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ŁOJASIEWICZ EXPONENTS AND SINGULARITIES AT INFINITY OF POLYNOMIALS IN TWO COMPLEX VARIABLES

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Abstract. For every polynomial F in two complex variables we define the Łojasiewicz exponents $\mathcal{L}_{p,t}(F)$ measuring the growth of the gradient ∇F on the branches centered at points p at infinity such that F approaches t along γ. We calculate the exponents $\mathcal{L}_{p,t}(F)$ in terms of the local invariants of singularities of the pencil of projective curves associated with F.

Introduction. The notion of Łojasiewicz exponent was introduced and studied by Lejeune-Jalabert and Teissier [LJ-T]. In the case of isolated singularities of hypersurfaces Teissier [T] showed that the Łojasiewicz exponent of the gradient can be calculated by means of polar invariants. Then Hà [H] defined the Łojasiewicz exponents at infinity $\mathcal{L}_{\infty,t}(F)$ and $\mathcal{L}_{\infty}(F)$ for every polynomial F of two complex variables and applied these notions to the singularities at infinity. His results were completed by Chądzyński and Krasiński [ChK1], [ChK2]. Moreover Cassou-Noguès and Hà [CN-H] gave a formula for $\mathcal{L}_{\infty}(F)$ using the Eisenbud and Neumann diagrams.

In this note we define the Łojasiewicz exponents $\mathcal{L}_{p,t}(F)$ at points (p,t) at infinity. To be more specific, consider a polynomial $F:\mathbb{C}^2\to\mathbb{C}$ of degree $d>0$. To study the growth of the gradient $\nabla F=(\partial F/\partial X,\partial F/\partial Y)$ at infinity we extend $\mathbb{C}^2$ to the projective plane $\mathbb{P}^2(\mathbb{C})=\mathbb{C}^2\cup\mathbb{L}_\infty$ and consider the Łojasiewicz exponents $\mathcal{L}_{p,t}(F)$ for every pair $(p,t)\in\mathbb{L}_\infty\times\mathbb{C}$. Roughly speaking (see Section 1 for the precise definition), $\mathcal{L}_{p,t}(F)$ measures the growth of ∇F on the branches γ of $\mathbb{P}^2(\mathbb{C})$ centered at p such that F approaches $t\in\mathbb{C}$ along γ. Let $F^*(X,Y,Z)=Z^dF(X/Z,Y/Z)$ be the homogeneous form corresponding to $F=F(X,Y)$. We will show how to calculate $\mathcal{L}_{p,t}(F)$ in terms of the local invariants of singularities at infinity of the pencil of projective curves $F^*(X,Y,Z)-tZ^d$, $t\in\mathbb{C}$. It turns out that if (p,t) is a critical point at infinity for F (see [D] and Preliminaries 0.3) then $\mathcal{L}_{p,t}(F)$ can be calculated by means of the polar invariants, as in the local case (see [T]). If (p,t) is a regular point at infinity then we need another invariant of singu-

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larity to calculate $\mathcal{L}_{p,t}(F)$ (see Preliminaries 0.2 and Theorem 1.2). Putting $\mathcal{L}_{\infty,t}(F) = \inf\{\mathcal{L}_{p,t}(F) : p \in \mathbb{L}_\infty\}$ we get the Łojasiewicz exponent of the gradient ∇F along the fiber $F = t$. This notion was studied by Hà [H] and recently by Chądzyński and Krasiński in [ChK2] in a global affine context. In particular in [ChK2] it was proved that there is a constant $\ell_\infty(F) \geq 0$ such that $\mathcal{L}_{\infty,t}(F) = \ell_\infty(F)$ for all regular values t of the mapping F. In general $\ell_\infty(F) \neq \mathcal{L}_\infty(F)$. The constant $\ell_\infty(F)$ characterizes the growth of the gradient ∇F on the regular fibers $F^{-1}(t)$. In this paper we give a description of polynomials F with $\ell_\infty(F) = 0$ (Theorem 1.3(i)) and we calculate $\ell_\infty(F)$ in the case of one branch at infinity (Proposition 1.9). Our main theorems (Theorem 1.2 and 1.3) improve the results obtained in [H] and [ChK2] and show that the Łojasiewicz exponent at infinity is a purely local notion.

0. Preliminaries. In this section we fix our notation and recall some useful notions and results.

0.1. *Branches at infinity.* We use the notions of the classical theory of plane algebraic curves. Let $\mathbb{P}^2(\mathbb{C}) = \mathbb{C}^2 \cup \mathbb{L}_\infty$, where $\mathbb{L}_\infty$ is the line at infinity. A plane branch γ will be called a *branch at infinity* if it is centered at a point $p \in \mathbb{L}_\infty$ and it is not a branch of $\mathbb{L}_\infty$. We denote by $\mathcal{B}_{\infty,p}$ the set of all branches at infinity centered at p and put $\mathcal{B}_\infty = \bigcup_p \mathcal{B}_{\infty,p}$.

Consider a projective coordinate system $(X : Y : Z)$ such that $Z = 0$ is the equation of $\mathbb{L}_\infty$. If $F = F(X,Y) \in \mathbb{C}[X,Y]$ is a polynomial of degree $d > 0$ and $F^* = F^*(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is the homogeneous form corresponding to F, then we put

$$\deg_\gamma F = -\operatorname{ord}_\gamma \frac{F^*}{Z^d} = d \operatorname{ord}_\gamma Z - \operatorname{ord}_\gamma F^*, \qquad \deg \gamma = \operatorname{ord}_\gamma Z$$

for every branch $\gamma \in \mathcal{B}_\infty$.

Here $\operatorname{ord}_\gamma F^*$ stands for the order of vanishing of the homogeneous form $F^* = F^*(X,Y,Z)$ at the branch γ. We adopt the usual conventions on the symbol ∞. Note that $\deg_\gamma F \in \mathbb{Z} \cup \{-\infty\}$ with $\deg_\gamma F = -\infty$ if and only if γ is a branch of the projective curve $F^* = 0$.

For every branch $\gamma \in \mathcal{B}_\infty$ we define the value $F(\gamma) \in \mathbb{C} \cup \{\infty\}$ as follows: if $\deg_\gamma F \leq 0$ then $F(\gamma)$ is the unique $t \in \mathbb{C}$ such that $\deg_\gamma(F - t) < 0$; if $\deg_\gamma F > 0$ then $F(\gamma) = \infty$.

Let $\gamma \in \mathcal{B}_\infty$. We say that a pair $p(T) = (x(T), y(T))$ of Laurent series $x(T)$, $y(T) \in \mathbb{C}((T))$ is a *meromorphic parametrization* of γ if $\operatorname{ord} p(T) := \min\{\operatorname{ord} x(T), \operatorname{ord} y(T)\}$ is negative and γ is given in projective coordinates by $(T^k x(T) : T^k y(T) : T^k)$, where $k = -\operatorname{ord} p(T)$. It is easy to check that $\deg \gamma = -\operatorname{ord} p(T)$, $\deg_\gamma F = -\operatorname{ord} F(p(T))$ and $F(\gamma) = F(p(T))|_{T=0}$.

For every nonconstant polynomial $F \in \mathbb{C}[X,Y]$ we consider its gradient $\nabla F = (\partial F/\partial X, \partial F/\partial Y)$. For every $\gamma \in \mathcal{B}_\infty$ we put

$$\deg_\gamma \nabla F = \sup\left\{\deg_\gamma \frac{\partial F}{\partial X}, \deg_\gamma \frac{\partial F}{\partial Y}\right\}.$$

Using meromorphic parametrizations we check that $\deg_\gamma \nabla F = -\infty$ if and only if γ is a branch at infinity of a multiple component of the curve $F^* = 0$.

Moreover for every $\gamma \in \mathcal{B}_\infty$ if $\deg_\gamma F \neq 0$ then $\deg_\gamma F \le \deg_\gamma \nabla F + \deg \gamma$.

0.2. *Germs of curves.* Our main reference is [Ca]. We will consider the germs $\gamma, \gamma', \ldots$ of analytic curves at a given point p of a complex nonsingular surface. We denote by $(\gamma \cdot \gamma')_p$ the intersection multiplicity of γ and γ' at p and by $\mu_p(\gamma)$ the Milnor number of γ. If γ is a branch then $S(\gamma)$ denotes the semigroup of γ generated by all intersection numbers $(\gamma \cdot \gamma')_p$. Let λ be a smooth branch at p. We say that two reduced germs γ, γ' are λ-*equisingular* if $\lambda \not\subset \gamma \cup \gamma'$ and there are decompositions $\gamma = \bigcup_{i=1}^r \gamma_i$ and $\gamma' = \bigcup_{i=1}^r \gamma_i'$ as unions of the same number $r > 0$ of branches such that

- $S(\gamma_i) = S(\gamma_i')$,
- $(\gamma_i \cdot \gamma_j)_p = (\gamma_i' \cdot \gamma_j')_p$,
- $(\gamma_i \cdot \lambda)_p = (\gamma_i' \cdot \lambda)_p$

for all $i, j = 1, \ldots, r$.

The first two conditions define the equisingularity of germs γ and γ'. Note that the equisingularity of the germs $\gamma \cup \lambda$ and $\gamma' \cup \lambda$ does not imply that γ and γ' are λ-equisingular. Take for example the germs at the origin of the curves $x(y - x^2) = 0$, $x(x - y^2) = 0$ and $y = 0$ as γ, γ' and λ respectively.

Let U be an open and connected subset of $\mathbb{C}$. Using the Zariski discriminant criterion [Z] we get the

Equisingularity Criterion. *Let $(\gamma^t : t \in U)$ be an analytic family of germs such that:*

(i) *there is an integer $n > 0$ such that $(\gamma^t \cdot \lambda)_p = n$ for all $t \in U$,*
(ii) *there is an integer $\mu \ge 0$ such that $\mu_0(\gamma^t) = \mu$ for all $t \in U$.*

Then any two germs of the family $(\gamma^t : t \in U)$ are λ-equisingular.

Note here that in [Z] the discriminant criterion is proved in the case where γ^t and λ are transverse. The proof in the general case needs some rather obvious modifications.

Let λ be a smooth branch at a point p of a complex surface. For any germ γ of an analytic curve we consider the maximal polar quotient $\eta_p(\gamma, \lambda)$ (cf. [T] and [Pł2]). To recall the definition of $\eta_p(\gamma, \lambda)$ choose a system of local coordinates (X, Y) such that $X(p) = Y(p) = 0$ and identify the local ring at 0 with the ring of convergent power series $\mathbb{C}\{X, Y\}$. Let $f(X, Y) = 0$ and $l(X, Y) = 0$ be the local reduced equations of γ and λ, respectively. Let

$$j(f, l) = \frac{\partial f}{\partial X}\frac{\partial l}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial l}{\partial X}$$

and put

$$\eta_p(\gamma,\lambda) = \sup\left\{\frac{(f,g)_0}{(l,g)_0} : g \text{ is an irreducible factor of } j(f,l)\right\}.$$

In [Pł2] an explicit formula for $\eta_p(\gamma,\lambda)$ is given which shows that $\eta_p(\gamma,\lambda)$ is a λ-invariant of equisingularity.

Now, define

$$\Theta_p(\gamma,\lambda) = \sup\left\{\frac{(h,j(f,l))_0}{(l,h)_0} : h \text{ is an irreducible factor of } f\right\}.$$

By the well known properties of intersection numbers

$$\begin{aligned}(f_i, j(f,l))_0 &= (f_i, j(f_i,l))_0 + \sum_{j\neq i}(f_i,f_j)_0 \\ &= \mu_0(f_i) + (f_i,l)_0 - 1 + \sum_{j\neq i}(f_i,f_j)_0\end{aligned}$$

we get the following formula for $\Theta_p(\gamma,\lambda)$:

Let $\gamma = \bigcup_{i=1}^r \gamma_i$ with branches γ_i pairwise different. Let μ_i be the Milnor number of γ_i. Then

$$\Theta_p(\gamma,\lambda) = \sup_{i=1}^{r}\left\{\frac{\mu_i - 1}{(\gamma_i\cdot\lambda)_p} + 1 + \frac{1}{(\gamma_i\cdot\lambda)_p}\sum_{j\neq i}(\gamma_i\cdot\gamma_j)_p\right\}.$$

The properties listed below are useful.

1. If $(\gamma\cdot\lambda)_p = 2$ (for γ irreducible or not) then $\Theta_p(\gamma,\lambda) = (\mu_p(\gamma)+1)/2$.
2. $\Theta_p(\gamma,\lambda) \geq 0$ with equality if and only if γ and λ are smooth and transverse.
3. $\Theta_p(\gamma,\lambda) \geq \frac{\mu_p(\gamma)-1}{(\gamma\cdot\lambda)_p} + 1$ with equality if γ is a branch.

0.3. *Critical points at infinity.* For any projective plane curve C we denote by $|C|$ the support of C. We identify C and $|C|$ if C has no multiple components. For any two projective plane curves C, C' we denote by $(C\cdot C')_p$ the intersection multiplicity of the germs (C,p) and (C',p), and by $\mu_p(C)$ the Milnor number of the germ (C,p). Note that $\mu_p(C) < +\infty$ if and only if there is no multiple component of C passing through p.

The following construction is due to Broughton [B]. Let $F = F(X,Y)$ be a polynomial of degree $d > 0$ and let $F^* = F^*(X,Y,Z)$ be the homogeneous form corresponding to F. Consider the pencil $C^t : F^*(X,Y,Z) - tZ^d = 0$, $t \in \mathbb{C}$, of projective curves. The set C_∞ given by $F^*(X,Y,Z) = Z = 0$ is the set of base points of the pencil $(C^t : t \in \mathbb{C})$. Fix a point $p \in C_\infty$ and let

$$\mu_p^{\min} = \inf\{\mu_p(C^t) : t \in \mathbb{C}\}, \qquad \Lambda_p(F) = \{t \in \mathbb{C} : \mu_p(C^t) > \mu_p^{\min}\}.$$

Thus $\mu_p^{\min}$ is an integer and $\Lambda_p(F)$ is a finite subset of $\mathbb{C}$. Clearly the integer $d_p = (C^t \cdot \mathbb{L}_\infty)_p$ does not depend on t. Applying the Equisingularity Criterion to the family (C^t, p), $t \in \mathbb{C} \setminus \Lambda_p(F)$, and to $\lambda = (\mathbb{L}_\infty, p)$ we get the

Equisingularity at Infinity Property. *For every $p \in C_\infty$ any two germs of the family (C^t, p), $t \in \mathbb{C} \setminus \Lambda_p(F)$, are $(\mathbb{L}_\infty, p)$-equisingular.*

The pairs $(p,t) \in \mathbb{L}_\infty \times \mathbb{C}$, where $p \in C_\infty$ and $t \in \Lambda_p(F)$, are called *critical points at infinity* of the polynomial F (see [D] and [GwP] for other definitions and examples).

1. Results. We keep the notation introduced in the Preliminaries. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ and let $\nabla F = (\partial F/\partial X, \partial F/\partial Y)$ be its gradient. For every pair $(p,t) \in \mathbb{L}_\infty \times \mathbb{C}$ we put

$$\mathcal{L}_{p,t}(F) = \inf \left\{ \frac{\deg_\gamma \nabla F}{\deg \gamma} : \gamma \in \mathcal{B}_{\infty,p} \text{ and } F(\gamma) = t \right\}$$

and call $\mathcal{L}_{p,t}(F)$ the *Łojasiewicz exponent* of the polynomial F at (p,t).

Let C be the projective closure of the affine curve $F(X,Y) = 0$ and let $C_\infty = |C| \cap \mathbb{L}_\infty$.

Property 1.1. *Let $(p,t) \in \mathbb{L}_\infty \times \mathbb{C}$. If $p \notin C_\infty$ then $\mathcal{L}_{p,t}(F) = +\infty$. If $p \in C_\infty$ and a multiple component of C^t passes through p then $\mathcal{L}_{p,t}(F) = -\infty$.*

Proof. If $p \notin C_\infty$ then the set $\{\gamma \in \mathcal{B}_{\infty,p} : F(\gamma) = t\}$ is empty and consequently $\mathcal{L}_{p,t}(F) = \inf \emptyset = +\infty$. If $p \in C_\infty$ and a multiple component of C^t passes through p then $\deg_\gamma \nabla F = -\infty$ for a branch γ of C^t centered at p and $\mathcal{L}_{p,t}(F) = -\infty$.

In what follows we assume that $p \in C_\infty$. We say that the Łojasiewicz exponent $\mathcal{L}_{p,t}(F)$ is *attained* on an affine curve Γ if there is a branch γ of Γ centered at p such that $F(\gamma) = t$ and $\frac{\deg_\gamma \nabla F}{\deg \gamma} = \mathcal{L}_{p,t}(F)$.

Recall that C^t is the projective closure of the fiber $F(X,Y) - t = 0$ (C^t may have multiple components). We put $\nabla_q F = a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y}$ for every $q = (a : b : 0) \in \mathbb{L}_\infty$ and we call $\nabla_q F = 0$ a *polar curve*. Our main results are:

Theorem 1.2. *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping of degree $d > 1$ and let $(C^t : t \in \mathbb{C})$ be the pencil of projective curves associated with F. Then*

(i) *$\mathcal{L}_{p,t}(F) = d - 1 - \Theta_p(C^t, \mathbb{L}_\infty)$ if $t \in \mathbb{C} \setminus \Lambda_p(F)$. Moreover $\mathcal{L}_{p,t}(F)$ is attained on the fiber $F = t$.*

(ii) *$\mathcal{L}_{p,t}(F) = d - 1 - \eta_p(C^t, \mathbb{L}_\infty)$ if $t \in \Lambda_p(F)$ and $\mathcal{L}_{p,t}(F)$ is attained on every polar curve $\nabla_q F = 0$, $q \notin C_\infty$.*

THEOREM 1.3. *Assume additionally that* $(p,t) \in C_\infty \times \mathbb{C}$ *and there is no multiple component of* C^t *passing through* p. *Then the Łojasiewicz exponent* $\mathcal{L}_{p,t}(F)$ *is determined by the class of* $(\mathbb{L}_\infty, p)$*-equisingularity of the germ* (C^t, p) *and the following holds*:

(i) *There exists a rational number* $\ell_p(F) \geq 0$ *such that* $\mathcal{L}_{p,t}(F) = \ell_p(F)$ *for all* $t \in \mathbb{C} \setminus \Lambda_p(F)$. *For every* $t \in \mathbb{C} \setminus \Lambda_p(F)$ *the exponent* $\mathcal{L}_{p,t}(F)$ *is attained on the fiber* $F = t$.

(ii) $\ell_p(F) = 0$ *if and only if* C *is a pencil of lines passing through* p.

(iii) *If* $t \in \Lambda_p(F)$ *then* $\mathcal{L}_{p,t}(F) < -1$. *Let* $q \in \mathbb{L}_\infty \setminus C_\infty$. *Then the exponent* $\mathcal{L}_{p,t}(F)$ *is attained on the polar curve* $\nabla_q F = 0$.

Note that property (ii) is implicit in [K-P]. The proofs of the above theorems are given in Section 4. Now let us present some applications.

COROLLARY 1.4 (cf. [H], [D]). *The following conditions are equivalent*:

(M) *the pair* $(p,t) \in \mathbb{L}_\infty \times \mathbb{C}$ *is a critical point at infinity for the polynomial* F,

(Ł) $\mathcal{L}_{p,t}(F) < -1$,

(G) *there exists a branch* $\gamma \in \mathcal{B}_{\infty,p}$ *such that*

$$\frac{\partial F}{\partial X}(\gamma) = \frac{\partial F}{\partial Y}(\gamma) = 0 \quad \textit{and} \quad F(\gamma) = t.$$

Proof. Conditions (M) and (Ł) are equivalent by Theorem 1.3(i) and (iii). To check that (G) implies (Ł) take a branch $\gamma \in \mathcal{B}_{\infty,p}$ satisfying (G). Then $\deg_\gamma \nabla F < 0$ and by definition of the Łojasiewicz exponent, $\mathcal{L}_{p,t}(F) < 0$. Therefore $\mathcal{L}_{p,t}(F) < -1$ by Theorem 1.3. The implication (Ł)⇒(G) is obvious.

Following [ChK2] we put

$$\mathcal{L}_{\infty,t}(F) = \inf\left\{\frac{\deg_\gamma \nabla F}{\deg \gamma} : \gamma \in \mathcal{B}_\infty \text{ and } F(\gamma) = t\right\}$$

and call $\mathcal{L}_{\infty,t}(F)$ the *Łojasiewicz exponent* of F *along the fiber* $F = t$.

It is easy to see that $\mathcal{L}_{\infty,t}(F) = \inf\{\mathcal{L}_{p,t}(F) : p \in C_\infty\}$. We say that the exponent $\mathcal{L}_{\infty,t}(F)$ is *attained* on an affine curve Γ if there is a branch at infinity γ of Γ such that $\frac{\deg_\gamma \nabla F}{\deg \gamma} = \mathcal{L}_{\infty,t}(F)$. Let $\Lambda(F) = \bigcup_{p \in C_\infty} \Lambda_p(F)$.

COROLLARY 1.5 (cf. [ChK2] and [H]). *There exists a constant* $\ell_\infty(F) \geq 0$ *such that* $\mathcal{L}_{\infty,t}(F) = \ell_\infty(F)$ *for all* $t \in \mathbb{C} \setminus \Lambda(F)$. *For such* t *the exponent* $\mathcal{L}_{\infty,t}(F)$ *is attained on the fiber* $F = t$. *If* $t \in \Lambda(F)$ *then* $\mathcal{L}_{\infty,t}(F) < -1$ *and the exponent* $\mathcal{L}_{\infty,t}(F)$ *is attained on every polar curve* $\nabla_q F = 0$, $q \notin C_\infty$.

Proof. We put $\ell_\infty(F) = \inf\{\ell_p(F) : p \in C_\infty\}$ and use Theorem 1.3.

Finally, consider the total Łojasiewicz exponent $\mathcal{L}_\infty(F)$ (see [H] and [ChK1]):
$$\mathcal{L}_\infty(F) = \inf\left\{\frac{\deg_\gamma \nabla F}{\deg \gamma} : \gamma \in \mathcal{B}_\infty\right\}.$$
Using Corollary 1.5 we get easily

COROLLARY 1.6 (cf. [H]). *If $\Lambda(F) \neq \emptyset$ then $\mathcal{L}_\infty(F) = \inf_{t\in\Lambda(F)} \mathcal{L}_{\infty,t}(F)$.*

REMARK 1.7. If $\Lambda(F) = \emptyset$ and the projective closure of the affine curve $F(X,Y) = 0$ crosses the line at infinity $\mathbb{L}_\infty$ at $c \neq \deg F$ distinct points, then $\mathcal{L}_\infty(F) < \text{ł}_\infty(F)$.

In [CN-H] the authors calculated the total Łojasiewicz exponent $\mathcal{L}_\infty(F)$ in terms of the Eisenbud and Neumann diagrams. Here is a reformulation of their result for polynomials F with $\Lambda(F) \neq \emptyset$.

PROPOSITION 1.8 (cf. [CN-H, Proposition 6]). *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ such that $\Lambda(F) \neq \emptyset$. Put $C'_\infty = \{\, p \in C_\infty : (C, \mathbb{L}_\infty)_p > 1\,\}$. Then*
$$\mathcal{L}_\infty(F) = d - 1 - \sup\{\eta_p(C^t, \mathbb{L}_\infty) : (p,t) \in C'_\infty \times \mathbb{C}\}.$$
Proof. We use Corollary 1.6 and Theorem 1.2(ii).

Let $\mu(F)$ be the total Milnor number of F defined to be the sum of all Milnor numbers of the curves $F = t$. Then $\mu(F) < +\infty$ if and only if all the curves $F = t$ are reduced.

PROPOSITION 1.9. *Let F be a square-free polynomial of degree $d > 1$ such that the curve $F = 0$ has only one branch at infinity. Then*
$$\text{ł}_\infty(F) = \frac{\mu(F) - 1}{d} + 1.$$
In particular, if F is a component of a polynomial automorphism then $\text{ł}_\infty(F) = 1 - 1/d$.

Proof. According to the Ephraim–Moh theorem ([E, Theorem 3.4]) we have $\Lambda(F) = \emptyset$. Let p be the unique point at infinity of the curve C. Then all germs (C^t, p) are reduced and irreducible, $\mu_p^t \equiv \mu_p^{\min}$ and $\theta_p^t \equiv (\mu_p^{\min} - 1)/d + 1$ (see Preliminaries 0.2 and 0.3). By Theorem 1.2(i) we get $\text{ł}_\infty(F) = d - 2 - (\mu_p^{\min} - 1)/d$. Using [CN, Proposition 12] we have $d^2 - 3d + 2 = \mu_p^{\min} + \mu(F)$ and the proposition follows.

2. Local invariants of singularities. We keep the notation introduced in the Preliminaries. Both invariants $\eta_p(\gamma, \lambda)$ and $\Theta_p(\gamma, \lambda)$ can be calculated by means of Puiseux series. Let $f(X,Y) = 0$ and $l(X,Y) = 0$ be local reduced equations of γ and λ. Let $\mathbb{C}\{X\}^* = \bigcup_{n\geq 1} \mathbb{C}\{X^{1/n}\}$.

PROPOSITION 2.1. *Let (X,Y) be a system of coordinates such that λ has the equation $X=0$. Suppose that $f(X,Y)=U(X,Y)\prod_{i=1}^{n}(Y-y_i(X))$ with $y_i=y_i(X)\in\mathbb{C}\{X\}^*$ without constant terms and $U(X,Y)\in\mathbb{C}\{X,Y\}$ such that $U(0,0)\neq 0$. Then*

(i) $(\gamma\cdot\lambda)_p=n$,

(ii) $\Theta_p(\gamma,\lambda)=\sup_{i=1}^{n}\Big\{\sum_{j\neq i}\operatorname{ord}(y_i-y_j)\Big\}$,

(iii) $\eta_p(\gamma,\lambda)=\sup_{i=1}^{n}\Big\{\sum_{j\neq i}\operatorname{ord}(y_i-y_j)+\max_{j\neq i}\{\operatorname{ord}(y_i-y_j)\}\Big\}$.

Proof. Properties (i) and (ii) follow easily from the definitions. The proof of (iii) is given in [Pł2, Proposition 2.2].

PROPOSITION 2.2. *If $n=(\gamma\cdot\lambda)_p>1$ then $\Theta_p(\gamma,\lambda)\le\frac{n-1}{n}\eta_p(\gamma,\lambda)$.*

Proof. With the above notation, $\Theta_p(\gamma,\lambda)=\sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j)$ for an $i_0\in\{1,\ldots,n\}$. Therefore $\Theta_p(\gamma,\lambda)\le(n-1)\max_{j\neq i_0}\{\operatorname{ord}(y_{i_0}-y_j)\}$ and

$$\begin{aligned}\Theta_p(\gamma,\lambda)+\frac{1}{n-1}\Theta_p(\gamma,\lambda)&\le\sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j)+\max_{j\neq i_0}\{\operatorname{ord}(y_{i_0}-y_j)\}\\&\le\eta_p(\gamma,\lambda)\end{aligned}$$

by Proposition 2.1(iii).

PROPOSITION 2.3. *Let $n=(\gamma\cdot\lambda)_p>1$. If $\Theta_p(\gamma,\lambda)=n-1$ and $\eta_p(\gamma,\lambda)=n$ then $n=\operatorname{ord}\gamma$, that is, γ and λ are transverse.*

Proof. With the notations of Proposition 2.1 we get

$$\sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j)=n-1\quad\text{for an } i_0\in\{1,\ldots,n\},\tag{1}$$

$$\sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j)+\max_{j\neq i_0}\{\operatorname{ord}(y_{i_0}-y_j)\}\le n.\tag{2}$$

From (1) and (2) we get

$$\operatorname{ord}(y_{i_0}-y_j)=1\quad\text{for all } j\neq i_0.\tag{3}$$

Now we can check that $y_{i_0}=y_{i_0}(X)\in\mathbb{C}\{X\}$. In fact, let $f_1(X,Y)\in\mathbb{C}\{X,Y\}$ be the irreducible power series such that $f_1(X,y_{i_0}(X))=0$. If we had $(f_1,X)_0>1$ then there would exist a solution $y_{i_1}(X)\neq y_{i_0}(X)$ of the equation $f_1(X,Y)=0$ such that $\operatorname{ord}(y_{i_0}(X)-y_{i_1}(X))\neq 1$ (see, for example, [Pł2, Proposition 3.1]) and we would get a contradiction with (3). Therefore $y_{i_0}(X)\in\mathbb{C}\{X\}$ and consequently $\operatorname{ord}y_{i_0}(X)\ge 1$. Now, by (3) we get $\operatorname{ord}y_j(X)\ge 1$ for all $j=1,\ldots,n$ and $\operatorname{ord}f(X,Y)=\operatorname{ord}\prod_{j=1}^{n}(Y-y_j(X))=\sum_{j=1}^{n}\operatorname{ord}(Y-y_j(X))=n=(f,X)_0$.

The germ given by the equation $j(f,l)=0$ will be called a *local polar* of γ with respect to λ.

PROPOSITION 2.4. *Let γ' be a local polar of γ with respect to λ. Then for every branch ξ,*

$$\frac{(\gamma\cdot\xi)_p}{(\lambda\cdot\xi)_p}\le \eta_p(\gamma,\lambda) \quad or \quad \frac{(\gamma'\cdot\xi)_p}{(\lambda\cdot\xi)_p}\le \Theta_p(\gamma,\lambda).$$

Proof. Let $f=f_1\cdots f_r$ and $j(f,l)=g_1\cdots g_s$ be the decompositions into irreducible factors and let $h=0$ be the reduced equation of ξ. Then

$$\frac{(f,h)_0}{(l,h)_0}\le \max_{j=1}^{s}\left\{\frac{(f,g_j)_0}{(l,g_j)_0}\right\} \quad \text{or} \quad \frac{(j(f,l),h)_0}{(l,h)_0}\le \max_{i=1}^{r}\left\{\frac{(f_i,j(f,l))_0}{(l,f_i)_0}\right\}$$

by [ChP, Theorem 1.1 and Concluding Remarks]. Now we use the definitions of η_p and Θ_p.

3. Polar curves. Let $F:\mathbb{C}^2\to\mathbb{C}$ be a polynomial of degree $d>1$ and let C be the projective curve $F^*(X,Y,Z)=0$. The polar curve $\nabla_q C$ with the equation $a\partial F^*/\partial X+b\partial F^*/\partial Y=0$ will be called *generic at infinity* if $q=(a:b:0)\in\mathbb{L}_\infty\setminus|C|$, that is, if $q=(a:b:0)$ and $F^*(a,b,0)\neq 0$. The following is well known:

LEMMA 3.1. *Let D be a polar of C, generic at infinity. Suppose that the germ (C,p) is reduced and $(C\cdot\mathbb{L}_\infty)_p>1$. Then (D,p) is a local polar curve of (C,p) with respect to $(\mathbb{L}_\infty,p)$.*

Recall that $C'_\infty=\{p\in C_\infty : d_p>1\}$.

LEMMA 3.2. *Let D be a polar of C generic at infinity. Then*

(i) $|C|\cap|D|\cap\mathbb{L}_\infty=C'_\infty$.

(ii) *If $\gamma\in\mathcal{B}_\infty$ is a branch of D such that* $\deg_\gamma F\neq 0$ *then* $\deg_\gamma F=\deg_\gamma\nabla F+\deg\gamma$.

Proof. (i) It suffices to observe that $p=(x_0:y_0:0)\in|C|\cap|D|$ if and only if the linear form y_0X-x_0Y is a multiple factor of $F^*(X,Y,0)$.

(ii) Let $p(T)=(x(T),y(T))$ be a meromorphic parametrization of γ and let $l(X,Y)=bX-aY$. We get $\operatorname{ord} l(p(T))=-\deg_\gamma l=\deg\gamma$, for the line $l=0$ does not intersect the polar D at infinity. From

$$a\,\frac{\partial F}{\partial X}(p(T))+b\,\frac{\partial F}{\partial Y}(p(T))=0$$

and

$$\dot{x}(T)a\,\frac{\partial F}{\partial X}(p(T))+\dot{y}(T)b\,\frac{\partial F}{\partial Y}(p(T))=\frac{d}{dT}F(p(T))$$

we get

$$\frac{d}{dt}l(T)\,\frac{\partial F}{\partial X}(p(T))=-b\,\frac{d}{dT}F(p(T)).$$

Computing the orders along γ of both sides we get $\deg_\gamma \partial F/\partial X = \deg_\gamma F - \deg\gamma$ if $b \neq 0$, $\deg_\gamma \partial F/\partial X = -\infty$ if $b = 0$, and similarly for $\deg_\gamma \partial F/\partial Y$. Hence (ii) follows.

LEMMA 3.3. *Let D be a polar of C generic at infinity. Let $p \in C_\infty$ and suppose that the germ (C, p) is reduced.*

(i) *If $p \in C'_\infty$ then*

$$\eta_p(C, \mathbb{L}_\infty) = \sup\left\{\frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma \mathbb{L}_\infty} : \gamma \in \mathcal{B}_{\infty,p} \textit{ is a branch of } D\right\}.$$

(ii) $\displaystyle \Theta_p(C, \mathbb{L}_\infty) = \sup\left\{\frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} : \gamma \in \mathcal{B}_{\infty,p} \textit{ is a branch of } \right\}.$

Proof. Use Lemma 3.1 and the definitions of η_p and Θ_p.

PROPOSITION 3.4. *Let D be a generic polar of C. Let $p \in C_\infty$ be such that (C, p) is reduced. Then for every branch $\gamma \in \mathcal{B}_{\infty,p}$,*

$$\frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \le \eta_p(C, \mathbb{L}_\infty) \quad \textit{or} \quad \frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \le \Theta_p(C, \mathbb{L}_\infty).$$

Proof. Use Proposition 2.4 and Lemma 3.1.

4. Łojasiewicz exponents and invariants of singularities.

In this section we give the proofs of Theorems 1.2 and 1.3. The following proposition is well known.

PROPOSITION 4.1. *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping of degree $d > 1$.*

(i) *If $\Lambda_p(F) = \emptyset$ then $\eta_p(C^t, \mathbb{L}_\infty) = \eta_p(C, \mathbb{L}_\infty) < d$.*

(ii) *If $\Lambda_p(F) \neq \emptyset$ then $\eta(C^t, \mathbb{L}_\infty) = d$ for $t \in \mathbb{C} \setminus \Lambda_p(F)$ and $\eta_p(C^t, \mathbb{L}_\infty) > d$ for $t \in \Lambda_p(F)$ (if C^t is not reduced at p then $\eta_p(C^t, \mathbb{L}_\infty) = +\infty > d$ by convention).*

Proof. See [CaP, pp. 35–37], [Pł1, Corollary 1.3], and [GaP, Proposition 1.4].

Proof of Theorem 1.2. We may assume that $\deg_X F = \deg_Y F = d$. Let D_1 be the polar $\partial F^*/\partial X = 0$ and D_2 the polar $\partial F^*/\partial Y = 0$ (note that $(\partial F/\partial X)^* = \partial F^*/\partial X$ provided that $\deg_X F = d$). Then, by definitions, we get

$$\frac{\deg_\gamma \nabla F}{\deg\gamma} = d - 1 - \inf_i\left\{\frac{\operatorname{ord}_\gamma D_i}{\operatorname{ord}_\gamma \mathbb{L}_\infty}\right\}, \tag{1}$$

$$\pounds_{p,t}(F) = d - 1 - \sup\left\{\inf_i\left\{\frac{\operatorname{ord}_\gamma D_i}{\operatorname{ord}_\gamma \mathbb{L}_\infty}\right\} : \gamma \in \mathcal{B}_{\infty,p},\ \frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d\right\}. \tag{2}$$

The property "if $\deg_\gamma F \neq 0$ then $\deg_\gamma F \leq \deg_\gamma \nabla F + \deg \gamma$" can be reformulated as follows:

$$\text{(3)} \qquad \text{if} \quad \frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \neq d \quad \text{then} \quad \inf_i \{\operatorname{ord}_\gamma D_i\} \leq \operatorname{ord}_\gamma C.$$

Now, let us pass to the proof of the first part of Theorem 1.2. Fix $t \in \mathbb{C} \setminus \Lambda_p(F)$ and let $\gamma \in \mathcal{B}_{\infty,p}$ be a branch such that $\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d$. By Proposition 3.4 we get, for every polar D generic at infinity,

$$\text{(4)} \qquad \frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \eta_p(C^t, \mathbb{L}_\infty) \quad \text{or} \quad \frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \Theta_p(C^t, \mathbb{L}_\infty).$$

Since $t \in \mathbb{C} \setminus \Lambda_p(F)$ we have $\eta_p(C^t, \mathbb{L}_\infty) \leq d$ by Proposition 4.1. Therefore $\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d \geq \eta_p(C^t, \mathbb{L}_\infty)$ and by (4), we get

$$\text{(5)} \qquad \frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \Theta_p(C^t, \mathbb{L}_\infty) \quad \text{for every polar } D \text{ generic at infinity.}$$

In particular $\frac{\operatorname{ord}_\gamma D_i}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \Theta_p(C^t, \mathbb{L}_\infty)$ for $i = 1, 2$ and consequently, by (2), we get

$$\text{(6)} \qquad \mathcal{L}_{p,t}(F) \geq d - 1 - \Theta_p(C^t, \mathbb{L}_\infty).$$

It is easy to check that

$$\text{(7)} \qquad \operatorname{ord}_\gamma D_1 = \operatorname{ord}_\gamma D_2 \quad \text{for every branch } \gamma \in \mathcal{B}_{\infty,p} \text{ of } C^t.$$

Thus, for every branch γ of C^t,

$$\text{(8)} \qquad \frac{\deg_\gamma \nabla F}{\deg \gamma} = d - 1 - \frac{\operatorname{ord}_\gamma D_1}{\operatorname{ord}_\gamma \mathbb{L}_\infty}$$

by (1). By definition of Θ_p there is a branch $\gamma_0 \in \mathcal{B}_{\infty,p}$ of C^t such that $\frac{\operatorname{ord}_{\gamma_0} D_1}{\operatorname{ord}_{\gamma_0} \mathbb{L}_\infty} = \Theta_p(C^t, \mathbb{L}_\infty)$. Consequently,

$$\text{(9)} \qquad \frac{\deg_{\gamma_0} \nabla F}{\deg \gamma_0} = d - 1 - \Theta_p(C^t, \mathbb{L}_\infty)$$

and Theorem 1.2(i) follows from (6) and (9).

To prove the second part of Theorem 1.2 fix $t \in \Lambda_p(F)$. We may assume that the germ (C^t, p) is reduced. Let γ be a branch such that $\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d$. By Proposition 3.4 we get either

$$\text{(10)} \qquad \text{(a)}\ \frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \eta_p(C^t, \mathbb{L}_\infty) \quad \text{or} \quad \text{(b)}\ \frac{\operatorname{ord}_\gamma D_1}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \Theta_p(C^t, \mathbb{L}_\infty).$$

In case (a), by (3) we get

$$\text{(11)} \qquad \frac{\inf_i \{\operatorname{ord}_\gamma D_i\}}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leq \eta_p(C^t, \mathbb{L}_\infty).$$

In case (b), we have $\frac{\inf_i\{\mathrm{ord}_\gamma D_i\}}{\mathrm{ord}_\gamma \mathbb{L}_\infty} \le \varTheta_p(C^t,\mathbb{L}_\infty) < \eta_p(C^t,\mathbb{L}_\infty)$ by Proposition 2.2. Consequently, in both cases (11) holds and finally we get

$$\mathcal{L}_{p,t}(F) \ge d-1-\eta_p(C^t,\mathbb{L}_\infty). \tag{12}$$

Now, fix a polar D generic at infinity. Then there is a branch γ_0 of D such that $\frac{\mathrm{ord}_{\gamma_0} C^t}{\mathrm{ord}_{\gamma_0} \mathbb{L}_\infty} = \eta_p(C^t,\mathbb{L}_\infty)$. Since $t \in \varLambda_p(F)$, we have $\eta_p(C^t,\mathbb{L}_\infty) > d$ by Proposition 4.1 and consequently $\frac{\mathrm{ord}_{\gamma_0} C^t}{\mathrm{ord}_{\gamma_0} \mathbb{L}_\infty} > d$. On the other hand, using Lemma 3.2(ii) we check that $\inf\{\mathrm{ord}_{\gamma_0} D_i\} = \mathrm{ord}_{\gamma_0} C^t$.

Therefore

$$\mathcal{L}_{p,t}(F) \le d-1-\frac{\inf\{\mathrm{ord}_{\gamma_0} D_i\}}{\mathrm{ord}_{\gamma_0} \mathbb{L}_\infty} = d-1-\frac{\mathrm{ord}_{\gamma_0} C^t}{\mathrm{ord}_{\gamma_0} \mathbb{L}_\infty} = d-1-\eta_p(C^t,\mathbb{L}_\infty) \tag{13}$$

and Theorem 1.2(ii) follows from (12) and (13).

Now, we can give

Proof of Theorem 1.3. By the Equisingularity at Infinity Property (Preliminaries 0.3) there is a constant $\varTheta_p^{\mathrm{gen}}$ such that $\varTheta_p(C^t,\mathbb{L}_\infty) = \varTheta_p^{\mathrm{gen}}$ for all $t \in \mathbb{C} \setminus \varLambda_p(F)$. Put $\ell_p(F) = d-1-\varTheta_p^{\mathrm{gen}}$. For $t \notin \varLambda_p(F)$ we have $\eta_p(C^t,\mathbb{L}_\infty) \le d$ by Proposition 4.1. Let $d_p = (C\cdot\mathbb{L}_\infty)_p$. Proposition 2.2 yields

$$\varTheta_p^{\mathrm{gen}} \le \left(1-\frac{1}{d_p}\right)d \le d-1. \tag{14}$$

Therefore $\ell_p(F) = d-1-\varTheta_p^{\mathrm{gen}} \ge 0$ and Theorem 1.3(i) follows from Theorem 1.2(i).

To check the second part of Theorem 1.3 suppose that $\ell_p(F)=0$. Then from (14) we get $d_p = d$, i.e. p is the only point at infinity of C. Moreover $\varTheta_p(C^t,\mathbb{L}_\infty) = \varTheta_p^{\mathrm{gen}} = d-1$ and $\eta_p(C^t,\mathbb{L}_\infty) = d$ for $t \in \mathbb{C}\setminus\varLambda_p(F)$. By Proposition 2.3 we get $\mathrm{ord}_p C^t = \deg C^t = d$. Therefore C^t and consequently C are pencils of lines through p. This proves Theorem 1.3(ii).

Fix now $t \in \varLambda_p(F)$. If (C^t,p) is not reduced then $\mathcal{L}_{p,t}(F) = -\infty$ and $\eta_p(C^t,\mathbb{L}_\infty) = +\infty$. Thus we may assume that (C^t,p) is reduced. Using Theorem 1.2(ii) and Proposition 4.1(ii) we get $\mathcal{L}_{p,t}(F) = d-1-\eta_p(C^t,\mathbb{L}_\infty) < -1$. Moreover $\mathcal{L}_{p,t}(F)$ is attained on every polar $\nabla_q C = 0$, $q \notin C$ by Theorem 1.2(ii), and Theorem 1.3(iii) follows.

5. Growth of the gradient. Let $F:\mathbb{C}^2 \to \mathbb{C}$ be a nonconstant polynomial. Fix $(p,t) \in C_\infty \times \mathbb{C}$. For completeness we give an interpretation of $\mathcal{L}_{p,t}(F)$ as the exponent of growth of $\nabla F(z)$ near the fiber $F(z)=t$ for $z \to p$. We consider $\mathbb{P}^2(\mathbb{C})$ with the usual topology. If $z=(x,y)\in\mathbb{C}$ then $|z| = \max(|x|,|y|)$. We set $F^{-1}(t)_\delta = \{z \in \mathbb{C}^2 : |F(z)-t| \le \delta\}$ for every $\delta > 0$.

THEOREM 5.1. *Assume that the germ* (C^t, p) *is reduced. Let* $\delta > 0$ *be such that the set* $\{\tilde{t} \in \mathbb{C} : 0 < |\tilde{t} - t| \leq \delta\}$ *does not intersect* $\Lambda(F)$. *Then there is a constant* $c > 0$ *such that*

$$|\nabla F(z)| \geq c|z|^{\mathcal{L}_{p,t}(F)} \quad \text{on the set } F^{-1}(t)_\delta \text{ for } z \to p. \tag{1}$$

The exponent $\mathcal{L}_{p,t}(F)$ *in* (1) *is optimal*: *if* $\nabla F(z)| \geq c_\sigma |z|^\sigma$ *with some* $c_\sigma > 0$ *and* $\sigma \in \mathbb{R}$ *on* $F^{-1}(t)_\delta$ *for* $z \to p$ *then* $\sigma \leq \mathcal{L}_{p,t}(F)$.

Proof (see [ChK2, Section 5]). Fix $\delta > 0$ as above and let $\mathcal{L}^\delta_{p,t}(F)$ be the least upper bound of the set of all $\sigma \in \mathbb{R}$ such that $|\nabla F(z)| \geq c_\sigma |z|^\sigma$ with some $c_\sigma > 0$ on $F^{-1}(t)_\delta$ for $z \to p$. By the Curve Selection Lemma there is a meromorphic parameterization $p(T) = (x(T), y(T)) \in \mathbb{C}((T))$ with $\operatorname{ord} p(T) < 0$, convergent in a punctured disc, such that the mapping $\tau \mapsto p(\tau)$ defined for the real numbers $\tau \neq 0$ small enough has the following properties:

- $p(\tau) \to p$ for $\tau \to 0^+$,
- $|F(p(\tau)) - t| \leq \delta$ for $\tau \to 0^+$,
- $\dfrac{\operatorname{ord} \nabla F(p(T))}{\operatorname{ord} p(T)} = \mathcal{L}^\delta_{p,t}(F)$.

Let γ be the branch at infinity with meromorphic parameterization $p(T)$. Then

$$\frac{\deg_\gamma \nabla F}{\deg \gamma} = \frac{\operatorname{ord} \nabla F(p(T))}{\operatorname{ord} p(T)}$$

and $|F(\gamma) - t| \leq \delta$ for $F(\gamma) = \lim_{\tau \to 0^+} F(p(\tau))$. By the choice of δ we get $F(\gamma) = t$ or $F(\gamma) \notin \Lambda(F)$. Hence by Theorem 1.3 we have

$$\mathcal{L}^\delta_{p,t}(F) = \frac{\deg_\gamma \nabla F}{\deg \gamma} \geq \mathcal{L}_{p,F(\gamma)}(F) \geq \mathcal{L}_{p,t}(F).$$

This proves the first part of the theorem.

To show the second part take by Theorem 1.2 a branch $\gamma \in \mathcal{B}_{\infty,p}$ such that $F(\gamma) = t$ and $\mathcal{L}_{p,t}(F) = \frac{\deg_\gamma \nabla F}{\deg \gamma}$. Let $\Gamma \subset \mathbb{C}^2$ be the image of a small punctured disc centered at $0 \in \mathbb{C}$ under the meromorphic parameterization of γ. Since $F(z) \to t$ on Γ for $z \to p$ we may assume that $\Gamma \subset F^{-1}(t)_\delta$. It is easy to see that

$$|\nabla F(z)| \geq c|z|^{\mathcal{L}_{p,t}(F)} \quad \text{on } \Gamma \text{ for } z \to p$$

and the exponent $\mathcal{L}_{p,t}(F)$ is optimal. Thus $\mathcal{L}^\delta_{p,t}(F) \leq \mathcal{L}_{p,t}(F)$.

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