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COHOMOLOGICAL DIMENSION FILTRATION AND ANNIHILATORS OF TOP LOCAL COHOMOLOGY MODULES

BY

ALI ATAZADEH, MONIREH SEDGHI and
REZA NAGHIPOUR (Tabriz)

Abstract. Let $\mathfrak{a}$ denote an ideal in a Noetherian ring R, and M a finitely generated R-module. We introduce the concept of the cohomological dimension filtration $\mathscr{M} = \{M_i\}_{i=0}^c$, where $c = \operatorname{cd}(\mathfrak{a}, M)$ and M_i denotes the largest submodule of M such that $\operatorname{cd}(\mathfrak{a}, M_i) \leq i$. Some properties of this filtration are investigated. In particular, if $(R, \mathfrak{m})$ is local and $c = \dim M$, we are able to determine the annihilator of the top local cohomology module $H_{\mathfrak{a}}^c(M)$, namely $\operatorname{Ann}_R(H_{\mathfrak{a}}^c(M)) = \operatorname{Ann}_R(M/M_{c-1})$. As a consequence, there exists an ideal $\mathfrak{b}$ of R such that $\operatorname{Ann}_R(H_{\mathfrak{a}}^c(M)) = \operatorname{Ann}_R(M/H_{\mathfrak{b}}^0(M))$. This generalizes the main results of Bahmanpour et al. (2012) and Lynch (2012).

1. Introduction. Let R be an arbitrary commutative Noetherian ring (with identity), $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module. An important problem concerning local cohomology is to determine the annihilators of the ith local cohomology module $H_{\mathfrak{a}}^i(M)$. This problem has been studied by several authors (see for example [11], [12], [13], [15]–[17]) and has led to some interesting results. In particular, Bahmanpour et al. [2] proved an interesting result about the annihilator $\operatorname{Ann}_R(H_{\mathfrak{m}}^d(M))$ of the dth local cohomology module when $(R, \mathfrak{m})$ is a complete local ring.

The purpose of the present paper is to introduce the concept of the cohomological dimension filtration $\mathscr{M} = \{M_i\}_{i=0}^c$, where $c = \operatorname{cd}(\mathfrak{a}, M)$ and M_i denotes the largest submodule of M such that $\operatorname{cd}(\mathfrak{a}, M_i) \leq i$. Because M is a Noetherian R-module, it follows easily from [9, Theorem 2.2] that the submodules M_i are well-defined. They also form an increasing family of submodules. Some properties of this filtration are investigated. In particular, if $(R, \mathfrak{m})$ is local and $c = \dim M$, we are able to determine the annihilator of the top local cohomology module $H_{\mathfrak{a}}^c(M)$. In fact, it is shown that

$$\operatorname{Ann}_R(H_{\mathfrak{a}}^c(M)) = \operatorname{Ann}_R(M/M_{c-1}).$$

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As a consequence, there exists an ideal $\mathfrak{b}$ of R such that

$$\operatorname{Ann}_R(H^c_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M/H^0_{\mathfrak{b}}(M)).$$

This generalizes the main results of [2] and [12].

As a main result in the second section, we describe in more detail the structure of the cohomological dimension filtration $\mathscr{M} = \{M_i\}_{i=0}^c$ in terms of the reduced primary decomposition of 0 in M. Namely, if $0 = \bigcap_{j=1}^n N_j$ denotes a reduced primary decomposition of the zero submodule in M such that N_j is a $\mathfrak{p}_j$-primary submodule of M and $\mathfrak{a}_i := \prod_{\operatorname{cd}(\mathfrak{a},R/\mathfrak{p}_j)\leq i} \mathfrak{p}_j$, we shall show:

THEOREM 1.1. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$ with respect to $\mathfrak{a}$ and let $\mathscr{M} = \{M_i\}_{i=0}^c$ be the cohomological dimension filtration of M. Then, for all integers $0 \leq i \leq c$:*

(i) $M_i = H^0_{\mathfrak{a}_i}(M) = \bigcap_{\operatorname{cd}(\mathfrak{a},R/\mathfrak{p}_j)>i} N_j$,
(ii) $\operatorname{Ass}_R M_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq i\}$,
(iii) $\operatorname{Ass}_R M/M_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) > i\}$,
(iv) $\operatorname{Ass}_R M_i/M_{i-1} = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = i\}$.

Pursuing this point of view further we establish some results about the annihilator of top local cohomology modules. More precisely, as a main result of the third section, we derive the following consequence of Theorem 1.1, which will describe the annihilator of the top local cohomology module $H^{\dim M}_{\mathfrak{a}}(M)$.

THEOREM 1.2. *Let $\mathfrak{a}$ denote an ideal of a local (Noetherian) ring R and let M be a finitely generated R-module of dimension c such that $H^c_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}_R(H^c_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M/M_{c-1}).$$

Several corollaries of this result are given. A typical one is the following generalization of [2, Theorem 2.6] and [12, Theorem 2.4] for an ideal $\mathfrak{a}$ in an arbitrary local ring R.

COROLLARY 1.3. *Let R be a local (Noetherian) ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M/H^0_{\mathfrak{b}}(M)) = \operatorname{Ann}_R\Big(M/\bigcap_{\operatorname{cd}(\mathfrak{a},R/\mathfrak{p}_j)=d} N_j\Big).$$

Here $0 = \bigcap_{j=1}^n N_j$ denotes a reduced primary decomposition of the zero submodule in M, N_j is a $\mathfrak{p}_j$-primary submodule of M for all $j = 1, \ldots, n$, and $\mathfrak{b} := \prod_{\operatorname{cd}(\mathfrak{a},R/\mathfrak{p}_j)\neq d} \mathfrak{p}_j$.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ will be an ideal of R. For any R-module L, the *ith local cohomology module* of L with support in $V(\mathfrak{a})$ is defined by

$$H^i_{\mathfrak{a}}(L) := \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, L).$$

For each R-module L, we denote by $\operatorname{Assh}_R L$ (resp. $\operatorname{mAss}_R L$) the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$ (resp. the set of minimal primes of $\operatorname{Ass}_R L$). For an Artinian R-module A, we shall use $\operatorname{Att}_R A$ to denote the set of attached prime ideals of A. Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. In addition, for any ideal $\mathfrak{b}$ of R, the *radical* of $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Finally, if $(R, \mathfrak{m})$ is a local (Noetherian) ring and M a finitely generated R-module, then $\hat{R}$ (resp. $\hat{M}$) denotes the completion of R (resp. M) with respect to the $\mathfrak{m}$-adic topology. For any unexplained notation and terminology we refer the reader to [4] and [14].

2. Cohomological dimension filtration. For an R-module M, the *cohomological dimension* of M with respect to an ideal $\mathfrak{a}$ of R is defined as

$$\operatorname{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

If $(R, \mathfrak{m})$ is local and $\mathfrak{a} = \mathfrak{m}$, it is known that $\operatorname{cd}(\mathfrak{a}, M) = \dim M$.

The purpose of this section is to introduce the notion of *cohomological dimension filtration* (abbreviated as cd-filtration) of M, which is a generalization of the concept of *dimension filtration* introduced by P. Schenzel [18]. Specifically, let $\mathfrak{a}$ be an ideal of R, and M a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$. For an integer $0 \leq i \leq c$, let M_i denote the largest submodule of M such that $\operatorname{cd}(\mathfrak{a}, M_i) \leq i$. In view of the maximal condition of a Noetherian R-module, it follows easily from [9, Theorem 2.2] that the submodules M_i of M are well-defined. Moreover, it is clear that $M_{i-1} \subseteq M_i$ for all $1 \leq i \leq c$.

Definition 2.1. The increasing filtration $\mathscr{M} = \{M_i\}_{i=0}^c$ of submodules of M is called the *cohomological dimension filtration* (abbreviated as cd-filtration) of M, where $c = \operatorname{cd}(\mathfrak{a}, M)$.

Before investigating some properties of the cohomological dimension filtration, we state the following lemma which plays a key role in this paper.

Lemma 2.2 (see [9, Theorem 2.2]). *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M and N be two finitely generated R-modules such that* $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. *Then*

$$\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M).$$

Proposition 2.3. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$ and let $\mathscr{M} = \{M_i\}_{i=0}^{c}$ be the cd-filtration of M. Then, for all integers $0 \leq i \leq c$, we have*

$$M_i = H^0_{\mathfrak{a}_i}(M) = \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) > i} N_j.$$

Here $0 = \bigcap_{j=1}^{n} N_j$ denotes a reduced primary decomposition of the zero submodule in M, N_j is a $\mathfrak{p}_j$-primary submodule of M, and $\mathfrak{a}_i = \prod_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) \leq i} \mathfrak{p}_j$.

Proof. First, we show that $H^0_{\mathfrak{a}_i}(M) = \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) > i} N_j$. The $\supseteq$ inclusion follows by easy arguments about the primary decomposition of the zero submodule of M. Suppose that there exists $x \in H^0_{\mathfrak{a}_i}(M)$ such that $x \notin \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) > i} N_j$. Then there exists an integer t such that $x \notin N_t$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_t) > i$. Now, as $x \in H^0_{\mathfrak{a}_i}(M)$, it follows that there is an integer $s_i \geq 1$ such that $\mathfrak{a}_i^{s_i} x = 0$, and so $\mathfrak{a}_i^{s_i} x \subseteq N_t$. Because $x \notin N_t$ and N_t is a $\mathfrak{p}_t$-primary submodule, it follows that $\mathfrak{a}_i \subseteq \mathfrak{p}_t$. Hence there is an integer j such that $\mathfrak{p}_j \subseteq \mathfrak{p}_t$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) \leq i$. Therefore, in view of Lemma 2.2,

$$\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_t) \leq \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) \leq i,$$

which is a contradiction.

Now we show that $M_i = H^0_{\mathfrak{a}_i}(M)$. Let $x \in M_i$. Then, in view of Lemma 2.2, $\operatorname{cd}(\mathfrak{a}, Rx) \leq i$. Now, let $\mathfrak{p}$ be a minimal prime ideal over $\operatorname{Ann}_R Rx$. Then, using Lemma 2.2, we see that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq i$. On the other hand, since $\mathfrak{p} \in \operatorname{Ass}_R Rx$, it follows that $\mathfrak{p} \in \operatorname{Ass}_R M$, and so there is $1 \leq j \leq n$ such that $\mathfrak{p}_j = \mathfrak{p}$. Hence

$$\mathfrak{a}_i \subseteq \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) \leq i} \mathfrak{p}_j \subseteq \bigcap_{\mathfrak{p} \in \operatorname{mAss}_R(Rx)} \mathfrak{p} = \operatorname{Rad}(\operatorname{Ann}_R Rx).$$

Therefore, there exists an integer $n_i \geq 1$ such that $\mathfrak{a}_i^{n_i} \subseteq \operatorname{Ann}_R Rx$, and hence $\mathfrak{a}_i^{n_i} x = 0$. That is, $x \in H^0_{\mathfrak{a}_i}(M)$, and so $M_i \subseteq H^0_{\mathfrak{a}_i}(M)$.

On the other hand, as $\operatorname{Supp} H^0_{\mathfrak{a}_i}(M) \subseteq V(\mathfrak{a}_i)$, it follows that for every $\mathfrak{p} \in \operatorname{Supp} H^0_{\mathfrak{a}_i}(M)$, there exists an integer $j \geq 1$ such that $\mathfrak{p}_j \subseteq \mathfrak{p}$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) \leq i$. Since $\operatorname{Supp} R/\mathfrak{p} \subseteq \operatorname{Supp} R/\mathfrak{p}_j$, Lemma 2.2 implies that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j) \leq i$. Therefore, in view of [19, Corollary 2.2], we have $\operatorname{cd}(\mathfrak{a}, H^0_{\mathfrak{a}_i}(M)) \leq i$. Now, the maximality of M_i yields $M_i = H^0_{\mathfrak{a}_i}(M)$, as required. ■

Definition 2.4. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$. We denote by $T_R(\mathfrak{a}, M)$ the largest submodule of M such that $\operatorname{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < c$.

Using Lemma 2.2, it is easy to check that $T_R(\mathfrak{a}, M) = \bigcup\{N \leq M \mid \operatorname{cd}(\mathfrak{a}, N) < c\}$. In particular, for a local ring $(R, \mathfrak{m})$, we denote $T_R(\mathfrak{m}, M)$ by $T_R(M)$. Thus

$$T_R(M) = \bigcup\{N \leq M \mid \dim N < \dim M\}.$$

Remark 2.5. Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R, and M a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$. Let $\{M_i\}_{i=0}^c$ be a cd-filtration of M. Then $T_R(\mathfrak{a}, M) = M_{c-1}$, and by Proposition 2.3 we have

$$T_R(\mathfrak{a}, M) = H^0_{\mathfrak{b}}(M) = \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=c} N_j,$$

where $0 = \bigcap_{j=1}^n N_j$ denotes a reduced primary decomposition of the zero submodule in M, N_j is a $\mathfrak{p}_j$-primary submodule of M, and $\mathfrak{b} = \prod_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)\neq c} \mathfrak{p}_j$.

The next proposition provides information about the associated primes of the cohomological dimension filtration of M.

Proposition 2.6. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$, and let $\{M_i\}_{i=0}^c$ be the cd-filtration of M. Then, for all integers $0 \leq i \leq c$:*

(i) $\operatorname{Ass}_R M_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq i\}$,
(ii) $\operatorname{Ass}_R M/M_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) > i\}$,
(iii) $\operatorname{Ass}_R M_i/M_{i-1} = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = i\}$.

Proof. In view of Proposition 2.3, $M_i = H^0_{\mathfrak{a}_i}(M)$, and so by [3, Section 2.1, Proposition 10],

$$\operatorname{Ass}_R M_i = \operatorname{Ass}_R M \cap V(\mathfrak{a}_i).$$

Now, (i) follows from Lemma 2.2. To show (ii), use [4, Exercise 2.1.12]. Finally, for (iii), as $M_i/M_{i-1} \subseteq M/M_{i-1}$, it follows from (ii) that $\operatorname{Ass}_R M_i/M_{i-1} \subseteq \operatorname{Ass}_R M$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \geq i$ for every $\mathfrak{p} \in \operatorname{Ass}_R M_i/M_{i-1}$. Furthermore, in view of the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

and Lemma 2.2, we have

$$\operatorname{cd}(\mathfrak{a}, M_i/M_{i-1}) \leq \operatorname{cd}(\mathfrak{a}, M_i) \leq i.$$

Hence, using again Lemma 2.2, we deduce that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq i$ for all $\mathfrak{p} \in \operatorname{Ass}_R M_i/M_{i-1}$. Therefore,

$$\operatorname{Ass}_R M_i/M_{i-1} \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = i\}.$$

Now, let $\mathfrak{p} \in \operatorname{Ass}_R M$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = i$. Then $\mathfrak{p} \in \operatorname{Ass}_R M_i$ by (i). As $\mathfrak{p} \notin \operatorname{Ass}_R M_{i-1}$, it follows from the exact sequence,

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

that $\mathfrak{p} \in \operatorname{Ass}_R M_i/M_{i-1}$, and so

$$\operatorname{Ass}_R M_i/M_{i-1} = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = i\}. \blacksquare$$

3. Annihilators of top local cohomology modules. The main point of this section is to determine the annihilator of top local cohomology modules in terms of the reduced primary decomposition of the zero submodule. Our main result is Theorem 3.5. The following lemmas and proposition play a key role in the proof.

LEMMA 3.1 (cf. [4, Lemma 7.3.1]). *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of finite dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Set $G := M/T_R(\mathfrak{a}, M)$. Then:*

(i) $\operatorname{cd}(\mathfrak{a}, G) = d$,
(ii) *G has no non-zero submodule of cohomological dimension (with respect to $\mathfrak{a}$) less than d,*
(iii) $\operatorname{Ass}_R G = \operatorname{Att}_R H^d_{\mathfrak{a}}(G) = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}$,
(iv) $H^d_{\mathfrak{a}}(G) \cong H^d_{\mathfrak{a}}(M)$.

Proof. The assertion follows easily from Proposition 2.6(ii), [8, Theorem 2.5], Lemma 2.2 and the exact sequence

$$0 \to T_R(\mathfrak{a}, M) \to M \to G \to 0. \blacksquare$$

Before stating the next lemma let us recall the important notion of a cofinite module with respect to an ideal. For an ideal $\mathfrak{a}$ of R, an R-module M is said to be $\mathfrak{a}$-*cofinite* if M has support in $V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for each i. This concept was introduced by R. Hartshorne [10].

LEMMA 3.2. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring such that $\hat{R}$ is integral over R. Let $\mathfrak{a}$ be an ideal of R and M a non-zero finitely generated R-module of dimension d. Then*

$$\operatorname{Att}_R(H^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \operatorname{Assh}_R M \mid \operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}\}.$$

Proof. Let $\mathfrak{p} \in \operatorname{Att}_R(H^d_{\mathfrak{a}}(M))$. Then [7, Theorem A] implies that $\mathfrak{p} \in \operatorname{Assh}_R M$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d$. Now, since by [6, Theorem 3] the R-module $H^d_{\mathfrak{a}}(M)$ is Artinian and $\mathfrak{a}$-cofinite, it follows from $\mathfrak{p} \in \operatorname{Att}_R(H^d_{\mathfrak{a}}(M))$ and [1, Theorem 2.2] that $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$. Hence

$$\operatorname{Att}_R(H^d_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \operatorname{Assh}_R M \mid \operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}\}.$$

To prove the reverse inclusion, let $\mathfrak{p} \in \operatorname{Assh}_R M$ be such that $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$. Then, as $\dim R/\mathfrak{p} = d$ and

$$H^d_{\mathfrak{a}}(R/\mathfrak{p}) \cong H^d_{\mathfrak{a}(R/\mathfrak{p})}(R/\mathfrak{p}) \cong H^d_{\mathfrak{m}}(R/\mathfrak{p}),$$

it follows that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d$, and so in view of [7, Theorem A], $\mathfrak{p} \in \operatorname{Att}_R(H^d_{\mathfrak{a}}(M))$. ∎

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a complete local (Noetherian) ring, $\mathfrak{a}$ an ideal of R, and M a non-zero finitely generated R-module of dimension d. Then*

$$\operatorname{Att}_R(H^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \operatorname{Assh}_R M \mid \operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}\}.$$

Proof. This follows from Lemma 3.2. ∎

The following proposition will serve to shorten the proof of the main theorem.

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a complete local (Noetherian) ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M/T_R(\mathfrak{a}, M)).$$

Proof. Let $G := M/T_R(\mathfrak{a}, M)$. In view of Lemma 3.1, it is enough to show that $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(G)) = \operatorname{Ann}_R(G)$. To do so, it follows easily from Lemma 3.1(iii) and Corollary 3.3 that $\mathfrak{m} = \operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R(G))$. Consequently, $H^d_{\mathfrak{a}}(G) \cong H^d_{\mathfrak{m}}(G)$, and hence, in view of [2, Theorem 2.6],

$$\operatorname{Ann}_R(H^d_{\mathfrak{a}}(G)) = \operatorname{Ann}_R(G/T_R(G)).$$

Now, since

$$\operatorname{cd}(\mathfrak{a}, T_R(G)) \leq \dim T_R(G) < \dim G,$$

Lemma 3.1(ii) shows that $T_R(G) = 0$, and so $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(G)) = \operatorname{Ann}_R(G)$, as required. ∎

We are now ready to prove the main theorem of this section, which generalizes all of the previous results concerning the annihilators of top local cohomology modules.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M/T_R(\mathfrak{a}, M)).$$

Proof. In view of Lemma 3.1, we may assume that $T_R(\mathfrak{a}, M) = 0$. Now, as

$$\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)),$$

it is enough to show that $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) \subseteq \operatorname{Ann}_R(M)$. To this end, we let $x \in \operatorname{Ann}_R(H^d_{\mathfrak{a}}(M))$ and show that $xM = 0$. Suppose that $xM \neq 0$. Then, as $T_R(\mathfrak{a}, M) = 0$, it follows that $\operatorname{cd}(\mathfrak{a}, xM) = d$. Hence $\operatorname{cd}(\mathfrak{a}\hat{R}, x\hat{M}) = d$. This implies $xH^d_{\mathfrak{a}\hat{R}}(\hat{M}) \neq 0$. Indeed, if $xH^d_{\mathfrak{a}\hat{R}}(\hat{M}) = 0$, then $x\hat{R} \subseteq \mathrm{Ann}_{\hat{R}}(H^d_{\mathfrak{a}\hat{R}}(\hat{M}))$. Hence, in view of Proposition 3.4,

$$x\hat{R} \subseteq \mathrm{Ann}_{\hat{R}}(\hat{M}/T_{\hat{R}}(\mathfrak{a}\hat{R}, \hat{M})),$$

and so $x\hat{M} \subseteq T_{\hat{R}}(\mathfrak{a}\hat{R}, \hat{M})$. Therefore, $\mathrm{cd}(\mathfrak{a}\hat{R}, x\hat{M}) < d$, a contradiction.

Consequently, $xH^d_{\mathfrak{a}}(M) \neq 0$, that is, $x \notin \mathrm{Ann}_R(H^d_{\mathfrak{a}}(M))$, contrary to assumption. ∎

The first application of Theorem 3.5 improves a result of Coung et al. [5, Lemma 3.2].

COROLLARY 3.6. *Let R be a local (Noetherian) ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then $V(\mathrm{Ann}_R(H^d_{\mathfrak{a}}(M))) = \mathrm{Supp}(M/T_R(\mathfrak{a}, M))$.*

Proof. In view of Theorem 3.5, we have

$$V(\mathrm{Ann}_R(H^d_{\mathfrak{a}}(M)) = V(\mathrm{Ann}_R(M/T_R(\mathfrak{a}, M))) = \mathrm{Supp}(M/T_R(\mathfrak{a}, M)). \ ∎$$

COROLLARY 3.7. *Let R be a local (Noetherian) ring and $\mathfrak{a}$ an ideal of R such that $H^{\dim R}_{\mathfrak{a}}(R) \neq 0$. Then $\mathrm{Ann}_R(H^{\dim R}_{\mathfrak{a}}(R))$ is the largest ideal of R such that*

$$\mathrm{cd}(\mathfrak{a}, \mathrm{Ann}_R(H^{\dim R}_{\mathfrak{a}}(R))) < \dim R.$$

Proof. The assertion follows from Theorem 3.5. ∎

PROPOSITION 3.8. *Let R be a local (Noetherian) ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\mathrm{Ann}_R(H^d_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/H^0_{\mathfrak{b}}(M)) = \mathrm{Ann}_R\Big(M/\bigcap_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=d} N_j\Big).$$

Here $0 = \bigcap_{j=1}^n N_j$ denotes a reduced primary decomposition of the zero submodule in M, N_j is a $\mathfrak{p}_j$-primary submodule of M for all $j = 1, \ldots, n$, and $\mathfrak{b} = \prod_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j)\neq d} \mathfrak{p}_j$.

Proof. The assertion follows easily from Theorem 3.5 and Remark 2.5. ∎

The following corollary is a generalization of the main result of [12, Theorem 2.4].

COROLLARY 3.9. *Let R be a local (Noetherian) ring of dimension d and $\mathfrak{a}$ an ideal of R such that $H^d_{\mathfrak{a}}(R) \neq 0$. Then*

$$\mathrm{Ann}_R(H^d_{\mathfrak{a}}(R)) = H^0_{\mathfrak{b}}(R) = \bigcap_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=d} \mathfrak{q}_j,$$

where $0 = \bigcap_{j=1}^n \mathfrak{q}_j$ is a reduced primary decomposition of the zero ideal of R, $\mathfrak{q}_j$ is a $\mathfrak{p}_j$-primary ideal of R for all $1 \leq j \leq n$, and $\mathfrak{b} = \prod_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j)\neq d} \mathfrak{p}_j$.

Proof. The result follows readily from Proposition 3.8. ∎

PROPOSITION 3.10. *Let R be a local (Noetherian) ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then*

(i) $\operatorname{Rad}(\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = \bigcap_{\mathfrak{p}\in \operatorname{Ass}_R M,\, \operatorname{cd}(\mathfrak{a},R/\mathfrak{p})=d} \mathfrak{p}$,
(ii) $\operatorname{Supp}(H^d_{\mathfrak{a}}(M)) \subseteq \bigcup_{\mathfrak{p}\in \operatorname{Ass}_R M,\, \operatorname{cd}(\mathfrak{a},R/\mathfrak{p})=d} V(\mathfrak{p}+\mathfrak{a})$.

Proof. Assertion (i) follows from Proposition 3.8. In order to prove (ii), by using (i) we have

$$\begin{aligned}\operatorname{Supp}(H^d_{\mathfrak{a}}(M)) &\subseteq V(\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M))) = V(\operatorname{Rad}(\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M))) \\ &= V\Big(\bigcap_{\mathfrak{p}\in \operatorname{Ass}_R M,\, \operatorname{cd}(\mathfrak{a},R/\mathfrak{p})=d} \mathfrak{p}\Big) = \bigcup_{\mathfrak{p}\in \operatorname{Ass}_R M,\, \operatorname{cd}(\mathfrak{a},R/\mathfrak{p})=d} V(\mathfrak{p}).\end{aligned}$$

Now, as $\operatorname{Supp}(H^d_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$, it follows that

$$\operatorname{Supp}(H^d_{\mathfrak{a}}(M)) \subseteq \Big(\bigcup_{\mathfrak{p}\in \operatorname{Ass}_R M,\, \operatorname{cd}(\mathfrak{a},R/\mathfrak{p})=d} V(\mathfrak{p})\Big) \cap V(\mathfrak{a}),$$

and the desired result follows. ∎

COROLLARY 3.11. *Let R be a local (Noetherian) ring, $\mathfrak{a}$ an ideal of R, and $x \in R$. Let M be a non-zero finitely generated R-module of dimension d such that $H^d_{\mathfrak{a}}(M) \neq 0$. Then $H^d_{\mathfrak{a}}(xM) = 0$ if and only if $xH^d_{\mathfrak{a}}(M) = 0$. In particular, $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = 0$ if and only if $\operatorname{cd}(\mathfrak{a}, rM) = d$ for every non-zero element r of R.*

Proof. The assertion follows readily from Theorem 3.5. ∎

The following result is a generalization of [12, Corollary 2.5] and [2, Corollary 2.9].

COROLLARY 3.12. *Let $(R,\mathfrak{m})$ be a local (Noetherian) ring of dimension d, and $\mathfrak{a}$ an ideal of R. Then the following conditions are equivalent:*

(i) $\operatorname{Ann}_R H^d_{\mathfrak{a}}(R) = 0$,
(ii) $\operatorname{Ass}_R R = \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$.

Proof. (i)⇒(ii). Let $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(R))=0$ and $\mathfrak{p}\in\operatorname{Ass}_R R$. Then $R/\mathfrak{p} \cong Rx$ for some x ($\neq 0$) $\in R$. Thus Corollary 3.11 yields $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \operatorname{cd}(\mathfrak{a}, Rx) = d$, and so by [7, Theorem A], $\mathfrak{p} \in \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$, as required.

(ii)⇒(i). In view of Theorem 3.5 and Corollary 3.11, it is enough to show that $\operatorname{cd}(\mathfrak{a}, Rx) = d$ for every non-zero element x of R. In view of [19, Corollary 2.2], there exists $\mathfrak{p} \in \operatorname{Ass}_R Rx$ such that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \operatorname{cd}(\mathfrak{a}, Rx)$. By (ii), $\mathfrak{p} \in \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$, and so $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d$. Therefore $\operatorname{cd}(\mathfrak{a}, Rx) = d$, as required. ∎

COROLLARY 3.13. *Let* $(R, \mathfrak{m})$ *be a local* (*Noetherian*) *domain of dimension* d *and* $\mathfrak{a}$ *an ideal of* R *such that* $H^d_{\mathfrak{a}}(R) \neq 0$. *Then* $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(R)) = 0$.

Proof. Since $\operatorname{Ass}_R R = 0$, the assertion follows immediately from Corollary 3.12. ∎

COROLLARY 3.14. *Let* R *be a Noetherian domain and* $\mathfrak{a}$ *an ideal of* R *with* $\operatorname{ht} \mathfrak{a} = n$. *Then* $\operatorname{Ann}_R(H^n_{\mathfrak{a}}(R)) = 0$.

Proof. Suppose that $\operatorname{Ann}_R(H^n_{\mathfrak{a}}(R)) \neq 0$. Then there exists a non-zero element r in $\operatorname{Ann}_R(H^n_{\mathfrak{a}}(R))$. Hence, $rH^n_{\mathfrak{a}}(R) = 0$. Now, let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{a}$ such that $\operatorname{ht} \mathfrak{q} = n$. Then $R_{\mathfrak{q}}$ is a local (Noetherian) domain of dimension n and $rH^n_{\mathfrak{q}R_{\mathfrak{q}}}(R_{\mathfrak{q}}) = 0$. Thus $r/1 \ (\neq 0) \in \operatorname{Ann}_{R_{\mathfrak{q}}}(H^n_{\mathfrak{q}R_{\mathfrak{q}}}(R_{\mathfrak{q}}))$, and so by Corollary 3.13, we achieve a contradiction. ∎

COROLLARY 3.15. *Let* $(R, \mathfrak{m})$ *be a local* (*Noetherian*) *ring of dimension* d *and* $\mathfrak{a}$ *an ideal of* R *such that* $\operatorname{grade} \mathfrak{a} = d$. *Then* $\operatorname{Ass}_R R = \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$.

Proof. The assertion follows from [12, Theorem 3.3] and Corollary 3.12. ∎

COROLLARY 3.16. *Let* R *be a local* (*Noetherian*) *ring and* $\mathfrak{a}$ *an ideal of* R. *Let* M *be a non-zero finitely generated* R*-module of dimension* d *such that* $\operatorname{Ass}_R M = \operatorname{Att}_R H^d_{\mathfrak{a}}(M)$. *Then* $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R M$.

Proof. Let $0 = \bigcap_{j=1}^n N_j$ denote a reduced primary decomposition of the zero submodule in M such that N_j is a $\mathfrak{p}_j$-primary submodule of M for all $j = 1, \ldots, n$. Then, as $\operatorname{Ass}_R M = \operatorname{Att}_R H^d_{\mathfrak{a}}(M)$ it follows that $\bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=d} N_j = 0$, and so by Proposition 3.8 $\operatorname{Ann}_R(H^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R M$, as required. ∎

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Ali Atazadeh, Monireh Sedghi
Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, Iran
E-mail: aalzp2002@yahoo.com
m_sedghi@tabrizu.ac.ir
sedghi@azaruniv.ac.ir

Reza Naghipour
Department of Mathematics
University of Tabriz
Tabriz, Iran
E-mail: naghipour@ipm.ir
naghipour@tabrizu.ac.ir

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