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SUBSEQUENCE SUMS OF ZERO-SUM FREE SEQUENCES OVER FINITE ABELIAN GROUPS

BY

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Abstract. Let G be a finite abelian group of rank r and let X be a zero-sum free sequence over G whose support $\mathrm{supp}(X)$ generates G. In 2009, Pixton proved that $|\varSigma(X)| \geq 2^{r-1}(|X| - r + 2) - 1$ for $r \leq 3$. We show that this result also holds for abelian groups G of rank 4 if the smallest prime p dividing $|G|$ satisfies $p \geq 13$.

1. Introduction. Let G be a finite abelian group. By a *sequence over* G we mean a finite sequence of terms from G where repetition is allowed and the order is disregarded. In 1972, Eggleton and Erdős [1] first tackled the problem of determining the minimal cardinality of the set $\varSigma(X)$ of subsums of zero-sum free sequences over G. In 1977, Olson and White [**?**] obtained a lower bound for $|\varSigma(X)|$, where X is a zero-sum free sequence over G and $\mathrm{supp}(X)$ generates a noncyclic group. Subsequently, several authors [2, 6, 7, 8] obtained a huge variety of results. In particular, we refer the reader to Part II of the recent monograph [5] by Grynkiewicz. In 2009, Pixton [6, Lemma 1.1 and Theorems 1.3 and 1.7] proved the following theorem.

Theorem A. *Let G be a finite abelian group of rank $r \leq 3$, and let X be a zero-sum free sequence over G whose support generates G. Then*

$$|\varSigma(X)| \geq 2^{r-1}(|X| - r + 2) - 1$$

where $|X|$ denotes the length of X.

In this paper, we show that the bound given in Theorem A also holds for a class of abelian groups of rank 4.

Main Theorem 1.1. *Let G be a finite abelian group of rank 4 and let X be a zero-sum free sequence over G whose support generates G. If the smallest prime p dividing $|G|$ satisfies $p \geq 13$, then*

$$|\varSigma(X)| \geq 8|X| - 17.$$

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Let G be a nontrivial finite abelian group, say $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. Then $r = \mathsf{r}(G)$ is the rank of G.

Suppose that $r = 4$. Then the lower bound given in Theorem 1.1 is best possible. Indeed, if (e_1, e_2, e_3, e_4) is a basis of G, then the sequence $S = e_1 e_2 e_3 e_4^{\mathrm{ord}(e_4)-1}$ is zero-sum free, $|S| = \mathrm{ord}(e_4) + 2$, and $|\Sigma(S)| = 2^3 \mathrm{ord}(e_4) - 1 = 8|S| - 17$. We do not know whether the result holds true for groups G having a prime divisor q which is smaller than 13. Note that we have no information on the maximal length or on the structure of zero-sum free sequences over groups of rank 4. We only want to recall that there are zero-sum free sequences S over groups of rank 4 whose lengths are strictly larger than $\sum_{i=1}^{4}(n_i - 1)$ (see [4]).

2. Preliminaries. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set of integers. For $a, b \in \mathbb{Z}$ with $a \le b$, we define $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let G be an additively written finite abelian group. Let $\mathscr{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathscr{F}(G)$ are called *sequences* over G.

We write a sequence $S \in \mathscr{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathrm{v}_g(S)} \quad \text{with } \mathrm{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathrm{v}_g(S)$ the *multiplicity* of g in S. We say that S *contains* g if $\mathrm{v}_g(S) > 0$. The unit element $1 \in \mathscr{F}(G)$ is called the *empty sequence*. A sequence S_1 is called a *subsequence* of S if $S_1 \mid S$ in $\mathscr{F}(G)$ (equivalently, $\mathrm{v}_g(S_1) \le \mathrm{v}_g(S)$ for all $g \in G$). Let $S_1, S_2 \in \mathscr{F}(G)$; we denote by $S_1 S_2$ the sequence

$$\prod_{g \in G} g^{\mathrm{v}_g(S_1) + \mathrm{v}_g(S_2)} \in \mathscr{F}(G),$$

and by $S_1 S_2^{-1}$ the sequence

$$\prod_{g \in G} g^{\mathrm{v}_g(S_1) - \min\{\mathrm{v}_g(S_1), \mathrm{v}_g(S_2)\}} \in \mathscr{F}(G).$$

If a sequence $S \in \mathscr{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. For

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathrm{v}_g(S)} \in \mathscr{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} \mathrm{v}_g(S) \in \mathbb{N}_0$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathrm{v}_g(S) g \in G$ the *sum* of S,
- $\mathrm{supp}(S) = \{g \in G \mid \mathrm{v}_g(S) > 0\} \subseteq G$ the *support* of S,
- $\Sigma(S) = \{\sum_{i \in I} g_i \mid \emptyset \ne I \subseteq [1, n]\}$ the *set of sub*(*sequence*) *sums* of S.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *zero-sum free sequence* if $0 \notin \Sigma(S)$,
- a *minimal zero-sum sequence* if $|S| \geq 1$, $\sigma(S) = 0$, and S contains no proper and nontrivial zero-sum subsequence.

If G_1 is a group and $\varphi : G \to G_1$ a map, then $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ is a sequence over G_1.

We set $\mathsf{D}(G) = \max\{|S| \mid S \text{ is a minimal zero-sum sequence over } G\}$, the *Davenport constant* of G.

For any integer-valued function $f : A \to \mathbb{Z}$ defined on a finite set A, we write $\min(f) = \min\{f(a) \mid a \in A\}$ and $\max(f) = \max\{f(a) \mid a \in A\}$.

Next we list some necessary lemmas.

Lemma 2.1 ([6, Lemma 4.4]). *Let G be a finite abelian group and $X \subseteq G \setminus \{0\}$ be a generating set for G. Suppose that $f : G \to \mathbb{Z}$ is a function on G. Then*

$$\sum_{x \in X,\, g \in G} \max\{f(g + x) - f(g), 0\} \geq (\max(f) - \min(f))|X|.$$

Lemma 2.2. *Let G be a finite abelian group, $H \subseteq G$ a subgroup, $S \subseteq G$ a subset, and let $f : G/H \to \mathbb{Z}$ be defined by $f(a + H) = |(a + H) \bigcap S|$ for all $a \in G$. Suppose that $X \subseteq G \setminus \{0\}$ is a generating set for G and satisfies $|(S + x) \setminus S| \leq 7$ for all $x \in X$. Then*

$$\min(f) \geq \max(f) - 7.$$

In particular, if there exists an element $b \in G$ such that $f(b + H) \geq 8$, then $f(a + H) \geq 1$ for all $a \in G$.

Proof. Obviously, the assertion holds for $H = \{0\}$ and for $H = G$. Suppose that $\{0\} \subsetneq H \subsetneq G$. Let $A \subseteq G$ be such that $G = \bigcup_{a \in A}(a + H)$ and $|A| = |G/H|$. Since $\{x + H \mid x \in X\}$ is a generating set for G/H, choose $X' \subseteq X$ such that $\{x + H \mid x \in X'\}$ is a generating set for G/H and $|\{x + H \mid x \in X'\}| = |X'|$.

From $|(S + x) \setminus S| \leq 7$ for all $x \in X$ we deduce

$$\begin{aligned} 7|X'| &\geq \sum_{x \in X'} |(S + x) \setminus S| = \sum_{x \in X'} \sum_{a \in A} \big|\big((S + x) \cap (a + H)\big) \setminus \big(S \cap (a + H)\big)\big| \\ &\geq \sum_{x \in X'} \sum_{a \in A} \max\{f(a - x + H) - f(a + H), 0\} \\ &= \sum_{x \in -X'} \sum_{a \in A} \max\{f(a + H + x + H) - f(a + H), 0\}. \end{aligned}$$

Since $\{x + H \mid x \in X'\}$ is a generating set for G/H, so is $\{x + H \mid x \in -X'\}$.

Therefore, by Lemma 2.1,

$$\begin{aligned} 7|X'| &\geq \sum_{x\in -X'}\sum_{a\in A} \max\{f(a+H+x+H)-f(a+H),0\} \\ &\geq (\max(f)-\min(f))|X'|. \end{aligned}$$

From $|X'|\neq 0$ we obtain

$$\min(f)\geq \max(f)-7.$$

In particular, if $f(b+H)\geq 8$ for some $b\in G$, then for all $a\in G$, $f(a+H)\geq \min(f)\geq \max(f)-7\geq f(b+H)-7\geq 1$. ■

We also need the following simple and well-known result.

Lemma 2.3. *Let G be a finite abelian group and S be a zero-sum free sequence over G. Then*

(1) $|\Sigma(S)|\geq |S|$,
(2) $\mathsf{D}(G)\leq |G|$.

Proof. (1) Suppose $S=g_1\cdot\ldots\cdot g_l$. Then $g_1, g_1+g_2,\ldots, g_1+\ldots+g_l$ are all distinct. It follows that $|\Sigma(S)|\geq l=|S|$.

(2) Assume to the contrary that X is a zero-sum free sequence over G with length $|G|$. Then by (1), $|\Sigma(X)|\geq |G|$, which implies that $0\in\Sigma(X)$, a contradiction. ■

Lemma 2.4. *Let G be a finite abelian group and $X=X_1X_2$ be a zero-sum free sequence over G. Then*

(1) $|\Sigma(X)|\geq |\Sigma(X_1)|+|\Sigma(X_2)|$.
(2) *Let $H=\langle \mathrm{supp}(X_1)\rangle$ and let $\varphi: G\to G/H$ be the canonical epimorphism. If $\varphi(X_2)$ is a zero-sum free sequence over G/H, then*

$$|\Sigma(X)|\geq (|\Sigma(X_1)|+1)(|X_2|+1)-1.$$

Proof. (1) This follows by [3, Theorem 5.3.1].

(2) Since $\varphi(X_2)$ is a zero-sum free sequence over G/H, we deduce that $H\cap\Sigma(X_2)=\emptyset$ and $|\Sigma(\varphi(X_2))|\geq |X_2|$ by Lemma 2.3(1). Thus for any a in $\Sigma(X_2)$,

$$|\Sigma(X_0)\cap(a+H)|\geq |(\Sigma(X_1)+a)\cup\{a\}|=|\Sigma(X_1)|+1.$$

Therefore

$$\begin{aligned} |\Sigma(X_0)| &\geq |\Sigma(X_0)\cap H|+\sum_{a+H\in\Sigma(\varphi(X_2))}|\Sigma(X_0)\cap(a+H)| \\ &\geq |\Sigma(X_1)|+(|\Sigma(X_1)|+1)|\Sigma(\varphi(X_2))|\geq (|\Sigma(X_1)|+1)(|X_2|+1)-1. \end{aligned}$$ ■

3. The proof of Theorem 1.1. For the simplicity of formulations, we define C-sequences and C-groups. To begin with, a sequence X over a finite abelian group G is called a *C-sequence* if:

(i) $\langle \mathrm{supp}(X)\rangle = G$,
(ii) X is zero-sum free,
(iii) $|\sum(X)| \le 8|X| - 18$.

Furthermore, a finite abelian group G is called a *C-group* if:

(i) $\mathsf{r}(G) = 4$,
(ii) the smallest prime p dividing $|G|$ satisfies $p \ge 13$,
(iii) there exists a C-sequence over G.

Proof of Theorem 1.1. If Theorem 1.1 does not hold, then there exists a C-group. Let G_0 be the C-group with minimal order and let X_0 be a C-sequence over G_0 with minimal length.

We proceed by the following four claims:

CLAIM A. *Let X be a zero-sum free sequence over G_0 and $H = \langle \mathrm{supp}(X)\rangle$ with $r = \mathsf{r}(H)$. If $|H| < |G_0|$ or $|X| < |X_0|$, then*

$$|\Sigma(X)| \ge 2^{r-1}(|X| - r + 2) - 1.$$

Proof. By Theorem A and the hypotheses about G_0 and X_0, this follows directly. ■

CLAIM B.

(1) *Let H be a subgroup of G_0. Then for any $a \in G_0$,*

$$|\Sigma(X_0) \cap (a + H)| \ge \max_{g\in G_0} |\Sigma(X_0) \cap (g + H)| - 7 \ge |\Sigma(X_0) \cap H| - 7.$$

(2) *Suppose that X_0 has a factorization $X_0 = X_1X_2$ such that $H = \langle \mathrm{supp}(X_1)\rangle$ is a proper subgroup of G. If $|\Sigma(X_1)| \ge 7$, then*

$$|\Sigma(X_0)| \ge (\Sigma(X_1) + 1)|G/H| - 1.$$

Proof. (1) For any $x \mid X_0$, denote $H_x = \langle \mathrm{supp}(X_0x^{-1})\rangle$. Then $\mathsf{r}(H_x) \ge \mathsf{r}(G_0) - 1 = 3$. By Claim A and $|X_0x^{-1}| < |X_0|$, we get

$$\begin{aligned}|\Sigma(X_0x^{-1})| &\ge \min\{4(|X_0| - 1 - 3 + 2) - 1, 8(|X_0| - 1 - 4 + 2) - 1\}\\ &= 4|X_0| - 9.\end{aligned}$$

If $H_x \ne G_0$, then $x \notin H_x$. Thus $|\Sigma(X_0)| \ge 2|\Sigma(X_0x^{-1})| + 1 \ge 8|X_0| - 17$ by Lemma 2.4(2), a contradiction to X_0 being a C-sequence.

Therefore $H_x = G_0$ and $\mathsf{r}(H_x) = 4$. By Claim A and $|X_0x^{-1}| < |X_0|$,

$$|\Sigma(X_0x^{-1})| \ge 8(|X_0| - 1) - 17 = 8|X_0| - 25.$$

Let $S = \Sigma(X_0)$. Then $|S| \le 8|X_0| - 18$ and for all $x \mid X_0$,

$$\begin{aligned}|(S + x) \setminus S| &= |S \setminus (S - x)| \le |S \setminus \Sigma(X_0 x^{-1})| \\ &\le |S| - |\Sigma(X_0 x^{-1})| \le (8|X_0| - 18) - (8|X_0| - 25) \le 7.\end{aligned}$$

By $\langle \mathrm{supp}(X_0)\rangle = G_0$ and Lemma 2.2, for any $a \in G$,

$$|\Sigma(X_0) \cap (a + H)| \ge \max_{g \in G_0} |\Sigma(X_0) \cap (g + H)| - 7 \ge |\Sigma(X_0) \cap H| - 7.$$

(2) Since H is a proper subgroup of G_0, there exists $x \mid X_2$ such that $x \notin H$. Then

$$|\Sigma(X_0) \cap (x + H)| \ge |(\Sigma(X_1) + x) \cup \{x\}| \ge |\Sigma(X_1)| + 1 \ge 8.$$

For any $a \in G_0 \setminus H$, we get $|\Sigma(X_0) \cap (a + H)| \ge |\Sigma(X_0) \cap (x + H)| - 7 \ge 1$ by (1), which implies that $\Sigma(X_2) \cap (a + H) \ne \emptyset$.

Choose $b \in \Sigma(X_2) \cap (a + H)$. Then we have $|\Sigma(X_0) \cap (a + H)| \ge |(\Sigma(X_1) + b) \cup \{b\}| = |\Sigma(X_1)| + 1$ for all $a \in G_0 \setminus H$. Therefore

$$\begin{aligned}|\Sigma(X_0)| &\ge |\Sigma(X_1)| + (|\Sigma(X_1)| + 1)(|G/H| - 1) \\ &\ge (|\Sigma(X_1)| + 1)(|G/H|) - 1. \ \blacksquare\end{aligned}$$

Claim C. *Let X be a subsequence of X_0. If $H = \langle \mathrm{supp}(X)\rangle$ is a proper subgroup of G_0, then $\mathsf{r}(H) \le 3$.*

Proof. Assume to the contrary that $\mathsf{r}(H) = 4$. Then $|X| \ge 4$.

Let $\varphi : G_0 \to G_0/H$ denote the canonical epimorphism from G_0 to G_0/H with $\ker(\varphi) = H$. Then $\varphi(X_0)$ is a sequence over G_0/H. We can get a factorization of X_0,

$$X_0 = X \cdot X_1 \cdot \ldots \cdot X_\alpha \cdot X',$$

such that for $1 \le i \le \alpha$, $\varphi(X_i)$ is a minimal zero-sum sequence over G_0/H and $\varphi(X')$ is a zero-sum free sequence over G_0/H. Thus $|\Sigma(\varphi(X'))| \ge |X'|$ and $|X_0| \le |X| + \alpha \mathsf{D}(G/H) + |X'| \le |X| + \alpha|G/H| + |X'|$ by Lemma 2.3.

Let $Y = X \cdot \sigma(X_1) \cdot \ldots \cdot \sigma(X_\alpha)$. Then Y is a zero-sum free sequence over H. From $H < G_0$ and Claim A we have

$$|\Sigma(X_0) \cap H| \ge |\Sigma(Y) \cap H| \ge 8|Y| - 17.$$

For any $a \in \Sigma(X')$, we get $a \notin H$ and

$$|\Sigma(X_0) \cap (a + H)| \ge |\Sigma(Y \cdot a) \cap (a + H)| \ge |\Sigma(Y) \cap H| + 1 \ge 8|Y| - 16.$$

Let $A' \subseteq \Sigma(X')$ satisfy $\{a + H \mid a \in \Sigma(X')\} = \{a + H \mid a \in A'\}$ and $|A'| = |\varphi(\Sigma(X'))|$. Let $A \subseteq G_0$ be a subset with $A \supseteq A'$ such that $G_0 = \bigcup_{a \in A}(a + H)$ and $|A| = |G_0/H|$. Then for any $b \in A \setminus (A' \cup H)$,

$$|\Sigma(X_0) \cap (b + H)| \ge |\Sigma(X_0) \cap H| - 7 \ge 8|Y| - 24,$$

by Claim B(1).

Therefore,

$$\begin{aligned}|\Sigma(X_0)| &= \sum_{a\in A} |\Sigma(X_0)\cap(a+H)| \\ &\ge 8|Y|-17+(8|Y|-16)|\Sigma(\varphi(X'))| \\ &\quad +(8|Y|-24)(|G/H|-1-|\Sigma(\varphi(X'))|) \\ &\ge (8|Y|-24)|G/H|+8|\Sigma(\varphi(X'))|+7 \\ &\ge 8(|X|-3)(|G|/|H|-1)+8(|X|+\alpha|G|/|H|+|X'|)-17 \\ &\ge 8|X_0|-17,\end{aligned}$$

a contradiction. ∎

Claim D. *Let Y be a subsequence of X_0 with length 4. Then* $\langle \mathrm{supp}(Y)\rangle = G_0$.

Proof. Let X be the longest subsequence of X_0 such that $\langle \mathrm{supp}(X)\rangle \neq G_0$. Denote $H = \langle \mathrm{supp}(X)\rangle$. Then $\mathsf{r}(H) = 3$ by Claim C and $|G_0/H| \ge 13$ since G_0 is a C-group. Let $\varphi : G_0 \to G_0/H$ denote the canonical epimorphism.

We only need to prove that $|X| \le 3$. Assume to the contrary that $|X| \ge 4$. We distinguish three cases.

Case 1: $|X_0| \le ((|X|-1)|G_0/H|+4)/2$. From $H < G_0$ and Claim A, we have $|\Sigma(X)| \ge 4(|X|-1)-1 \ge 11$. Then by Claim B(2),

$$|\Sigma(X_0)| \ge (|\Sigma(X)|+1)|G_0/H|-1 \ge 4(|X|-1)|G_0/H|-1,$$

which implies that $|\Sigma(X_0)| \ge 8|X_0|-17$ since $|X_0| \le ((|X|-1)|G_0/H|+4)/2$, a contradiction.

Case 2: There exists no zero-sum free subsequence of $\varphi(X_0X^{-1})$ with length 6. Since $\varphi(X_0)$ is a sequence over G_0/H, we can get a factorization of X_0,

$$X_0 = X\cdot X_1\cdot\ldots\cdot X_\alpha\cdot X',$$

such that for $1\le i\le\alpha$, $\varphi(X_i)$ is a minimal zero-sum sequence over G_0/H and $\varphi(X')$ is a zero-sum free sequence over G_0/H. Thus $|X_0| = |X|+|X_1|+\cdots+|X_\alpha|+|X'| \le |X|+|X'|+6\alpha$ and $|\Sigma(\varphi(X'))| \ge |X'|$ by Lemma 2.3.

Let $Y = X\cdot\sigma(X_1)\cdot\ldots\cdot\sigma(X_\alpha)$. Then Y is a zero-sum free sequence over H. By Claim A and $H < G_0$, we have

$$|\Sigma(X_0)\cap H| \ge |\Sigma(Y)\cap H| \ge 4|Y|-5.$$

For any $a\in\Sigma(X')$, we obtain $a\notin H$ and

$$|\Sigma(X_0)\cap(a+H)| \ge |\Sigma(Y\cdot a)\cap(a+H)| \ge |\Sigma(Y)\cap H|+1 \ge 4|Y|-4.$$

Let $A' \subseteq \Sigma(X')$ satisfy $\{a+H \mid a\in\Sigma(X')\} = \{a+H \mid a\in A'\}$ and $|A'| = |\varphi(\Sigma(X'))|$. Let $A\subseteq G_0$ be a subset with $A\supseteq A'$ such that

$G_0 = \bigcup_{a\in A}(a+H)$ and $|A| = |G_0/H|$. Then for any $b \in A \setminus (\Sigma(X') \cup H)$,

$$|\Sigma(X_0) \cap (b+H)| \geq |\Sigma(X_0) \cap H| - 7 \geq 4|Y| - 12,$$

by Claim B(1). Therefore,

$$\begin{aligned}|\Sigma(X_0)| &= \sum_{a\in A} |\Sigma(X_0) \cap (a+H)| \\ &\geq 4|Y| - 5 + (4|Y|-4)|\Sigma(\varphi(X'))| \\ &\quad + (4|Y|-12)(|G/H| - 1 - |\Sigma(\varphi(X'))|) \\ &\geq (4|Y|-12)|G/H| + 8|\Sigma(\varphi(X'))| + 7 \\ &\geq (4|X| + 4\alpha - 12)|G/H| + 8|X'| + 7.\end{aligned}$$

Since $|X| \geq 4$, $|G/H| \geq 13$, and $|X_0| \leq |X| + |X'| + 6\alpha$, we conclude that $|\Sigma(X_0)| \geq 8|X_0| - 17$, a contradiction.

Case 3: $|X_0| > ((|X|-1)|G/H|+4)/2$ and there exists a subsequence X_1 of X_0X^{-1} such that $\varphi(X_1)$ is a zero-sum free subsequence over G_0/H of length 6. Since $|X_0| > ((|X|-1)|G/H|+4)/2$, we obtain

$$|X_0| - 2|X| > \frac{|X|(|G/H|-4) - |G/H| + 4}{2} \geq 7.$$

Denote

$$X_2 = X_0(XX_1)^{-1}.$$

Then $|X_2| = |X_0| - |XX_1| > |X|+1$. Thus $\langle X_2\rangle = G_0$ since X is the longest subsequence of X_0 such that $\langle \mathrm{supp}(X)\rangle \neq G_0$. Hence $|\Sigma(X_2)| \geq 8|X_2| - 17$ from $|X_2| < |X_0|$ and Claim A.

By $H < G_0$ and Claim A, $|\Sigma(X)| \geq 4(|X|-1) - 1$. It follows that $|\Sigma(XX_1)| \geq 4(|X|-1)(|\Sigma(\varphi(X_1))|+1) - 1 \geq 8|XX_1|$ from Lemma 2.4(2), $|X| \geq 4$ and $|X_1| = 6$.

Therefore by Lemma 2.4(1),

$$|\Sigma(X_0)| \geq |\Sigma(XX_1)| + |\Sigma(X_2)| \geq 8|XX_1| + 8|X_2| - 17 = 8|X_0| - 17,$$

a contradiction. ■

Now we finish the proof of Theorem 1.1 by distinguishing the following two cases.

Suppose that $|X_0| \geq 13$. Denote $X_0 = x_1 \cdot \ldots \cdot x_n$. Then by Claim D, any four elements of X_0 are independent, which implies that $x_i, x_j + x_k$, $1 \leq i \leq n$, $1 \leq j < k \leq n$, are all different elements in G_0. Therefore,

$$|\Sigma(X_0)| \geq n + n(n-1)/2 \geq 8|X_0| - 17,$$

a contradiction.

Suppose that $|X_0| \leq 12$. Let X be a subsequence of X_0 of length 3 and $H = \langle \mathrm{supp}(X)\rangle$ be a proper subgroup of G_0. Then by Claim D, the three elements of X must be independent, which implies that $|\Sigma(X)| = 7$. It follows by Claim B(2) that

$$|\Sigma(X_0)| \geq (|\Sigma(X)| + 1)|G_0/H| - 1 \geq 8 \cdot 13 - 1 \geq 8|X_0| - 17,$$

a contradiction. ∎

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