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LIFTING OF LINEAR VECTOR FIELDS TO FIBER PRODUCT PRESERVING VERTICAL GAUGE BUNDLES

BY

J. KUREK (Lublin) and W. M. MIKULSKI (Kraków)

Abstract. We classify all natural operators lifting linear vector fields on vector bundles to vector fields on vertical fiber product preserving gauge bundles over vector bundles. We explain this result for some known examples of such bundles.

Introduction. Let m, n be fixed positive integers. The category of vector bundles with m-dimensional bases and vector bundle maps with embeddings as base maps will be denoted by $\mathcal{VB}_m$. The category of vector bundles with m-dimensional bases and n-dimensional fibers and vector bundle embeddings will be denoted by $\mathcal{VB}_{m,n}$.

Let $F : \mathcal{VB}_m \to \mathcal{FM}$ be a covariant functor. Let $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ and $B_{\mathcal{VB}_m} : \mathcal{VB}_m \to \mathcal{M}f$ be the base functors.

A *gauge bundle functor* on $\mathcal{VB}_m$ is a functor F as above satisfying:

(i) (*Base preservation*) $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}_m}$. Hence the induced projections form a functor transformation $\pi : F \to B_{\mathcal{VB}_m}$.

(ii) (*Localization*) For every inclusion of an open vector subbundle $i_{E|U} : E|U \to E$, $F(E|U)$ is the restriction $\pi^{-1}(U)$ of $\pi : FE \to B_{\mathcal{VB}_m}(E)$ to U and $Fi_{E|U}$ is the inclusion $\pi^{-1}(U) \to FE$.

(iii) (*Regularity*) F transforms smoothly parametrized systems of $\mathcal{VB}_m$-morphisms into smoothly parametrized families of $\mathcal{FM}$-morphisms.

A gauge bundle functor F on $\mathcal{VB}_m$ is *fiber product preserving* if for any fiber product projections $E_1 \xleftarrow{\mathrm{pr}_1} E_1 \times_M E_2 \xrightarrow{\mathrm{pr}_2} E_2$ in the category $\mathcal{VB}_m$, $FE_1 \xleftarrow{F\mathrm{pr}_1} F(E_1 \times_M E_2) \xrightarrow{F\mathrm{pr}_2} FE_2$ are fiber product projections in the category $\mathcal{FM}$. In other words, $F(E_1 \times_M E_2) = F(E_1) \times_M F(E_2)$.

A gauge bundle functor F on $\mathcal{VB}_m$ is called *vertical* if for any $\mathcal{VB}_m$-objects $E \to M$ and $E_1 \to M$ with the same basis, any $x \in M$ and any $\mathcal{VB}_m$-map $f : E \to E_1$ covering the identity of M the fiber restriction $F_x f : F_x E \to F_x E_1$ depends only on $f_x : E_x \to (E_1)_x$.

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From now on we are interested in vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$.

The most known example of vertical fiber product preserving gauge bundle functor F on $\mathcal{VB}_m$ is the so-called *vertical r-jet prolongation functor* $J^r_v : \mathcal{VB}_m \to \mathcal{FM}$, where for a $\mathcal{VB}_m$-object $p : E \to M$ we have a vector bundle $J^r_v E = \{j^r_x\gamma \mid \gamma \text{ is a local map } M \to E_x,\ x \in M\}$ and for a $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$ we have a vector bundle map $J^r_v f : J^r_v E_1 \to J^r_v E_2$, where $J^r_v f(j^r_x\gamma) = j^r_{\underline{f}(x)}(f \circ \gamma \circ \underline{f}^{-1})$ for $j^r_x\gamma \in J^r_v E_1$.

Another example is the *vertical Weil functor* V^A on $\mathcal{VB}_m$ corresponding to a Weil algebra A, where for a $\mathcal{VB}_m$-object $p : E \to M$ we have $V^A E = \bigcup_{x\in M} T^A(E_x)$ and for a $\mathcal{VB}_m$-map $f : E_1 \to E_2$ we have $V^A f = \bigcup_{x\in M_1} T^A(f_x) : V^A E_1 \to V^A E_2$. The functor V^A is equivalent to $E \otimes A$.

The fiber product $F_1 \times_{B_{\mathcal{VB}_m}} F_2 : \mathcal{VB}_m \to \mathcal{FM}$ of vertical fiber product preserving gauge bundle functors $F_1, F_2 : \mathcal{VB}_m \to \mathcal{FM}$ is again a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

In [6], we observed that any fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ has values in the category $\mathcal{VB}_m$. Hence we can compose fiber product preserving gauge bundle functors on $\mathcal{VB}_m$. The composition of vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$ is again a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

In [6], we classified all fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$ of finite order r in terms of triples (V, H, t), where V is a finite-dimensional vector space over $\mathbb{R}$, $H : G^r_m \to \mathrm{GL}(V)$ is a smooth group homomorphism from $G^r_m = \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ into $\mathrm{GL}(V)$ and $t : \mathcal{D}^r_m \to \mathrm{gl}(V)$ is a G^r_m-equivariant unity preserving associative algebra homomorphism from $\mathcal{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ into $\mathrm{gl}(V)$. Moreover, we proved that all fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$ are of finite order. Analyzing the construction on (V, H, t) one can easily see that the triple (V, H, t) corresponding to a vertical F in question has trivial t. This implies that all vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$ can be constructed (up to $\mathcal{VB}_m$-equivalence) as follows.

Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a vector natural bundle. For any $\mathcal{VB}_m$-object $p : E \to M$ we put $F^V E = E \otimes_M VM$ and for any $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$ we put $F^V f = f \otimes_{\underline{f}} V\underline{f} : F^V E_1 \to F^V E_2$. The correspondence $F^V : \mathcal{VB}_m \to \mathcal{FM}$ is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

In this note we study the problem how a linear vector field X on a vector bundle E from $\mathcal{VB}_{m,n}$ can induce canonically a vector field $A(X)$ on FE for some vertical fiber product preserving gauge bundle functor $F : \mathcal{VB}_m \to \mathcal{FM}$. This problem is reflected in the concept of $\mathcal{VB}_{m,n}$-natural operators $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$ in the sense of [3]. We can assume that

$F = F^V$ for some vector natural bundle $V : \mathcal{M}f_m \to \mathcal{VB}$. We prove that any such operator A is of the form

$$A(X) = \lambda \mathcal{F}^V X + L \otimes B(\underline{X})$$

for some real number λ and some $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ of vertical type transforming vector fields $\underline{X} \in \mathcal{X}(M)$ from m-dimensional manifolds M into linear vertical vector fields $B(\underline{X})$ on VM, where X denotes a linear vector field on a $\mathcal{VB}_{m,n}$-object E, $\underline{X}$ is the underlying vector field of X, $\mathcal{F}^V X$ is the flow lifting of X, and L is the Liouville vector field on E.

In Section 3, we explain this main result for some known examples of vertical fiber product preserving gauge bundle functors F on $\mathcal{VB}_m$. Thus for J^r_v we recover the result from [6] saying that any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TJ^r_v$ is a linear combination with real coefficients of the flow operator, the Liouville vector field on J^r_v and some explicitly defined natural operators V^s for $s = 0, 1, \dots, r$.

The trivial vector bundle $\mathbb{R}^m \times \mathbb{R}^n$ over $\mathbb{R}^m$ with standard fiber $\mathbb{R}^n$ will be denoted by $\mathbb{R}^{m,n}$. The coordinates on $\mathbb{R}^m$ will be denoted by $x^1, \dots, x^m$. The fiber coordinates on $\mathbb{R}^{m,n}$ will be denoted by $y^1, \dots, y^n$.

All manifolds are assumed to be finite-dimensional and smooth. Maps are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$.

1. The main result. From now on, let F be a fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. We can assume that $F = F^V$ for some natural vector bundle $V : \mathcal{M}f_m \to \mathcal{VB}$, where F^V is described in the Introduction.

Let $p : E \to M$ be a vector bundle. A projectable vector field X on E with the underlying vector field $\underline{X}$ on M is *linear* if $X : E \to TE$ is a vector bundle map from $p : E \to M$ into the vector bundle $Tp : TE \to TM$ covering $\underline{X} : M \to TM$. Equivalently, the flow of X is formed by vector bundle maps. The space of all linear vector fields on E will be denoted by $\mathcal{X}_{\mathrm{lin}}(E)$.

A *$\mathcal{VB}_{m,n}$-natural operator* $T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^V$ is a $\mathcal{VB}_{m,n}$-invariant family of regular operators $A : \mathcal{X}_{\mathrm{lin}}(E) \to \mathcal{X}(F^V E)$ for $\mathcal{VB}_{m,n}$-objects E. The invariance means that if linear vector fields X' and X'' on $\mathcal{VB}_{m,n}$-vector bundles are f-related then $A(X')$ and $A(X'')$ are $F^V f$-related. The regularity means that A transforms a smoothly parametrized family of linear vector fields into a smoothly parametrized family of vector fields.

We now present examples of $\mathcal{VB}_{m,n}$-natural operators $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^V$.

EXAMPLE 1 (*The flow operator*). Let X be a linear vector field on a $\mathcal{VB}_{m,n}$-object E. The flow Fl^X_t of X is formed by $\mathcal{VB}_{m,n}$-morphisms. Applying F^V we obtain a flow $F^V(Fl^X_t)$ on $F^V E$. The last flow generates a vector field $\mathcal{F}^V X$ on $F^V E$. The corresponding $\mathcal{VB}_{m,n}$-operator $\mathcal{F}^V : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^V$ is called the *flow operator*.

EXAMPLE 2. Suppose we have a vertical type $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ lifting vector fields $\underline{X}$ on m-manifolds M to linear vector fields $B(\underline{X})$ on VM. Let X be a linear vector field on a $\mathcal{VB}_{m,n}$-object $E \to M$ with the underlying vector field $\underline{X}$ on M. Applying B to $\underline{X}$ we produce the linear vertical vector field $B(\underline{X})$ on VM. On E we have the Liouville vector field L generated by fiber homotheties of E. Clearly, L is vertical and linear. Then we have the tensor product $L \otimes B(\underline{X}) \in \mathcal{X}_{\mathrm{lin}}(F^V E)$ (generated by the tensor product of the flows of L and $B(\underline{X})$). Thus we have the corresponding $\mathcal{VB}_{m,n}$-natural operator $L \otimes B : T_{\mathrm{lin}\mathcal{VB}_{m,n}} \rightsquigarrow TF^V$ of vertical type.

The main result of the present paper is the following classification theorem.

THEOREM 1. *Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a natural vector bundle. Any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^V$ is of the form*

$$A(X) = \lambda \mathcal{F}^V X + L \otimes B(\underline{X})$$

for some real number λ and some $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ lifting vector fields $\underline{X}$ on m-manifolds M to vertical linear vector fields $B(\underline{X})$ on VM, where X denotes a linear vector field on a $\mathcal{VB}_{m,n}$-object E, $\underline{X}$ is the underlying vector field of X, $\mathcal{F}^V X$ is the flow lifting of X, and L is the Liouville vector field on E.

2. Proof of Theorem 1. Since any linear vector field X with the non-vanishing underlying vector field is locally $\partial/\partial x^1$ in some $\mathcal{VB}_{m,n}$-trivialization, A is uniquely determined by $A(\partial/\partial x^1)$ over $0 \in \mathbb{R}^m$.

Consider $T\pi \circ A(\partial/\partial x^1)(y) \in T_0\mathbb{R}^m \cong \mathbb{R}^m$ for $u \in F^V_0\mathbb{R}^{m,n}$, where $\pi : F^V E \to M$ is the bundle projection. Using the invariance of $A(\partial/\partial x^1)$ with respect to the fiber homotheties we deduce that $T\pi \circ A(\partial/\partial x^1)(u) = T\pi \circ A(\partial/\partial x^1)(tu)$ for any $u \in F^V_0\mathbb{R}^{m,n}$, $t \neq 0$. Then $T\pi \circ A(\partial/\partial x^1)(u) = T\pi \circ A(\partial/\partial x^1)(0)$ for u as above. Thus using the invariance of $A(\partial/\partial x^1)$ with respect to the $\mathcal{VB}_{m,n}$-maps $(x^1, tx^2, \ldots, tx^m, y^1, \ldots, y^n)$ we deduce that $T\pi \circ A(\partial/\partial x^1)(u) = \lambda e_1$ for some real number λ, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m$. Now replacing A by $A - \lambda\mathcal{F}^V$ we have $T\pi \circ A(\partial/\partial x^1)(u) = 0$ for any u as above. Therefore A is of vertical type because of the first sentence of the proof.

We define an $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ as follows.

Let $\underline{X} \in \mathcal{X}(M)$ be a vector field on an m-manifold M. We consider $\underline{X}$ as the linear vector field $\underline{X}$ on the trivial vector bundle $M \times \mathbb{R}^n$ over M. Applying A we obtain the vertical vector field $A(\underline{X})$ on $F^V(M \times \mathbb{R}^n)$. Using the invariance of $A(\underline{X})$ with respect to the fiber homotheties of $M \times \mathbb{R}^n$ we easily see that $A(\underline{X})$ is linear. Define a fibered map $\widetilde{A}(\underline{X}) : VM \to VM$ covering id_M by

$$\widetilde{A}(\underline{X})(y) = \langle \mathrm{pr}_2 \circ A(\underline{X})(e_1(x) \otimes y), e_1^* \rangle ,$$

$y \in V_xM$, $x \in M$, where $\mathrm{pr}_2 : V(F^V(M\times\mathbb{R}^n)) \cong F^V(M\times\mathbb{R}^n)\times_M F^V(M\times\mathbb{R}^n) \to F^V(M \times \mathbb{R}^n)$ is the projection on the second (essential) factor, $\cong$ is the usual identification of the vertical bundle $V(F^V(M \times \mathbb{R}^n))$ of the vector bundle $F^V(M \times \mathbb{R}^n)$ with $F^V(M \times \mathbb{R}^n) \times_M F^V(M \times \mathbb{R}^n)$, $e_1, \dots, e_n$ is the usual basis of sections of $M \times \mathbb{R}^n$, and $e_1^*, \dots, e_n^*$ is the dual basis. Since $A(\underline{X})$ is linear, $\widetilde{A}(\underline{X})$ is a vector bundle map. Consequently, we have a linear vertical vector field $B(\underline{X})$ on VM such that

$$B(\underline{X})(y) = \frac{d}{dt}\Big|_0 (y + t\widetilde{A}(\underline{X})(y))$$

for $y \in V_xM$, $x \in M$. Let $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ be the corresponding $\mathcal{M}f_m$-natural operator.

It remains to show that $A = L \otimes B$. It is immediately seen that

$$\langle \mathrm{pr}_2 \circ (L \otimes B(\partial/\partial x^1))(e_1(0) \otimes y), e_1^*(0) \rangle = \langle \mathrm{pr}_2 \circ A(\partial/\partial x^1)(e_1(0) \otimes y), e_1^*(0) \rangle$$

for all $y \in V_0\mathbb{R}^m$. Let $i = 2, \dots, n$. Then by the invariance of $A(\partial/\partial x^1)$ and $L \otimes B(\partial/\partial x^1)$ with respect to a $\mathcal{VB}_{m,n}$-map of the form $\mathrm{id}_{\mathbb{R}^m} \times \psi$ for a linear isomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ preserving e_1 and sending e_1^* into $e_1^* + e_i^*$ we obtain

$$\langle \mathrm{pr}_2 \circ (L \otimes B(\partial/\partial x^1))(e_1(0) \otimes y), e_i^*(0) \rangle = \langle \mathrm{pr}_2 \circ A(\partial/\partial x^1)(e_1(0) \otimes y), e_i^*(0) \rangle$$

for all $i = 1, \dots, n$ and all $y \in V_0\mathbb{R}^m$. Then

$$\mathrm{pr}_2 \circ (L \otimes B(\partial/\partial x^1))(e_1(0) \otimes y) = \mathrm{pr}_2 \circ A(\partial/\partial x^1)(e_1(0) \otimes y)$$

for all $y \in V_0\mathbb{R}^m$. Now by the invariance of $L \otimes B(\partial/\partial x^1)$ and $A(\partial/\partial x^1)$ with respect to $\mathcal{VB}_{m,n}$-maps of the form $\mathrm{id}_{\mathbb{R}^m} \times \psi$ for linear isomorphisms $\psi : \mathbb{R}^n \to \mathbb{R}^n$ we deduce that

$$\mathrm{pr}_2 \circ (L \otimes B(\partial/\partial x^1))(e \otimes y) = \mathrm{pr}_2 \circ A(\partial/\partial x^1)(e \otimes y)$$

for all non-zero $e \in (\mathbb{R}^{m,n})_0 \cong \mathbb{R}^n$ and all $y \in V_0\mathbb{R}^m$. Hence

$$\mathrm{pr}_2 \circ A(\partial/\partial x^1)(u) = \mathrm{pr}_2 \circ (L \otimes B(\partial/\partial x^1))(u)$$

for all $u \in F_0^V\mathbb{R}^{m,n}$. (Indeed, the $e \otimes y$ as above generate $(\mathbb{R}^{m,n})_0 \otimes V_0\mathbb{R}^m$, and both sides of the last equality are linear in u because of the invariance of $A(\partial/\partial x^1)$ and $L \otimes B(\partial/\partial x^1)$ with respect to the fiber homotheties of $\mathbb{R}^{m,n}$ and the homogeneous function theorem.) Thus $A(\partial/\partial x^1) = L \otimes B(\partial/\partial x^1)$ over $0 \in \mathbb{R}^m$. Hence $A = L \otimes B$ because of the first sentence of the proof.

3. Applications. Let p be a positive integer. Let $T^{(p,0)} = \bigotimes^p T : \mathcal{M}f_m \to \mathcal{VM}$ be the natural vector bundle of tensor fields of type $(p,0)$ over m-manifolds. Let $F^{(p,0)} = F^{T^{(p,0)}} : \mathcal{VB}_m \to \mathcal{FM}$, $F^{(p,0)}E = E \otimes_M T^{(p,0)}M$, $F^{(p,0)}f = f \otimes_{\underline{f}} T^{(p,0)}\underline{f}$ be the corresponding vertical fiber product preserving gauge bundle functor (see Introduction). In [2], I. Kolář showed that any $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TT^{(p,0)}$ lifting vector fields $\underline{X}$ on m-manifolds M to linear vertical vector fields $B(\underline{X})$ on $T^{(p,0)}M$ is a constant multiple of the Liouville vector field on $T^{(p,0)}$. Thus we have the following corollary.

COROLLARY 1. *Any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^{(p,0)}$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on $F^{(p,0)}$.*

Let $T^{(r)} = (J^r(\cdot,\mathbb{R})_0)^* : \mathcal{M}f_m \to \mathcal{VB}$ be the linear r-tangent bundle functor. Let $T^{[r]fl} = F^{T^{(r)}} : \mathcal{VB}_m \to \mathcal{FM}$ be the corresponding vertical fiber product preserving gauge bundle functor. We call $T^{[r]fl}$ the $[r]$-*tangent gauge bundle functor* (see [4]). In [9], we showed that any $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TT^{(r)}$ lifting vector fields $\underline{X}$ on m-manifolds M to linear vertical vector fields $B(\underline{X})$ on $T^{(r)}M$ is a constant multiple of the Liouville vector field on $T^{(r)}$ (see also [1] in the case $r = 2$). Thus we recover the following result.

COROLLARY 2 ([4]). *Any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{[r]fl}$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on $T^{[r]fl}$.*

Let $T^{r*} = J^r(\cdot,\mathbb{R})_0 : \mathcal{M}f_m \to \mathcal{VB}$ be the r-cotangent bundle functor. Let $F^{r*} = F^{T^{r*}} : \mathcal{VB}_m \to \mathcal{FM}$ be the corresponding vertical fiber product preserving gauge bundle functor. From [8] it follows that any $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TT^{r*}$ sending vector fields on M to linear vector fields on $T^{r*}M$ is a constant multiple of the Liouville vector field on T^{r*}. Of course, $F^{r*}E = (T^{[r]fl}E^*)^*$. Thus we recover the following result.

COROLLARY 3 ([4]). *Any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^{r*}$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on F^{r*}.*

Let $E^{r*} = T^{r*} \times \mathbb{R} = J^r(\cdot,\mathbb{R}) : \mathcal{M}f_m \to \mathcal{VB}$ be the extended r-cotangent bundle functor. The corresponding vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ is equivalent to the vertical r-jet functor J^r_v (see [6], [5]).

LEMMA 1. *The vector space of all $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TE^{r*}$ transforming vector fields $\underline{X}$ on m-manifolds M into linear vertical vector fields $B(\underline{X})$ on $E^{r*}M$ is of dimension less than or equal to $r+2$.*

Schema of the proof. Any B in question is uniquely determined by the linear map $\widetilde{B} = \mathrm{pr}_2 \circ B(\partial/\partial x^1) : T_0^{r*}\mathbb{R}^m \times \mathbb{R} \to T_0^{r*}\mathbb{R}^m \times \mathbb{R}$, where $\mathrm{pr}_2 : V(E^{r*}\mathbb{R}^m) \cong E^{r*}\mathbb{R}^m \times E^{r*}\mathbb{R}^m \to E^{r*}\mathbb{R}^m$ is the projection onto the second (essential) factor. Using obvious inclusions and projections we see that $\widetilde{B}$ is determined by four obvious linear maps: $T^{r*}\mathbb{R}^m \to T^{r*}\mathbb{R}^m$, $\mathbb{R} \to T^{r*}\mathbb{R}^m$, $T^{r*}\mathbb{R}^m \to \mathbb{R}$ and $\mathbb{R} \to \mathbb{R}$. The first map corresponds to a natural operator $T_{\mathcal{M}f_m} \rightsquigarrow TT^{r*}$ with suitable properties. So, by the result from [9] mentioned before Corollary 2, the first map is a constant multiple of the identity. The last map is obviously a constant multiple of the identity. The second map is zero as is easy to show using the invariance of B with respect to homotheties and the homogeneous function theorem. The third map is determined by the evaluations at $j_0^r(x^1)^s$ for $s = 1, \ldots, r$ as is easy to see by using the invariance of $B(\partial/\partial x^1)$ with respect to the $\mathcal{M}f_m$-maps $(x^1, tx^2, \ldots, tx^m)$ for $t \neq 0$. ■

Thus using the dimension argument we recover the following result.

COROLLARY 4 ([7]). *Any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TJ_v^r$ is a linear combination with real coefficients of the flow operator, the Liouville vector field on J_v^r and some operators V^s for $s = 0, \ldots, r$ defined by $V^s(X)_{j_x^r\sigma} = \frac{d}{dt}{}_0(j_x^r\sigma + tj_x^r(\underline{X}^s\sigma(x)))$, where X is a linear vector field on $E \to M$ with the underlying vector field $\underline{X}$, $\sigma : M \to E_x$ is a map, $x \in M$, and where $\underline{X}^s\sigma(x) = \underline{X} \circ \cdots \circ \underline{X}\sigma(x)$ (with $\underline{X}$ repeated s times) is treated as the constant map $M \to E_x$.*

We have the following obvious lemma.

LEMMA 2. *Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a natural vector bundle. Let $V^* : \mathcal{M}f_m \to \mathcal{VB}$ be the vector natural bundle dual to V, $V^*M = (VM)^*$, $V^*f = (V(f^{-1}))^*$. Any $\mathcal{M}f_m$-natural operator $C : T_{\mathcal{M}f_m} \rightsquigarrow TV^*$ lifting vector fields $\underline{X}$ on m-manifolds M to vertical linear vector fields $C(\underline{X})$ on V^*M is of the form $C(\underline{X}) = (B(\underline{X}))^*$ for some (uniquely determined by B) $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ lifting vector fields $\underline{X}$ on m-manifolds M to vertical linear vector fields $B(\underline{X})$ on VM, where given a linear vertical vector field X on a vector bundle E, X^* is the linear vertical vector field on E^* dual to X (if f_t is the flow of X, then $(f_t^{-1})^*$ is the flow of X^*).*

Using Lemma 2 and Theorem 1 we immediately obtain

COROLLARY 5. *Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a natural vector bundle. Any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^{V^*}$ is of the form $A(X) = \lambda\mathcal{F}^{V^*}X + L \otimes (B(\underline{X}))^*$ for some real number λ and some $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TV$ lifting vector fields $\underline{X}$ on m-manifolds M to vertical linear vector fields $B(\underline{X})$ on VM, where X denotes a linear vector field over $\underline{X}$.*

In particular, any $\mathcal{VB}_{m,n}$-natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF^{E^{(r)}}$ is a linear combination with real coefficients of the flow operator, the Liouville vector field on $F^{E^{(r)}}$ and the natural operators $(V^s)^*$ for $s = 1, \dots, r$, $(V^s)^*(X) = (V^s(X))^*$, where $E^{(r)} = (J^r(\cdot, \mathbb{R}))^* : \mathcal{M}f_m \to \mathcal{VB}$ is the extended linear r-tangent bundle and where the operators V^s are as in Corollary 4.

Since $F^{E^{(r)}}E \cong (J^r_v E^*)^*$, the second part of Corollary 5 is Theorem 8 in [7].

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Institute of Mathematics
Maria Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: kurek@golem.umcs.lublin.pl

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: mikulski@im.uj.edu.pl

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