COLLOQUIUM MATHEMATICUM

VOL. 103 2005 NO. 1

ON (a,b,c,d)-ORTHOGONALITY IN NORMED LINEAR SPACES

BY

C.-S. LIN (Lennoxville)

Dedicated to Professor Chair Wu on her retirement

Abstract. We first introduce a notion of (a,b,c,d)-orthogonality in a normed linear space, which is a natural generalization of the classical isosceles and Pythagorean orthogonalities, and well known α- and (α,β)-orthogonalities. Then we characterize inner product spaces in several ways, among others, in terms of one orthogonality implying another orthogonality.

1. Introduction. In what follows we denote by $\mathbb{R}$ the real number field, by $\mathbf{X}$ a normed linear space over $\mathbb{R}$, and by $\|\cdot\|$ a norm on $\mathbf{X}$. So far as norm characterization of an inner product space among normed linear spaces is concerned, Carlsson's theorem [3] is perhaps the most celebrated result, which will be repeated in Theorem C below. On the other hand, one can characterize an inner product space by means of orthogonalities. James [6] defined and studied two types of orthogonalities, namely the isosceles and Pythagorean orthogonalities. By using relations among orthogonal vectors he was able to prove that in $\mathbf{X}$ if either orthogonality has the homogeneity or additivity property, then $\mathbf{X}$ is an inner product space. This has been extended to (α,β)-orthogonality in [1]. On the other hand Day's result [4] amounts to saying that if isosceles orthogonality implies Pythagorean orthogonality, or vice versa, then $\mathbf{X}$ is again an inner product space.

In this article we first define a notion of (a,b,c,d)-orthogonality in a normed linear space, which is a natural extension of isosceles and Pythagorean orthogonalities, and of well known α- and (α,β)-orthogonalities [5]. We prove that $\mathbf{X}$ is an inner product space if and only if one orthogonality implies another, if and only if (a,b,c,d)-orthogonality has the homogeneity or additivity property. Consequently, we offer short proofs of some main results in [1, 4, 6]. Aside from Day's condition above, it is the author's belief

2000 *Mathematics Subject Classification*: Primary 46C05, 46C10.

Key words and phrases: normed linear space, isosceles orthogonality, Pythagorean orthogonality, α-orthogonality, (α,β)-orthogonality, (a,b,c,d)-orthogonality, inner product space.

that this type of characterizing an inner product space does not seem to appear in the literature. The main device used in this paper is Carlsson's norm characterization mentioned above, and our unified approach is different from James', Day's and others.

2. Definition of (a, b, c, d)-orthogonality. Let us repeat from [6] the two most familiar definitions of orthogonality for $x, y, z \in \mathbf{X}$, which do not require the existence of an inner product.

(1) *Isosceles orthogonality*: $(x \perp y)(\mathrm{I})$ whenever

$$\|x - y\| = \|x + y\|.$$

(2) *Pythagorean orthogonality*: $(x \perp y)(\mathrm{P})$ whenever

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

Also recall properties of orthogonality from [6]:

(1) *Homogeneity*: $(x \perp y)(\cdot)$ implies $(ax \perp by)(\cdot)$ for all $a, b \in \mathbb{R}$.
(2) *Additivity*: $(z \perp x)(\cdot)$ and $(z \perp y)(\cdot)$ imply $(z \perp (x + y))(\cdot)$,

where $(\cdot)$ means either (I) or (P) (but the same throughout (1) or (2)).

Definition. Let $x, y \in \mathbf{X}$. Let $a, b, c, d \in \mathbb{R}$ be such that at least two of them are nonzero, $a \neq c$ and $b \neq d$. We say that x is (a, b, c, d)-*orthogonal* to y, denoted by $(x \perp y)(a, b, c, d)$, if

$$\|ax - by\|^2 + \|cx - dy\|^2 = \|ax - dy\|^2 + \|cx - by\|^2.$$

For a constant $k \in \mathbb{R}$, the notation $k(x \perp y)(a, b, c, d)$ means

$$k\|ax - by\|^2 + k\|cx - dy\|^2 = k\|ax - dy\|^2 + k\|cx - by\|^2.$$

Remark 1. (a) Just as for isosceles or Pythagorean orthogonality it is clear that $(x \perp x)(a, b, c, d)$ implies $x = 0$, and in an inner product space we have $(x \perp y)(a, b, c, d)$ if and only if $(x, y) = 0$.

(b) According to our definition, $(x \perp y)(1, 1, -1, -1)$ means $(x \perp y)(\mathrm{I})$, and $(x \perp y)(1, 1, 0, 0)$ means $(x \perp y)(\mathrm{P})$. Notice that there are many other expressions of $(x \perp y)(\mathrm{I})$ and $(x \perp y)(\mathrm{P})$ by means of (a, b, c, d)-orthogonality; e.g., $(x \perp y)(1, 1, 0, -1)$ for the former and $(x \perp y)(1, 0, 0, 1)$ for the latter.

(c) In particular, $(x \perp y)(1, 1, \alpha, \alpha)$ for fixed $\alpha \neq 1$ means

$$(1 + \alpha^2)\|x - y\|^2 = \|x - \alpha y\|^2 + \|\alpha x - y\|^2,$$

which is called α-*orthogonality* in [5] and denoted by $(x \perp y)(\alpha)$. This is extended in (d) below.

(d) For fixed $\alpha, \beta \neq 1$, (α, β)-*orthogonality*, denoted by $(x \perp y)(\alpha, \beta)$, is the relation

$$\|x - y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha x - y\|^2$$

according to [1]. This is precisely our $(x \perp y)(1, 1, \alpha, \beta)$.

(e) There are many other expressions of $(x \perp y)(\alpha)$ and $(x \perp y)(\alpha,\beta)$ by means of (a,b,c,d)-orthogonality; e.g., $(x \perp y)(\alpha,1,1,\alpha)$ for the former and $(x \perp y)(\alpha,\beta,1,1)$ for the latter.

3. Norm characterization of inner product spaces. We begin by repeating [3, Theorem].

THEOREM C. *Let $a_i \neq 0$, b_i, c_i, $i = 1,\dots,n$, be a fixed collection in $\mathbb{R}$ satisfying*

$$\sum_{i=1}^{n} a_i b_i^2 = \sum_{i=1}^{n} a_i b_i c_i = \sum_{i=1}^{n} a_i c_i^2 = 0. \tag{$\sharp$}$$

If (b_i,c_i) and (b_j,c_j) are linearly independent in $\mathbb{R}^2$ for $i \neq j$, and if

$$\sum_{i=1}^{n} a_i \|b_i x + c_i y\|^2 \sim 0 \tag{$*$}$$

for all $x, y \in \mathbb{X}$, where $\sim$ stands for $\geq$ or $\leq$ (the same throughout $\mathbf{X}$), then $\mathbf{X}$ is an inner product space.

The conclusion of Theorem C remains valid if $\sim$ is replaced by $=$, and in this case it is not necessary to assume the restrictions in ($\sharp$) (see [2]).

4. Orthogonality characterization of inner product spaces. We are ready to consider the main results. Our work in this section will be concerned primarily with characterizing inner product spaces among normed linear spaces in terms of various types of orthogonalities mentioned in Section 2. For example, $\mathbf{X}$ is an inner product space if and only if one orthogonality implies another orthogonality. It follows that this will be the case if and only if each orthogonality is either homogeneous or additive. Consequently, many well known results in the literature will be proved by our methods in a shorter way.

THEOREM 1. *For any fixed nonzero $q \in \mathbb{R}$, the following are equivalent (where it is understood that each of conditions (2)–(7) and (12)–(15) starts with "for all $x, y \in \mathbf{X}$") (1):*

(1) $\mathbf{X}$ *is an inner product space*;
(2) $(x \perp y)(\mathrm{I})$ *implies* $(x \perp y)(\mathrm{P})$ [4, Theorem 5.1];
(3) $(x \perp y)(\mathrm{P})$ *implies* $(x \perp y)(\mathrm{I})$ [4, Theorem 5.2];
(4) $(x \perp y)(\mathrm{I})$ *implies* $(qx \perp y)(\mathrm{I})$ *for* $q > 1$;
(5) $(x \perp y)(\mathrm{P})$ *implies* $(qx \perp y)(\mathrm{P})$ *for* $q > 1$;
(6) $(x \perp y)(\mathrm{I})$ *and* $(x \perp qy)(\mathrm{I})$ *imply* $(x \perp (1+q)y)(\mathrm{I})$ *for* $q \neq \pm 1$, -2, $-1/2$;

(1) The same convention will be tacitly assumed in further statements.

(7) $(x \perp y)$(P) *and* $(x \perp qy)$(P) *imply* $(x \perp (1+q)y)$(P) *for* $q \neq \pm 1$;
(8) *Isosceles orthogonality is homogeneous* [6, Theorem 4.7];
(9) *Isosceles orthogonality is additive* [6, Theorem 4.8];
(10) *Pythagorean orthogonality is homogeneous* [6, Theorem 5.2];
(11) *Pythagorean orthogonality is additive* [6, Corollary 5.3];
(12) $(qx \perp y)$(I) *implies* $(x \perp y)$(I) *for* $q < -1$;
(13) $(qx \perp y)$(P) *implies* $(x \perp y)$(P) *for* $q < 1$;
(14) $q(x \perp y)$(I) *implies* $(qx \perp y)$(I) *for* $q \neq \pm 1$;
(15) $q(x \perp y)$(P) *implies* $(qx \perp y)$(P) *for* $q \neq 1$.

Proof. First of all, it is a straightforward verification that (1) implies all the other assertions. In fact this follows by merely writing the norm of a vector in terms of the inner product, i.e., $\|x\|^2 = (x, x)$ for $x \in \mathbf{X}$. So, we need only prove that each assertion implies (1).

(2)⇒(1). (2) means that $\|x-y\|^2 - \|x+y\|^2 = 0$ implies $\|x-y\|^2 - \|x\|^2 - \|y\|^2 = 0$, i.e.,

$$\|x-y\|^2 - \|x+y\|^2 \geq \|x-y\|^2 - \|x\|^2 - \|y\|^2,$$

or

$$\|x+y\|^2 + \|x-y\|^2 \leq 2[\|x\|^2 + \|y\|^2].$$

This inequality is also obtained from Theorem C by replacing $\sim$ by $\leq$, and letting $a_1 = a_2 = 1$, $a_3 = a_4 = -2$, $b_1 = b_2 = b_3 = c_1 = -c_2 = c_4 = 1$, $b_4 = c_3 = 0$. Clearly, $\{(b_i, c_i)\}_{i=1}^4 = \{(1,1), (1,-1), (1,0), (0,1)\}$ is a linearly independent set, and $\sum_{i=1}^4 a_i b_i^2 = \sum_{i=1}^4 a_i b_i c_i = \sum_{i=1}^4 a_i c_i^2 = 0$. Hence all conditions in Theorem C are satisfied and the desired conclusion follows.

(3)⇒(1). Similar arguments as above. Here the symbol $\sim$ means $\geq$.

Since the arguments are basically the same, we shall provide only the outlines of the rest of the proofs.

(4)⇒(1). By assumption,

$$\begin{aligned} q[\|x-y\|^2 - \|x+y\|^2] &> \|x-y\|^2 - \|x+y\|^2 \\ &\geq \|qx-y\|^2 - \|qx+y\|^2. \end{aligned}$$

So, $a_1 = -a_2 = q$, $a_3 = -a_4 = -1$, $b_1 = b_2 = 1$, $b_3 = b_4 = q$, $c_1 = -c_2 = c_3 = -c_4 = -1$, and $\{(b_i, c_i)\}_{i=1}^4 = \{(1,-1), (1,1), (q,-1), (q,1)\}$ is a linearly independent set.

(5)⇒(1). By assumption,

$$\begin{aligned} q[\|x-y\|^2 - \|x\|^2 - \|y\|^2] &> \|x-y\|^2 - \|x\|^2 - \|y\|^2 \\ &\geq \|qx-y\|^2 - \|qx\|^2 - \|y\|^2, \end{aligned}$$

or

$$q\|x-y\|^2 + (q^2-q)\|x\|^2 + (1-q)\|y\|^2 > \|qx-y\|^2.$$

So, $a_1 = q$, $a_2 = q^2 - q$, $a_3 = 1 - q$, $a_4 = -1$, $b_1 = b_2 = 1$, $b_3 = 0$, $b_4 = q$, $c_1 = -c_3 = c_4 = -1$, $c_2 = 0$, and $\{(b_i, c_i)\}_{i=1}^4 = \{(1,-1), (1,0), (0,1), (q,-1)\}$ is a linearly independent set.

(6)$\Rightarrow$(1). By assumption,

$$\|x - qy\|^2 - \|x + qy\|^2 + \|x - y\|^2 - \|x + y\|^2 \geq \|x - (q+1)y\|^2 - \|x + (q+1)y\|^2.$$

So, $a_1 = -a_2 = a_3 = -a_4 = -a_5 = a_6 = 1$, $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 1$, $c_1 = -c_2 = -q$, $c_3 = -c_4 = -1$, $c_5 = -c_6 = -(q+1)$, and $\{(b_i, c_i)\}_{i=1}^6 = \{(1,-q), (1,q), (1,-1), (1,1), (1,-q-1), (1,q+1)\}$ is a linearly independent set.

(7)$\Rightarrow$(1). By assumption,

$$\|x - qy\|^2 - \|x\|^2 - \|qy\|^2 + \|x - y\|^2 - \|x\|^2 - \|y\|^2 \geq \|x - (q+1)y\|^2 - \|x\|^2 - \|(q+1)y\|^2,$$

or

$$\|x - qy\|^2 - \|x\|^2 + 2q\|y\|^2 + \|x - y\|^2 \geq \|x - (q+1)y\|^2.$$

So, $a_1 = -a_2 = a_4 = -a_5 = 1$, $a_3 = 2q$, $b_1 = b_2 = b_4 = b_5 = 1$, $b_3 = 0$, $c_1 = -q$, $c_2 = 0$, $c_3 = -c_4 = 1$, $c_5 = -(q+1)$, and $\{(b_i, c_i)\}_{i=1}^5 = \{(1,-q), (1,0), (0,1), (1,-1), (1,-q-1)\}$ is a linearly independent set.

(8)$\Rightarrow$(1). If (8) holds, so does (4) in particular, which implies (1).

(9)$\Rightarrow$(1). Since (9)$\Rightarrow$(6)$\Rightarrow$(1).

(10)$\Rightarrow$(1). Since (10)$\Rightarrow$(5)$\Rightarrow$(1).

(11)$\Rightarrow$(1). Since (11)$\Rightarrow$(7)$\Rightarrow$(1).

((12) or (14)) $\Rightarrow$ (1). The proof is similar to that of (4)$\Rightarrow$(1).

((13) or (15)) $\Rightarrow$ (1). This is similar to that of (5)$\Rightarrow$(1) and thus Theorem 1 is proved.

THEOREM 2. *For any fixed* $\alpha \in \mathbb{R}$ *such that* $\alpha \neq 1$, *the following are equivalent*:

(1) $\mathbf{X}$ *is an inner product space*;
(2) $(x \perp y)(\alpha)$ *implies* $(x \perp y)(\mathrm{P})$ *for* $\alpha \neq 0$ *and* $\alpha(2 - \alpha) \geq 0$;
(3) $(x \perp y)(\mathrm{P})$ *implies* $(x \perp y)(\alpha)$ *for* $\alpha \neq 0$ *and* $\alpha(\alpha - 2) \geq 0$;
(4) $(x \perp y)(\alpha)$ *implies* $(x \perp y)(\mathrm{I})$ *for* $1 + 2\alpha - \alpha^2 \geq 0$;
(5) $(x \perp y)(\mathrm{I})$ *implies* $(x \perp y)(\alpha)$ *for* $\alpha \neq -1$ *and* $1 + 2\alpha - \alpha^2 \leq 0$;
(6) $(x \perp y)(\alpha)$ *implies* $(1 - \alpha)^2(x \perp y)(\mathrm{P})$ *for* $\alpha \neq 0$;
(7) $(1 - \alpha)^2(x \perp y)(\mathrm{P})$ *implies* $(x \perp y)(\alpha)$ *for* $\alpha \neq 0$;
(8) $2(x \perp y)(\alpha)$ *implies* $(1 - \alpha)^2(x \perp y)(\mathrm{I})$ *for* $\alpha \neq -1$;
(9) $(1 - \alpha)^2(x \perp y)(\mathrm{I})$ *implies* $2(x \perp y)(\alpha)$ *for* $\alpha \neq -1$;
(10) α*-orthogonality is homogeneous*;
(11) α*-orthogonality is additive.*

Proof. As in the proof of Theorem 1 we will prove that each assertion implies (1), and we give outlines only.

(2)⇒(1). The assumption means that

$$\begin{aligned}(1+\alpha^2)\|x-y\|^2 - \|x-\alpha y\|^2 - \|\alpha x - y\|^2 \\ \geq \|x-y\|^2 - \|x\|^2 - \|y\|^2 \geq (1-\alpha)^2[\|x-y\|^2 - \|x\|^2 - \|y\|^2]\end{aligned}$$

if $1 \geq (1-\alpha)^2$, or $\alpha(2-\alpha) \geq 0$. Rewrite the above inequality as

$$-2\alpha\|x-y\|^2 - (1-\alpha)^2(\|x\|^2 + \|y\|^2) + \|x-\alpha y\|^2 + \|\alpha x - y\|^2 \leq 0.$$

So, $a_1 = -2\alpha$, $a_2 = a_3 = -(1-\alpha)^2$, $a_4 = a_5 = 1$, $b_1 = b_2 = b_4 = 1$, $b_3 = 0$, $b_5 = \alpha$, $c_1 = -c_3 = c_5 = -1$, $c_2 = 0$, $c_4 = -\alpha$, and $\{(b_i, c_i)\}_{i=1}^5 = \{(1,-1), (1,0), (0,1), (1,-\alpha), (\alpha,-1)\}$ is a linearly independent set.

(3)⇒(1). Similar to the proof above. In this case $\sim$ means $\geq$.

(4)⇒(1). By assumption,

$$\begin{aligned}2[(1+\alpha^2)\|x-y\|^2 - \|x-\alpha y\|^2 - \|\alpha x - y\|^2] \\ \geq 2[\|x-y\|^2 - \|x+y\|^2] \geq (1-\alpha)^2[\|x-y\|^2 - \|x+y\|^2]\end{aligned}$$

if $2 \geq (1-\alpha)^2$, i.e., $1 + 2\alpha - \alpha^2 \geq 0$, and so $\alpha \neq \pm 1$ (this is used to check linear independence). Rewrite the above inequality as

$$2(\|x-\alpha y\|^2 + \|\alpha x - y\|^2) - (1-\alpha)^2\|x+y\|^2 \leq (1+\alpha)^2\|x-y\|^2.$$

So, $a_1 = a_2 = 2$, $a_3 = -(1-\alpha)^2$, $a_4 = -(1+\alpha)^2$, $b_1 = b_3 = b_4 = 1$, $b_2 = \alpha$, $c_1 = -\alpha$, $c_2 = -c_3 = c_4 = -1$, and $\{(b_i, c_i)\}_{i=1}^4 = \{(1,-\alpha), (\alpha,-1), (1,1), (1,-1)\}$ is a linearly independent set.

(5)⇒(1). Similar to the proof above. The symbol $\sim$ means $\geq$.

((6), (7), (8) or (9)) ⇒ (1). The proof of each one is similar to the above. More precisely, from the proof that (2), (3), (4) or (5) implies (1), we deduce that (6), (7), (8) or (9) implies (1), respectively.

(10)⇒(1). If α-orthogonality is homogeneous, then in particular isosceles orthogonality is homogeneous (let $\alpha = -1$, see Remark 1(b),(c)). So (1) is true by (8) of Theorem 1.

(11)⇒(1). Indeed, (11) implies (9) of Theorem 1, and the proof is finished.

THEOREM 3. *For any fixed $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta \neq 1$, the following are equivalent*:

(1) $\mathbf{X}$ *is an inner product space*;
(2) $(x \perp y)(\alpha, \beta)$ *implies* $(x \perp y)(\mathrm{P})$ *for* $\alpha, \beta \neq 0$ *and* $\alpha + \beta - \alpha\beta \geq 0$;
(3) $(x \perp y)(\mathrm{P})$ *implies* $(x \perp y)(\alpha, \beta)$ *for* $\alpha, \beta \neq 0$ *and* $\alpha + \beta - \alpha\beta \leq 0$;
(4) $(x \perp y)(\alpha, \beta)$ *implies* $(x \perp y)(\mathrm{I})$ *for* $\alpha, \beta \neq -1$ *and* $1 + \alpha + \beta - \alpha\beta \geq 0$;
(5) $(x \perp y)(\mathrm{I})$ *implies* $(x \perp y)(\alpha, \beta)$ *for* $\alpha, \beta \neq -1$ *and* $1 + \alpha + \beta - \alpha\beta \leq 0$;

(6) $(x \perp y)(\alpha, \beta)$ *implies* $(1-\alpha)(1-\beta)(x \perp y)(\mathrm{P})$ *for* $\alpha, \beta \neq 0$;
(7) $(1-\alpha)(1-\beta)(x \perp y)(\mathrm{P})$ *implies* $(x \perp y)(\alpha, \beta)$ *for* $\alpha, \beta \neq 0$;
(8) $2(x \perp y)(\alpha, \beta)$ *implies* $(1-\alpha)(1-\beta)(x \perp y)(\mathrm{I})$ *for* $\alpha, \beta \neq -1$;
(9) $(1-\alpha)(1-\beta)(x \perp y)(\mathrm{I})$ *implies* $2(x \perp y)(\alpha, \beta)$ *for* $\alpha, \beta \neq -1$;
(10) (α, β)*-orthogonality is homogeneous* [1, Theorems 3.1 and 3.2];
(11) (α, β)*-orthogonality is additive* [1, Theorem 3.2].

Proof. As in the preceding proof, we shall prove that each statement implies (1) without going into details.

(2)⇒(1). By assumption,

$$\begin{aligned} &\|x-y\|^2 + \|\alpha x - \beta y\|^2 - \|x - \beta y\|^2 - \|\alpha x - y\|^2 \\ &\qquad\qquad \geq \|x-y\|^2 - \|x\|^2 - \|y\|^2 \\ &\qquad\qquad \geq (1-\alpha)(1-\beta)[\|x-y\|^2 - \|x\|^2 - \|y\|^2] \end{aligned}$$

if $(1-\alpha)(1-\beta) \leq 1$, i.e., $\alpha + \beta - \alpha\beta \geq 0$. Rewrite the above inequality as

$$\begin{aligned} &(\alpha\beta - \alpha - \beta)\|x-y\|^2 - (1-\alpha)(1-\beta)(\|x\|^2 + \|y\|^2) \\ &\qquad\qquad \leq \|\alpha x - \beta y\|^2 - \|x - \beta y\|^2 - \|\alpha x - y\|^2. \end{aligned}$$

Hence, $a_1 = \alpha\beta - \alpha - \beta$, $a_2 = a_3 = (1-\alpha)(\beta - 1)$, $a_4 = -a_5 = -a_6 = -1$, $b_1 = b_2 = b_5 = 1$, $b_3 = 0$, $b_4 = b_6 = \alpha$, $c_1 = -c_3 = c_6 = -1$, $c_2 = 0$, $c_4 = c_5 = -\beta$, and $\{(b_i, c_i)\}_{i=1}^6 = \{(1,-1), (1,0), (0,1), (\alpha,-\beta), (1,-\beta), (\alpha,-1)\}$ is a linearly independent set.

(3)⇒(1). Similar to the proof above, and the symbol $\sim$ means $\geq$.

(4)⇒(1). By assumption,

$$\begin{aligned} &2[\|x-y\|^2 + \|\alpha x - \beta y\|^2 - \|x - \beta y\|^2 - \|\alpha x - y\|^2] \\ &\qquad \geq 2[\|x-y\|^2 - \|x+y\|^2] \geq (1-\alpha)(1-\beta)[\|x-y\|^2 - \|x+y\|^2] \end{aligned}$$

if $2 \geq (1-\alpha)(1-\beta)$, i.e., $1 + \alpha + \beta - \alpha\beta \geq 0$. Rewrite the above inequality as

$$\begin{aligned} &2[\|\alpha x - \beta y\|^2 - \|x - \beta y\|^2 - \|\alpha x - y\|^2] \\ &\qquad\qquad \geq (\alpha\beta - \alpha - \beta - 1)\|x-y\|^2 + (\alpha + \beta - \alpha\beta - 1)\|x+y\|^2. \end{aligned}$$

So, $a_1 = -a_2 = -a_3 = 2$, $a_4 = 1 - \alpha\beta + \alpha + \beta$, $a_5 = 1 - \alpha - \beta + \alpha\beta$, $b_1 = b_3 = \alpha$, $b_2 = b_4 = b_5 = 1$, $c_1 = c_2 = -\beta$, $c_3 = c_4 = -c_5 = -1$, and $\{(b_i, c_i)\}_{i=1}^5 = \{(\alpha,-\beta), (1,-\beta), (\alpha,-1), (1,-1), (1,1)\}$ is a linearly independent set.

(5)⇒(1). Similar to the proof above. Here $\sim$ means $\geq$.

((6), (7), (8) or (9)) ⇒ (1). By the proofs above. In fact, the proofs that (2), (3), (4) or (5) implies (1) show that (6), (7), (8) or (9) implies (1), respectively.

(10)$\Rightarrow$(1). If (α, β)-orthogonality is homogeneous, so is α-orthogonality (let $\beta = \alpha$, see Remark 1(c),(d)). Hence we have (1) due to (10) in Theorem 2.

(11)$\Rightarrow$(1). Indeed, (11) implies (11) of Theorem 2, and the proof is complete.

Corollary 1. *For fixed $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta \neq 1$, if (α, β)-orthogonality is homogeneous, then*

(1) (α, β)-*orthogonality implies Pythagorean orthogonality* [1, Theorem 2.1];
(2) *Pythagorean orthogonality implies* (α, β)-*orthogonality*;
(3) (α, β)-*orthogonality implies isosceles orthogonality* [1, Theorem 2.2];
(4) *Isosceles orthogonality implies* (α, β)-*orthogonality*;
(5) $\mathbf{X}$ *is a strictly convex space* [1, Theorem 2.3].

Proof. If (α, β)-orthogonality is homogeneous, then clearly $\mathbf{X}$ is an inner product space by (10) of Theorem 3. Hence (1) through (4) follow trivially by expressing the norm of a vector in terms of the inner product. As for (5), it is well known that every inner product space is strictly convex.

Corollary 2. *For fixed $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta \neq 1$, if (α, β)-orthogonality is additive, then the five statements in Corollary* 1 *still hold true.*

Proof. $\mathbf{X}$ is an inner product space by (11) of Theorem 3.

Theorem 4. *For any fixed $a, b, c, d \in \mathbb{R}$ such that at least two of them are nonzero, $a \neq c$ and $b \neq d$, and for any fixed nonzero $s, t, p \in \mathbb{R}$, the following are equivalent*:

(1) $\mathbf{X}$ *is an inner product space*;
(2) $st(x \perp y)(a, b, c, d)$ *implies* $(a - c)(b - d)(sx \perp ty)(\mathrm{P})$ *whenever* $\{(a, -b), (c, -d), (a, -d), (c, -b), (s, -t), (s, 0), (0, t)\}$ *is a linearly independent set*;
(3) $(a - c)(b - d)(sx \perp ty)(\mathrm{P})$ *implies* $st(x \perp y)(a, b, c, d)$ *whenever* $\{(a, -b), (c, -d), (a, -d), (c, -b), (s, -t), (s, 0), (0, t)\}$ *is a linearly independent set*;
(4) $2st(x \perp y)(a, b, c, d)$ *implies* $(a - c)(d - b)(sx \perp ty)(\mathrm{I})$ *whenever* $\{(a, -b), (c, -d), (a, -d), (c, -b), (s, t), (s, -t)\}$ *is a linearly independent set*;
(5) $(a - c)(d - b)(sx \perp ty)(\mathrm{I})$ *implies* $2st(x \perp y)(a, b, c, d)$ *whenever* $\{(a, -b), (c, -d), (a, -d), (c, -b), (s, t), (s, -t)\}$ *is a linearly independent set*;
(6) $p(x \perp y)(a, b, c, d)$ *implies* $(px \perp y)(a, b, c, d)$ *whenever* $\{(ap, -b), (cp, -d), (ap, -d), (cp, -b), (a, -b), (c, -d), (a, -d), (c, -b)\}$ *is a linearly independent set*;

(7) $(px \perp y)(a, b, c, d)$ *implies* $p(x \perp y)(a, b, c, d)$ *whenever* $\{(ap, -b), (cp, -d), (ap, -d), (cp, -b), (a, -b), (c, -d), (a, -d), (c, -b)\}$ *is a linearly independent set*;

(8) $(x \perp y)(a, b, c, d)$ *and* $(x \perp py)(a, b, c, d)$ *imply* $(x \perp (p+1)y)(a, b, c, d)$ *whenever* $\{(a, -bp), (c, -dp), (a, -dp), (c, -bp), (a, -b), (c, -d), (a, -d), (c, -b), (a, -b(p+1)), (c, -d(p+1)), (a, -d(p+1)), (c, -b(p+1))\}$ *is a linearly independent set*;

(9) (a, b, c, d)*-orthogonality is homogeneous*;

(10) (a, b, c, d)*-orthogonality is additive.*

Proof. As before, it suffices to prove that each assertion implies (1).

(2)⇒(1). In Theorem C let $a_1 = a_2 = -a_3 = -a_4 = st$, $a_5 = -a_6 = -a_7 = -(a-c)(b-d)$, $b_1 = b_3 = a$, $b_2 = b_4 = c$, $b_5 = b_6 = s$, $b_7 = 0$, $c_1 = c_4 = -b$, $c_2 = c_3 = -d$, $c_5 = -c_7 = -t$ and $c_6 = 0$. Then it is easily checked that all conditions in Theorem C, apart from the assumption that the set $\{(b_i, c_i)\}_{i=1}^{7}$ is linearly independent, are satisfied. In other words, we have $a_i \neq 0$, $i = 1, \ldots, 7$, $\sum_{i=1}^{7} a_i b_i^2 = \sum_{i=1}^{7} a_i b_i c_i = \sum_{i=1}^{7} a_i c_i^2 = 0$, and $\sum_{i=1}^{7} a_i \|b_i x + c_i y\|^2 \sim 0$ for all $x, y \in \mathbf{X}$. Hence $\mathbf{X}$ is an inner product space. The last relation is precisely

$$st[\|ax - by\|^2 + \|cx - dy\|^2 - \|ax - dy\|^2 - \|cx - by\|^2] \\ - (a-c)(b-d)[\|sx - ty\|^2 - \|sx\|^2 - \|ty\|^2] \sim 0$$

for all $x, y \in \mathbf{X}$, i.e., the relation (1).

(3)⇒(1). Same as above.

(4)⇒(1). In Theorem C let $a_1 = a_2 = -a_3 = -a_4 = 2st$, $a_5 = -a_6 = -(a-c)(d-b)$, $b_1 = b_3 = a$, $b_2 = b_4 = c$, $b_5 = b_6 = s$, $c_1 = c_4 = -b$, $c_2 = c_3 = -d$ and $c_5 = -c_6 = t$. Then similar to (2)⇒(1) above we conclude that $\mathbf{X}$ is an inner product space.

(5)⇒(1). Same as above.

(6)⇒(1). In Theorem C let $a_1 = a_2 = -a_3 = -a_4 = 1$, $a_5 = a_6 = -a_7 = -a_8 = -p$, $b_1 = b_3 = ap$, $b_2 = b_4 = cp$, $b_5 = b_7 = a$, $b_6 = b_8 = c$, $c_1 = c_4 = c_5 = c_8 = -b$ and $c_2 = c_3 = c_6 = c_7 = -d$. It follows as in (2)⇒(1) that $\mathbf{X}$ is an inner product space.

(7)⇒(1). Same as above.

(8)⇒(1). In Theorem C let $a_1 = a_2 = -a_3 = -a_4 = a_5 = a_6 = -a_7 = -a_8 = -a_9 = -a_{10} = a_{11} = a_{12} = 1$, $b_1 = b_3 = b_5 = b_7 = b_9 = b_{11} = a$, $b_2 = b_4 = b_6 = b_8 = b_{10} = b_{12} = c$, $c_1 = c_4 = -bp$, $c_2 = c_3 = -dp$, $c_5 = c_8 = -b$, $c_6 = c_7 = -d$, $c_9 = c_{12} = -b(p+1)$ and $c_{10} = c_{11} = -d(p+1)$. Then we conclude that $\mathbf{X}$ is an inner product space.

(9)⇒(1). The assumption implies that (α, β)-orthogonality is homogeneous (let $a = b = 1$, $c = \alpha$, $d = \beta$ and see Remark 1(d)). So, $\mathbf{X}$ is an inner product space by (10) of Theorem 3.

(10)$\Rightarrow$(1). The assumption implies that (α,β)-orthogonality is additive (let $a=b=1$, $c=\alpha$, $d=\beta$ and see Remark 1(d)). Hence we have (1) by (11) of Theorem 3. In fact, we have a simple alternative proof by observing that (10)$\Rightarrow$(8)$\Rightarrow$(1).

REMARK 2. In each result above we have chosen only typical orthogonalities. If different orthogonalities are selected, other similar characterizations of inner product spaces can be obtained. In other words, our approach has a unifying aspect.

REFERENCES

[1] E. Z. Andalafte, C. R. Diminnie, and R. W. Freese, *(α,β)-orthogonality and a characterization of inner product spaces*, Math. Japon. 30 (1985), 341–349.

[2] S. O. Carlsson, *Orthogonality in normed linear spaces*, Ark. Mat. 4 (1961), 297–318.

[3] —, *A characteristic property of Euclidean spaces*, ibid. 5 (1965), 327–330.

[4] M. M. Day, *Some characterizations of inner-product spaces*, Trans. Amer. Math. Soc. 62 (1947), 320–337.

[5] C. R. Diminnie, R. W. Freese and E. Z. Andalafte, *An extension of pythagorean and isosceles orthogonality and characterization of inner product spaces*, J. Approx. Theory 39 (1983), 295–298.

[6] R. C. James, *Orthogonality in normed linear spaces*, Duke Math. J. 12 (1945), 291–302.

[7] C.-S. Lin, *Sesquilinear and quadratic forms on modules over $*$-algebras*, Publ. Inst. Math. (Beograd) 51 (65) (1992), 81–86.

Department of Mathematics
Bishop's University
Lennoxville, Quebec
Canada J1M 1Z7
E-mail: plin@ubishops.ca

Received 14 February 2003 (4311)