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AN ANTI-KÄHLERIAN EINSTEIN STRUCTURE ON THE TANGENT BUNDLE OF A SPACE FORM

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Abstract. In [11] we have considered a family of almost anti-Hermitian structures (G, J) on the tangent bundle TM of a Riemannian manifold (M, g), where the almost complex structure J is a natural lift of g to TM interchanging the vertical and horizontal distributions VTM and HTM and the metric G is a natural lift of g of Sasaki type, with the property of being anti-Hermitian with respect to J. Next, we have studied the conditions under which (TM, G, J) belongs to one of the eight classes of anti-Hermitian structures obtained in the classification in [2]. In this paper, we study some geometric properties of the anti-Kählerian structure obtained in [11]. In fact we prove that it is Einstein. This result offers nice examples of anti-Kählerian Einstein manifolds studied in [1].

1. Introduction. Let (M, g) be an n-dimensional Riemannian manifold and denote by $\tau : TM \to M$ its tangent bundle. There are several Riemannian and semi-Riemannian metrics induced on TM from the Riemannian metric g on M. Among them, we may quote the Sasaki metric and the complete lift of the metric g. On the other hand, there are natural lifts of g to TM, leading to several new geometric structures with many nice geometric properties (see [3], [4], [13]).

In [11] we have considered an almost anti-Hermitian structure (G, J), defined on TM by using some natural lifts of the Riemannian metric g. The vertical distribution VTM and the horizontal distribution HTM are interchanged by the almost complex structure J, and the metric G is a natural lift of g of Sasaki type with the property of being anti-Hermitian with respect to J. We have studied the conditions under which the almost anti-Hermitian structure (TM, G, J) belongs to one of the eight classes obtained in [2].

In this paper we consider the case when the almost anti-Hermitian structure considered in [11] is anti-Kählerian, and we show that in this case (TM, G, J) is an Einstein manifold.

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The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l being always $\{1, \ldots, n\}$.

2. An anti-Kählerian structure on TM. Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \to M$. Recall that TM has a structure of a $2n$-dimensional smooth manifold, induced from the structure of smooth n-dimensional manifold of M. Every local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n,\ y^1, \ldots, y^n)$, on TM, as follows. For a tangent vector $y \in \tau^{-1}(U) \subset TM$, the first local coordinates $x^1, \ldots, x^n$ are the local coordinates $x^1, \ldots, x^n$ of its base point $x = \tau(y)$ in the local chart (U, φ) (in fact we abuse notation, identifying x^i with $\tau^* x^i = x^i \circ \tau,\ i = 1, \ldots, n$). The last n local coordinates $y^1, \ldots, y^n$ of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis $((\partial/\partial x^1)_{\tau(y)}, \ldots, (\partial/\partial x^1)_{\tau(y)})$, defined by the local chart (U, φ), i.e. $y = y^i (\partial/\partial x^i)_{\tau(y)}$.

We shall use the horizontal distribution HTM, defined by the Levi-Civita connection $\dot\nabla$ of g, in order to define some natural lifts to TM of the Riemannian metric g on M. Denote by $VTM = \operatorname{Ker} \tau_* \subset TTM$ the vertical distribution on TM. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM. \tag{1}$$

If $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)$ is a local chart on TM, induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, then the local vector fields $\partial/\partial y^1, \ldots, \partial/\partial y^n$ on $\tau^{-1}(U)$ define a local frame for VTM over $\tau^{-1}(U)$ and the local vector fields $\delta/\delta x^1, \ldots, \delta/\delta x^n$ define a local frame for HTM over $\tau^{-1}(U)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\partial y^h}, \qquad \Gamma^h_{0i} = y^k \Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $(\partial/\partial y^1, \ldots, \partial/\partial y^n, \delta/\delta x^1, \ldots, \delta/\delta x^n)$ defines a local frame on TM, adapted to the direct sum decomposition (1). Note that

$$\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^V, \qquad \frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^H,$$

where X^V and X^H denote the vertical and horizontal lifts of the vector field X on M.

Let $C = y^i\partial/\partial y^i$ be the Liouville vector field on TM and consider the horizontal vector field $\widetilde{C} = y^i\delta/\delta x^i$ on TM, defined in a similar way.

Since we work in a fixed local chart (U, φ) on M and in the corresponding induced local chart $(\tau^{-1}(U), \Phi)$ on TM, we shall use the following simpler notations:

$$\frac{\partial}{\partial y^i} = \partial_i, \qquad \frac{\delta}{\delta x^i} = \delta_i.$$

Denote by

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^iy^k, \qquad y \in \tau^{-1}(U), \tag{2}$$

the energy density defined by g at the tangent vector y. We have $t \in [0,\infty)$ for all $y \in TM$. Consider real-valued smooth functions a_1, a_2, b_1, b_2 defined on $[0,\infty) \subset \mathbb{R}$ and define a tensor field J of type $(1,1)$ on TM, obtained as a natural lift, by using these coefficients and the Riemannian metric g, just like the first order natural lifts of g to TM are obtained in [3]. If the coefficients a_1, a_2, b_1, b_2 are related by some specific algebraic relations, the tensor field J will define an almost complex structure on TM. The expression of J is given by (see [5], [14], [12], [6], [7], [9])

$$\begin{cases} JX_y^H = a_1(t)X_y^V + b_1(t)g_{\tau(y)}(y,X)C_y, \\ JX_y^V = -a_2(t)X_y^H - b_2(t)g_{\tau(y)}(y,X)\widetilde{C}_y. \end{cases} \tag{3}$$

The expression of J in adapted local frames is given by

$$J\delta_i = a_1(t)\partial_i + b_1(t)g_{0i}C,$$
$$J\partial_i = -a_2(t)\delta_i - b_2(t)g_{0i}\widetilde{C}.$$

Now, we consider the following particular first order natural lift G of g to TM, defined by four real-valued smooth functions $c_1, d_1, c_2, d_2 : [0,\infty) \to \mathbb{R}$:

$$\begin{cases} G_y(X^H, Y^H) = c_1(t)g_{\tau(y)}(X,Y) + d_1(t)g_{\tau(y)}(y,X)g_{\tau(y)}(y,Y), \\ G_y(X^V, Y^V) = c_2(t)g_{\tau(y)}(X,Y) + d_2(t)g_{\tau(y)}(y,X)g_{\tau(y)}(y,Y), \\ G_y(X^H, Y^V) = G_y(Y^V, X^H) = 0. \end{cases} \tag{4}$$

The expression of G in local adapted frames is

$$G^{(1)}_{ij} = G(\delta_i, \delta_j) = c_1g_{ij} + d_1g_{0i}g_{0j}, \qquad G^{(2)}_{ij} = G(\partial_i, \partial_j) = c_2g_{ij} + d_2g_{0i}g_{0j},$$
$$G(\partial_i, \delta_j) = G(\delta_i, \partial_j) = 0.$$

If the coefficients $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are related by some specific algebraic relations, the tensor fields G, J will define an almost anti-Hermitian structure on TM.

Proposition 1. *The Levi-Civita connection ∇ of the pseudo-Riemannian metric G on TM has the following expression in the local adapted frame*

$(\partial_i, \dots, \partial_n, \delta_i, \dots, \delta_n)$:

$$\nabla_{\partial_i}\partial_j = Q^h_{ij}\partial_h, \quad \nabla_{\delta_i}\partial_j = \Gamma^h_{ij}\partial_h + P^h_{ji}\delta_h,$$
$$\nabla_{\partial_i}\delta_j = P^h_{ij}\delta_h, \quad \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S^h_{ij}\partial_h,$$

where the M-tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij} are given by

$$\begin{aligned}
P^h_{ji} &= \frac{c_1'}{2c_1}\, g_{0j}\delta^h_i + \frac{d_1}{2c_1}\, g_{0i}\delta^h_j - \frac{c_1'd_1 + (d_1)^2 - c_1d_1'}{2c_1(c_1+2td_1)}\, g_{0i}g_{0j}y^h \\
&\quad + \frac{d_1}{2(c_1+2td_1)}\, g_{ij}y^h - \frac{c_2}{2c_1}\, R^h_{ij0} - \frac{c_2d_1}{2c_1(c_1+2td_1)}\, R_{j0i0}y^h, \\
Q^h_{ij} &= \frac{c_2'}{2c_2}\,(g_{0i}\delta^h_j + g_{0j}\delta^h_i) + \frac{2d_2 - c_2'}{2(c_2+2td_2)}\, g_{ij}y^h + \frac{c_2d_2' - 2d_2c_2'}{2c_2(c_2+2td_2)}\, g_{0i}g_{0j}y^h, \\
S^h_{ij} &= -\frac{d_1}{2c_2}\,(g_{0i}\delta^h_j + g_{0j}\delta^h_i) - \frac{c_1'}{2(c_2+2td_2)}\, g_{ij}y^h \\
&\quad + \frac{2d_1d_2 - c_2d_1'}{2c_2(c_2+2td_2)}\, g_{0i}g_{0j}y^h - \frac{1}{2}\, R^h_{0ij},
\end{aligned}$$

R_{likj} denoting the local coordinate components of the Riemann–Christoffel tensor of $\dot\nabla$ on M, and $R_{0ikj} = R_{likj}y^l$, $R_{0ij0} = R_{lijk}y^ly^k$.

The condition under which the almost anti-Hermitian structure defined by (G, J) on TM is anti-Kählerian is expressed by the relation $F = 0$, where the tensor field F of type $(0,3)$ is given by

$$F(X,Y,Z) = G((\nabla_X J)Y, Z), \quad X, Y, Z \in \Gamma(TM).$$

An equivalent condition is $\nabla J = 0$ (see [2], [8], [10], [11]). By using the expression of F, presented above, we have shown in [11] that (TM, G, J) is anti-Kählerian if and only if the base manifold (M, g) has constant sectional curvature c and the functions $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are given by

$$(5) \quad \begin{cases} a_1 = \sqrt{B+2ct}, \quad b_1 = 0, \quad c_1 = A(B+2ct), \quad d_1 = -cA, \\ a_2 = \dfrac{1}{\sqrt{B+2ct}}, \quad b_2 = 0, \quad c_2 = -A, \qquad d_2 = \dfrac{cA}{B+2ct}, \end{cases}$$

where A is a nonzero real constant and B is a positive constant (see Theorem 7 in [11]).

Note that if the sectional curvature c of (M, g) is positive, we obtain an anti-Kählerian structure on the whole TM, but if $c < 0$, then the anti-Kählerian structure is defined only in the tube $t < -B/2c$ around the zero section in TM.

From now on, we assume that (TM, G, J) is an anti-Kählerian manifold, i.e. the base manifold has constant sectional curvature c and the functions $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are given by (5). Then the expressions of the

M-tensor fields $P_{ij}^h,\ Q_{ij}^h, S_{ij}^h$ which appear in Proposition 1 become

$$(6)\qquad \begin{cases} P_{ij}^h = \dfrac{c}{B+2ct}\, g_{0i}\delta_j^h - \dfrac{c}{B}\, g_{ij}y^h + \dfrac{c^2}{B(B+2ct)}\, g_{0i}g_{0j}y^h, \\ Q_{ij}^h = -\dfrac{c}{B}\, g_{ij}y^h + \dfrac{c^2}{B(B+2ct)}\, g_{0i}g_{0j}y^h, \\ S_{ij}^h = -cg_{0j}\delta_i^h + \dfrac{c(B+2ct)}{B}\, g_{ij}y^h - \dfrac{c^2}{B}\, g_{0i}\, g_{0j}y^h. \end{cases}$$

The main result obtained in this paper is that the anti-Kählerian manifold (TM, G, J) is an Einstein manifold. Recall that the curvature tensor field K of the Levi-Civita connection ∇ of G is defined by

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X,Y,Z \in \Gamma(TM).$$

By a straightforward computation using Proposition 1 and formula (6), we obtain

$$\begin{aligned} K(\delta_i,\delta_j)\delta_k &= K_{kij}^h\delta_h, & K(\delta_i,\delta_j)\partial_k &= K_{kij}^h\partial_h, \\ K(\partial_i,\partial_j)\delta_k &= L_{kij}^h\delta_h, & K(\partial_i,\partial_j)\partial_k &= L_{kij}^h\partial_h, \\ K(\partial_i,\delta_j)\delta_k &= K_{kij}^h\partial_h, & K(\partial_i,\delta_j)\partial_k &= L_{kij}^h\delta_h, \end{aligned}$$

where

$$K_{kij}^h = \frac{c}{AB}\,(G_{jk}^{(1)}\delta_i^h - G_{ik}^{(1)}\delta_j^h), \qquad L_{kij}^h = \frac{c}{AB}\,(G_{jk}^{(2)}\delta_i^h - G_{ik}^{(2)}\delta_j^h).$$

From these expressions we deduce that the Ricci tensor $S(Y,Z) = \mathrm{trace}(X \to K(X,Y)Z)$ is expressed in the local adapted frame by

$$(7)\qquad \begin{cases} S(\partial_j,\partial_k) = \dfrac{2c(n-1)}{B}\left[-g_{jk} + \dfrac{c}{B+2ct}\, g_{0j}g_{0k}\right], \\ S(\delta_j,\delta_k) = \dfrac{2c(n-1)}{B}\,[(B+2ct)g_{jk} - cg_{0j}g_{0k}], \\ S(\partial_j,\delta_k) = S(\delta_k,\partial_j) = 0. \end{cases}$$

By using (5) and the expression of G in the local adapted frame, we get from (7)

$$(8)\quad \begin{cases} S(\delta_j,\delta_k) = \dfrac{2c(n-1)}{AB}\, G(\delta_j,\delta_k), \quad S(\partial_j,\partial_k) = \dfrac{2c(n-1)}{AB}\, G(\partial_j,\partial_k), \\ S(\partial_j,\delta_k) = S(\delta_k,\partial_j) = G(\partial_j,\delta_k) = G(\delta_k,\partial_j) = 0. \end{cases}$$

Hence

$$S(X,Y) = \frac{2c(n-1)}{AB}\, G(X,Y), \qquad X,Y \in \Gamma(TM),$$

i.e. the metric G is Einstein with the factor $2c(n-1)/AB$. Thus we may state

THEOREM 2. *The anti-Kählerian structure* (G,J) *on* TM *defined by* (3), (4) *with the coefficients* $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ *given by* (5) *is Einstein.*

Remark. This result is quite interesting since it yields a large class of Einstein anti-Kählerian manifolds studied recently in [1]. However, these manifolds are not compact.

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