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THE MULTIPLICATIVE AND FUNCTIONAL INDEPENDENCE OF DEDEKIND ZETA FUNCTIONS OF ABELIAN FIELDS

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Abstract. It is shown that the multiplicative independence of Dedekind zeta functions of abelian fields is equivalent to their functional independence. We also give all the possible multiplicative dependence relations for any set of Dedekind zeta functions of abelian fields.

In this paper we consider the problem of the multiplicative and, more generally, functional independence of Dedekind zeta functions for abelian fields. As in the paper of Voronin [6], we say that the complex functions f_j $(j = 1, \ldots, m)$ of a complex variable are *functionally independent* if for any continuous functions $F_l : \mathbb{C}^m \to \mathbb{C}$, for $l = 0, 1, \ldots, q$, not all identically zero, the function

$$f(s) = \sum_{l=0}^{q} s^l F_l(f_1(s), \ldots, f_m(s))$$

is not identically zero.

On the other hand, the *multiplicative independence* of $f_1, \ldots, f_m$ means that the equality

$$\prod_{k=1}^{m} f_k^{a_k} = 1$$

with rational integers $a_1, \ldots, a_m$ holds only if $a_1 = \cdots = a_m = 0$.

Using the functional independence of Dirichlet L-functions for non-equivalent characters, proved by Voronin (see Corollary 1 in [4]), we shall prove the equivalence of the multiplicative and functional independence for Dedekind zeta functions.

We shall give necessary and sufficient conditions for a set of Dedekind zeta functions of abelian fields to be functionally independent. Also, for any set of Dedekind zeta functions of abelian fields, we give all the possible integers $a_1, \ldots, a_n$ appearing in any multiplicative dependence relation for those functions.

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Following the book of L. C. Washington [7] we shall use the correspondence between subfields of an abelian field and subgroups of the group of Dirichlet characters associated with this field.

Let $L_1, \dots, L_m$ be abelian fields and L their compositum. Then there exists a minimal integer r such that $L \subseteq \mathbb{Q}(\zeta_r)$, where ζ_r is a primitive rth root of unity. Let $X = \{\chi_1, \dots, \chi_n\}$ be the group of Dirichlet characters associated with the field L and $X_1, \dots, X_m$ the subgroups of X associated with $L_1, \dots, L_m$ respectively.

Now, for any $j = 1, \dots, m$ and $k = 1, \dots, n$, we put $\varepsilon_{jk} = 1$ when $\chi_k \in X_j$ and $\varepsilon_{jk} = 0$ otherwise. Thus, for all j, we have

$$\zeta_{L_j}(s) = \prod_{k=1}^{n} L(s, \chi_k^*)^{\varepsilon_{jk}},$$

where χ_k^* is the primitive Dirichlet character inducing χ_k, $L(s, \chi_k^*)$ the L-series attached to χ_k^*, and ζ_{L_j} is the Dedekind zeta function for L_j.

Now we prove

Theorem 1. *The following conditions are equivalent*:

(i) *The rank of the $m \times n$ matrix $[\varepsilon_{jk}]$ equals m, and $m \le n$.*

(ii) *$\zeta_{L_1}, \dots, \zeta_{L_m}$ are multiplicatively independent.*

(iii) *$\zeta_{L_1}, \dots, \zeta_{L_m}$ are functionally independent.*

Proof. (i)$\Leftrightarrow$(ii). Suppose that $\operatorname{rank}[\varepsilon_{jk}] = m$ and for some integers a_j,

$$\prod_{j=1}^{m} \zeta_{L_j}(s)^{a_j} = 1. \tag{1}$$

Then we have

$$1 = \prod_{j=1}^{m} \prod_{k=1}^{n} L(s, \chi_k^*)^{\varepsilon_{jk} a_j} = \prod_{k=1}^{n} L(s, \chi_k^*)^{\sum_{j=1}^{m} \varepsilon_{jk} a_j}$$

and, by the functional independence of Dirichlet L-functions, we obtain

$$\sum_{j=1}^{m} \varepsilon_{jk} a_j = 0 \tag{2}$$

for $k = 1, \dots, n$. Since the rank of the matrix $[\varepsilon_{jk}]$ equals m, the system (2) has only one solution, i.e. $a_1 = \dots = a_m = 0$.

Conversely, if the rank of $[\varepsilon_{jk}]$ were less than m or $m > n$, then (2) would have non-trivial solutions and this would lead to (1) with a_j's not all zero.

(ii)$\Rightarrow$(iii). Now assume that $\zeta_{L_1}, \dots, \zeta_{L_m}$ are multiplicatively independent and suppose that for continuous functions $F_l : \mathbb{C}^m \to \mathbb{C}$, $l = 0, 1, \dots, q$, one has

$$\sum_{l=0}^{q} s^l F_l(\zeta_{L_1}(s), \ldots, \zeta_{L_m}(s)) = 0 \tag{3}$$

for any s. For any l we put $G_l = F_l \circ \boldsymbol{P} : \mathbb{C}^n \to \mathbb{C}$, where $\boldsymbol{P} = (P_1, \ldots, P_m) : \mathbb{C}^n \to \mathbb{C}^m$ is defined by

$$P_j(s_1, \ldots, s_n) = \prod_{k=1}^{n} s_k^{\varepsilon_{kj}}.$$

Thus we may rewrite (3) as

$$\sum_{l=0}^{q} s^l G_l(L(s, \chi_1^*), \ldots, L(s, \chi_1^*)) = 0.$$

Hence, again by the functional independence of L-functions, G_l is identically zero for each l. This implies that so is F_l. Indeed, if $F_l(z_1, \ldots, z_m) \neq 0$ for some $(z_1, \ldots, z_m) \in \mathbb{C}^m$ then, by continuity of F_l, we may assume that $(z_1, \ldots, z_m) \neq (0, \ldots, 0)$. In order to find $(s_1 \ldots, s_n)$ such that $(z_1, \ldots, z_m) = \boldsymbol{P}(s_1 \ldots, s_n)$ we put

$$z_j = r_j e^{it_j}, \qquad s_k = x_k e^{iy_k} \qquad \text{with } r_j > 0,$$

where x_k, y_k satisfy

$$\sum_{k=1}^{n} \varepsilon_{kj} \log x_k = \log r_j, \qquad \sum_{k=1}^{n} \varepsilon_{kj} y_k = t_j, \qquad j = 1, \ldots, m.$$

Thus $G_l(s_1, \ldots, s_n) = F_l(z_1, \ldots, z_m) \neq 0$, contrary to the functional independence of L-functions. ∎

REMARK 1. If we assume that the functions F_j in the definition of functional independence are entire of minimum type of order 1, then the implication (ii)⇒(iii) of Theorem 1 holds for a wider class of functions, the Selberg class containing the Dedekind zeta functions of all algebraic fields (see Theorem 1 in [2]).

Now as an application of the above theorem, we have

COROLLARY 1. *The Dedekind zeta functions $\zeta_{L_1}, \ldots, \zeta_{L_m}$ are functionally independent if they satisfy one of the following conditions*:

(i) $X_j \nsubseteq \bigcup_{i=1,\, i\neq j}^{m} X_i$ *for any* $j \in \{1, \ldots, m\}$.
(ii) *There is a prime p such that* $(L_j : \mathbb{Q}) = p$ *for any* $j \in \{1, \ldots, m\}$.
(iii) $L_j = \mathbb{Q}(\zeta_{l_j})$, *where l_j is an odd integer not dividing* $\operatorname{lcm}\{l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_m\}$.

Proof. (i) For any $j \in \{1, \ldots, m\}$ choose

$$\chi_{n_j} \in X_j \setminus \bigcup_{i=1,\, i\neq j}^{m} X_i$$

and observe that $[\varepsilon_{kn_j}]$ is the identity $m \times m$ matrix, after some permutation of rows. This, by Theorem 1(i), proves (i).

(ii) As the associated groups of characters X_j satisfy $X_i \cap X_j = \{1\}$ for any $i \neq j$, the condition of (i) of our corollary is satisfied.

(iii) The condition $l_j \nmid \operatorname{lcm}\{l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_m\}$ means that

$$\mathbb{Q}(\zeta_{l_J}) \not\subseteq \mathbb{Q}(\zeta_{l_1}) \cdots \mathbb{Q}(\zeta_{l_{j-1}}) \mathbb{Q}(\zeta_{l_{j+1}}) \cdots \mathbb{Q}(\zeta_{l_m}),$$

whence $X_j \not\subseteq X_1 \cdots X_{j-1} X_{j+1} \cdots X_m$ and again using (i) we obtain (iii). ■

In order to give all the possible integers a_j occurring in a dependence relation (1) we introduce some notation.

Let G be the Galois group of an extension $L \supseteq \mathbb{Q}$, and L^H the subfield of L corresponding to a subgroup H of G by Galois theory. We also put

$$H^{\perp} = \{\chi \in X : \forall h \in H \;\; \chi(h) = 1\}, \qquad \widetilde{H} = \sum_{h\in H} h \in \mathbb{Z}G,$$

where H is a subgroup of G and $\mathbb{Z}G$ denotes the group ring of G over $\mathbb{Z}$.

Now we prove

THEOREM 2. *The dependence relation*

$$\prod_{H\leq G} \zeta_{L^H}(s)^{a_H} = 1 \quad \textit{holds if and only if} \quad \sum_{H\leq G} a_H (G:H)\widetilde{H} = 0.$$

Proof. Assume that for some integers a_H one has

$$\sum_{H\leq G} a_H (G:H)\widetilde{H} = 0. \tag{4}$$

If 1_H^G denotes the character of G induced by the unit character on H then the equation

$$\sum_{g\in G} 1_H^G(g) g = |H|^{-1} \sum_{g\in G} g\widetilde{H}g^{-1}$$

may be used to convert (4) into the character relation

$$\sum_{H\leq G} a_H 1_H^G = 0.$$

Since $\zeta_{L^H}(s) = L(s, 1_H^G)$, where $L(s, 1_H^G)$ denotes the Artin L-function (see e.g. the book of J. Neukirch [4]), the above equation gives us the required dependence relation.

Conversely, suppose that

$$\prod_{H\le G} \zeta_{L^H}(s)^{a_H} = 1. \tag{5}$$

Since the group of characters $H^\perp$ is associated with the field L^H one has

$$\zeta_{L^H}(s) = \prod_{\chi\in H^\perp} L(s,\chi^*) = \prod_{\chi\in X} L(s,\chi^*)^{\varepsilon_{H\chi}},$$

where $\varepsilon_{H\chi} = 1$ when $\chi \in H$ and $\varepsilon_{H\chi} = 0$ otherwise.

Applying the last equality to (5) we obtain

$$1 = \prod_{H\le G} \Big(\prod_{\chi\in X} L(s,\chi^*)^{\varepsilon_{H\chi}}\Big)^{a_H} = \prod_{\chi\in X} L(s,\chi^*)^{S_{H_\chi}},$$

where

$$S_{H_\chi} = \sum_{H\le G} \varepsilon_{H\chi} a_H,$$

and, by the independence of L-functions, we get

$$\sum_{H\le G} \varepsilon_{H\chi} a_H = 0 \quad \text{for any } \chi\in X.$$

Since $1^X_{H^\perp}(\chi) = \frac{\#X}{\#H^\perp}\varepsilon_{H\chi}$ the above equalities imply

$$\sum_{H\le G} a_H \#H^\perp\, 1^X_{H^\perp} = 0. \tag{6}$$

If we identify $\widehat{X}$ with G and observe that the characters $1^X_{H^\perp}$ are linear combinations of irreducible characters of X, we have

$$1^X_{H^\perp} = \sum_{g\in G} \langle 1^X_{H^\perp}, g\rangle g \quad \text{for any } H\le G. \tag{7}$$

Note that for any $H\le G$ and $g\in G$ we obtain

$$\begin{aligned}\langle 1^X_{H^\perp}, g\rangle &= \frac{1}{\#X}\sum_{\chi\in X} 1^X_{H^\perp}(\chi)\,\overline{g(\chi)} = \frac{1}{\#X}\sum_{\chi\in H^\perp} \frac{\#X}{\#H^\perp}\,\overline{\chi(g)}\\ &= \frac{1}{\#H^\perp}\sum_{\chi\in H^\perp}\overline{\chi(g)} = \alpha_{Hg},\end{aligned}$$

where $\alpha_{Hg} = 1$ when $g\in H$ and $\alpha_{Hg} = 0$ otherwise, and hence

$$1^X_{H^\perp} = \sum_{g\in G} \alpha_{Hg} g = \widetilde{H}.$$

The above equality, (6) and (7) finally give

$$0 = \sum_{H \le G} a_H \# H \, 1^X_{H^\perp} = \sum_{H \le G} a_H \# H^\perp \, \widetilde{H},$$

and $\# H^\perp = (G : H)$ we obtain $\sum_{H \le G} a_H (G : H) \widetilde{H} = 0$. ■

REMARK 2. (i) Relations of the form (5) are important because they lead to the so-called class number relations (see e.g. [1], [3]).

(ii) Equalities in $\mathbb{Z}G$ of the form (4) are called *norm relations* and have been studied by Rehm in [5], where he gives a base for the $\mathbb{Z}$-module of norm relations for any finite group.

(iii) It seems that analogous assertions should be true for nonabelian fields.

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