at $0.929837 < 8/(k+4)$, which is impossible because $k > 1670$.

We now deal with the case when $n > 2k+2$ and $\ell \geq k+2$. By using Lemma 3.3 once again, we write

$$F_n^{(k)} = 2^{n-2}\left(1 + \frac{k-n}{2^{k+1}} + \zeta_1\right) \quad \text{and} \quad F_m^{(\ell)} = 2^{n-2}\left(1 + \frac{\ell-n}{2^{\ell+1}} + \zeta_2\right),$$

with $\zeta_1 \neq 0$ and $|\zeta_i| < 4n^2/2^{2k+2}$ for $i = 1, 2$. So, (1.4) can be rewritten as

$$\left((n-k)2^{n-k-3} - (n-\ell)2^{n-\ell-3}\right) - |c| = 2^{n-2}(\zeta_1 - \zeta_2).$$

Dividing through by $(n-k)2^{n-k-3} - (n-\ell)2^{n-\ell-3} > 0$ (because $\ell > k$), and taking absolute values, we get

$$\left|1 - \frac{|c|}{(n-k)2^{n-k-3} - (n-\ell)2^{n-\ell-3}}\right| < \frac{4}{3 \times 2^{0.002k}}, \tag{3.11}$$

where we have used

$$(3.12)\qquad (n-k)2^{n-k-3}-(n-\ell)2^{n-\ell-3} = (n-k)2^{n-k-3}\left(1-\frac{n-\ell}{n-k}2^{k-\ell}\right) > (3/4)(n-k)2^{n-k-3},$$

as well as the facts that $n^2 < 2^{0.998k}$ and $n-k \geq 4$. On the other hand, the absolute value of the left-hand side of (3.11) is nonzero. In addition, we saw that $2^{n-k} > (2|c|+18)^{3\log 2}$ since $n-k > k+2 > 3\log(2|c|+18)$. From the above, we can lower bound the absolute value in (3.11). Indeed,

$$\frac{|c|}{(n-k)2^{n-k-3}-(n-\ell)2^{n-\ell-3}} < \frac{2^5|c|}{3(n-k)2^{n-k}} < \frac{2^5}{18\log(2|c|+18)(2|c|+18)^{3\log 2-1}} \leq \frac{2^5}{18\log(20)20^{3\log 2-1}}.$$

Thus,

$$\left|1-\frac{|c|}{(n-k)2^{n-k-3}-(n-\ell)2^{n-\ell-3}}\right| > 0.97,$$

which, combined with (3.11), gives $2^{0.002k} < 4/(3\times 0.97)$. So, $k < 500$, which is a contradiction.

We record what we have just proved.

THEOREM 3.4. *Let $c < 0$ be an integer and consider the Diophantine equation $F_n^{(k)} - F_m^{(\ell)} = c$ with $n \geq k+2$, $m \geq \ell+2$, $\ell \geq k+1 > \max\{1670, 3\log(2|c|+18)\}$ and $n = m$. If $k+2 \leq n \leq 2k+2$, then there are infinitely many solutions of the above equation given by*

$$(m,n,\ell,k) = (k+s,k+s,k+s-r,k) \quad \text{for } k \geq s-2.$$

If, on the contrary, $n > 2k+2$, then the equation has no solutions.

3.2. The cases when either $n \leq k+1$ and $m \geq \ell+2$, or $n \geq k+2$ and $m \leq \ell+1$. We note that if $n \leq k+1$ and $m \geq \ell+2$, then $n < m$. Here, by using similar arguments to those in Subsection 3.1, we obtain an upper bound for $m = \max\{m,n,\ell,k\}$, namely

$$m \leq \max\{8.8\times 10^{24}, 5.4\times 10^{15}\log^4(2|c|+4)(\log\log(2|c|+4)+2)^3\}.$$

On the other hand, for $n \geq k+2$ and $m \leq \ell+1$, we distinguish the cases $n \neq m$ and $n = m$ with $c = s2^{s-3}$ and $c \neq s2^{s-3}$, respectively, where $s \geq 2$ is an integer. Indeed, after using linear forms in logarithms, we conclude that

$$\delta \leq 7.7\times 10^{13}k^4(\log k)^3.$$

If $k < \max\{170, k_1\}$ with $k_1 := 3\log(2|c|+10)$, then

$$\delta < \max\{8.8\times 10^{24}, 7.7\times 10^{13}k_1^4(\log k_1)^3\} := H_3.$$

We next deal with $\delta \geq H_3$ and obtain $k \geq \max\{170, k_1\}$. So, $n \leq \delta < 2^{k/2}$. Considering $n \neq m$ and taking into account (1.8), we get

$$\frac{1}{2} \leq |1 - 2^{-|n-m|}| < \frac{|c| + 5}{2^{k/2}},$$

which leads to the contradiction $k < k_1$. Now, when $n = m$, we use (1.3) and argue as before to deduce that, if $k + 2 \leq n \leq 2k + 2$, then $c = -s2^{s-3}$ for a positive integer s. In addition, the solutions of (1.4) are given by

$$(n, m, k) = (k + s, k + s, k) \quad \text{for } k \geq s - 2 \text{ and } \ell \geq k + s - 1.$$

For $n > 2k + 2$ and $c = -s2^{s-3}$, (1.4) has no solutions. Indeed, in view of Lemma 3.2 and some calculations, we get

$$\begin{aligned} \frac{1}{2} &< \left|1 - \left(\frac{s}{n-k}\right)^{\epsilon} 2^{-|n-k-s|}\right| < \frac{4n^2 2^{n-2}}{\max\{n-k, s\} 2^{\max\{n-k,s\}+2k-1}} \\ &\leq \frac{4n^2 2^{n-2}}{(n-k)2^{n+k-1}} < \frac{2}{n-k}. \end{aligned}$$

In the above, we have used $\max\{n - k, s\} \geq n - k$ and $n < 2^{k/2}$. Thus, $n - k < 4$, which is not the case.

Finally, for $n > k + 2$ and $c \neq -s2^{s-3}$, we use a similar argument to that used in (3.11) to get an upper bound on k which contradicts $k > \max\{170, k_1\}$.

4. On differences between k-Fibonacci numbers: Final remark. Consider the equation $F_n^{(k)} - F_m^{(k)} = c$ and suppose that it has two integer solutions. That is, suppose that $n > m$ and $u > v$ are positive integers such that

$$F_n^{(k)} - F_m^{(k)} = c = F_u^{(k)} - F_v^{(k)}. \tag{4.1}$$

Assume that $c > 0$, since the case $c < 0$ can be handled in the same way. Note that there is no loss of generality in assuming that $n > u$. Hence, $n \geq \max\{u, m\} + 1$.

If $u \neq m$, then $\min\{u, m\} \leq \max\{u, m\} - 1$, so that

$$F_n^{(k)} + F_v^{(k)} = F_u^{(k)} + F_m^{(k)} \leq F_{\max\{u,m\}-1}^{(k)} + F_{\max\{u,m\}}^{(k)} < F_{\max\{u,m\}+1}^{(k)} \leq F_n^{(k)},$$

which is a contradiction. Hence, $u = m$ and so (4.1) becomes $F_n^{(k)} + F_v^{(k)} = 2F_u^{(k)}$. We now use inequalities (1.7) to deduce that $n = u + 1$ or $u + 2$. But $n \neq u + 2$ because $F_{u+2}^{(k)} > 2F_u^{(k)}$. Now, if $n = u + 1$, then we recall the identity $F_{u+1}^{(k)} = 2F_u^{(k)} - F_{u-k}^{(k)}$, which holds for all $u \geq 2$, to conclude that $F_v^{(k)} = F_{u-k}^{(k)}$ and that $(n, m) = (u + 1, u)$ are the only other solutions to (4.1) if and only if $u - v = k$.

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