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GROUPS SATISFYING THE MAXIMAL CONDITION ON SUBNORMAL NON-NORMAL SUBGROUPS

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Abstract. The structure of (generalized) soluble groups for which the set of all subnormal non-normal subgroups satisfies the maximal condition is described, taking as a model the known theory of groups in which normality is a transitive relation.

1. Introduction. A group G is called a *T-group* if all its subnormal subgroups are normal, i.e. if normality is a transitive relation in G. The structure of soluble T-groups has been described by W. Gaschütz [12] in the finite case and by D. J. S. Robinson [13] for arbitrary groups. It turns out that soluble groups with the property T are metabelian and hypercyclic, and that finitely generated soluble T-groups are either finite or abelian; moreover, Sylow properties of periodic soluble T-groups have been studied. In recent years, many papers deal with the structure of (generalized) soluble groups in which normality is imposed only on certain systems of subnormal subgroups (see [8], [9], [11]). Other classes of generalized T-groups can be introduced by imposing that the set of all subnormal non-normal subgroups of the group is small in some sense; this point of view was for instance adopted in [1], [6] and [10]. Groups satisfying the minimal condition on subnormal non-normal subgroups have recently been investigated (see [7]), and the aim of this paper is to study soluble groups satisfying the maximal condition on subnormal non-normal subgroups.

We shall say that a group G is a *$\widehat{T}$-group* (or that G has the *property $\widehat{T}$*) if the set of all subnormal non-normal subgroups of G satisfies the maximal condition, i.e. if there does not exist in G an infinite properly ascending chain

$$X_1 < X_2 < \cdots < X_n < \cdots$$

of subnormal non-normal subgroups. Any group satisfying the maximal condition on subnormal subgroups (in particular, any polycyclic-by-finite group) is obviously a $\widehat{T}$-group; notice also that groups with finitely many subnormal non-normal subgroups, as well as groups in which every sub-

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normal subgroup of infinite index is normal, have the property $\widehat{T}$. Observe finally that nilpotent $\widehat{T}$-groups must satisfy the maximal condition on non-normal subgroups. Locally soluble groups with this latter property have been completely described by G. Cutolo [5]; in particular, they either satisfy the maximal condition or are nilpotent of class at most 2. For instance, it turns out that the direct product $H = X \times Y$ of a non-abelian group X of order p^3 and exponent p and a group Y of type p^∞ is not a $\widehat{T}$-group, while the property $\widehat{T}$ holds for the factor group $H/\langle xy\rangle$, where $Z(X) = \langle x\rangle$ and y is an element of order p of Y.

We take the known theory of T-groups as our model, and show that a number of properties of such groups have an analogue in the class of groups satisfying $\widehat{T}$.

Most of our notation is standard and can be found in [14].

We wish to thank the referee for his useful comments.

2. General properties of $\widehat{T}$-groups. In the first part of this section we describe the behaviour of the Fitting subgroup of a $\widehat{T}$-group. Recall that the *Baer radical* of a group G is the subgroup generated by all abelian subnormal subgroups of G, and that G is a *Baer group* if it coincides with its Baer radical. It is easy to prove that G is a Baer group if and only if all its cyclic subgroups are subnormal.

LEMMA 2.1. *Let G be a $\widehat{T}$-group, and let X be a subnormal non-normal subgroup of G. If X is a Baer group, then the normal closure X^G of X satisfies the maximal condition on subgroups.*

Proof. Since X is contained in the Baer radical of G, also its normal closure X^G is a Baer group. Assume by contradiction that X is not finitely generated. Since all finitely generated subgroups of X are subnormal in G, we may consider a maximal element M of the set of all finitely generated subgroups of X which are not normal in G. Thus $\langle M, x\rangle$ is a normal subgroup of G for each $x \in X \setminus M$, and hence also

$$X = \langle \langle M, x\rangle \mid x \in X \setminus M\rangle$$

is normal in G. This contradiction shows that X is finitely generated, and so it satisfies the maximal condition on subgroups. Clearly, the set of all subnormal subgroups H of G such that $X \leq H \leq X^G$ satisfies the maximal condition. It follows that for each positive integer i the locally nilpotent group $X^{G,i}/X^{G,i+1}$ has the maximal condition on subnormal subgroups, and so also the maximal condition on subgroups (see [14, Part 1, Theorem 5.37]). Therefore X^G satisfies the maximal condition on subgroups. ∎

LEMMA 2.2. *Let G be a locally nilpotent group whose derived subgroup G' is finitely generated. Then G is nilpotent.*

Proof. As G' is nilpotent, it is enough to prove that G/G'' is nilpotent, so that without loss of generality it can be assumed that G' is abelian. Moreover, since the subgroup T consisting of all elements of finite order of G' is finite, we may also replace G by the factor group G/T and suppose that G' is a finitely generated torsion-free abelian group. If G' has rank r, it follows that $[G',_r G]$ is contained in $(G')^p$ for each prime number p, so that

$$[G',_r G] \leq \bigcap_p (G')^p = \{1\}$$

and the group G is nilpotent. ■

LEMMA 2.3. *Let G be a Baer group with the property $\widehat{T}$. Then the derived subgroup G' of G satisfies the maximal condition on subgroups.*

Proof. We may obviously suppose that G is not a Dedekind group. Let X be a maximal subnormal non-normal subgroup of G; then G/X^G is a Dedekind group, and so $G'X^G/X^G$ is finite. On the other hand, X^G satisfies the maximal condition on subgroups by Lemma 2.1, so that in particular $G' \cap X^G$ has this property, and hence G' itself satisfies the maximal condition on subgroups. ■

COROLLARY 2.4. *If G is a $\widehat{T}$-group, then the Baer radical of G is nilpotent. In particular, the Baer radical and the Fitting subgroup of G coincide.*

Proof. Let B be the Baer radical of G. By Lemma 2.3 the derived subgroup B' of B satisfies the maximal condition on subgroups, and hence B is nilpotent by Lemma 2.2. ■

COROLLARY 2.5. *Let G be a $\widehat{T}$-group, and let F be the Fitting subgroup of G. If X is any subgroup of F which is not finitely generated, then X is normal in G. In particular, all infinite periodic subgroups of F are normal in G.*

Proof. As F is nilpotent by Corollary 2.4, the subgroup X is subnormal in G and does not satisfy the maximal condition on subgroups. Thus it follows from Lemma 2.1 that X is normal in G. ■

A group G is said to be *subsoluble* if it has an ascending series with abelian factors consisting of subnormal subgroups. Clearly, all hyperabelian groups are subsoluble, and for groups satisfying $\widehat{T}$ subsolubility is equivalent to solubility, as the following result shows.

THEOREM 2.6. *Let G be a subsoluble $\widehat{T}$-group. Then G is soluble.*

Proof. Suppose first that the group G is hyperabelian, and let

$$\{1\} = G_0 < G_1 < \cdots < G_\alpha < G_{\alpha+1} < \cdots < G_\tau = G$$

be an ascending normal series of G with abelian factors. Assume that G is not soluble, and consider the least ordinal $\mu \leq \tau$ such that G_μ is not soluble. Clearly, μ is a limit ordinal and G_α is soluble for each $\alpha < \mu$. Let Λ be the set of all ordinals $\alpha < \mu$ for which there exists a subnormal non-normal subgroup X of G such that $G_\alpha < X < G_\beta$ for some $\beta < \mu$; since G satisfies $\widehat{T}$, the set Λ must be finite, and so there is an ordinal $\delta < \mu$ such that if X is any subnormal subgroup of G with $G_\delta \leq X \leq G_\beta$ and $\delta < \beta < \mu$, then X is normal in G. In particular, G_β/G_δ is a T-group for $\delta < \beta < \mu$ and so it is metabelian. As

$$G_\mu = \bigcup_{\delta<\beta<\mu} G_\beta,$$

it follows that G_μ/G_δ is likewise metabelian, contradicting the assumption that G_μ is insoluble.

In the general case, let

$$\{1\} = X_0 < X_1 < \cdots < X_\alpha < X_{\alpha+1} < \cdots < X_\tau = G$$

be an ascending subnormal series of G with abelian factors. The set of all ordinals α such that the subgroup X_α is not normal in G is obviously finite, of order k say. If $k = 0$, then G is hyperabelian, and so even soluble by the first part of the proof. Assume that $k > 0$, and let $\varrho < \tau$ be the largest ordinal such that X_ϱ is not normal in G. Then $X_{\varrho+1}$ is normal in G, and $G/X_{\varrho+1}$ is a hyperabelian $\widehat{T}$-group, so that it is soluble. Moreover, $X_{\varrho+1}$ has an ascending subnormal series with abelian factors in which there are at most $k-1$ non-normal terms, so that by induction on k the subgroup $X_{\varrho+1}$ is soluble, and hence G itself is soluble. ∎

LEMMA 2.7. *Let G be a soluble $\widehat{T}$-group which is not polycyclic, and let F be the Fitting subgroup of G. Then $F/Z(F)$ is a finite abelian group. In particular, if F is torsion-free, then it is abelian.*

Proof. Since soluble groups of automorphisms of polycyclic groups are likewise polycyclic (see [14, Part 1, Theorem 3.27]), the subgroup F is not polycyclic. On the other hand, the nilpotent group F satisfies the maximal condition on non-normal subgroups, and so $F/Z(F)$ is finite and abelian (see [5, Corollary 2.5]). In particular, F' is finite by Schur's theorem and hence F is abelian, provided that it is torsion-free. ∎

A *power automorphism* of a group G is an automorphism mapping every subgroup of G onto itself, and the set $\mathrm{PAut}\, G$ of all power automorphisms of G is an abelian residually finite normal subgroup of the full automorphism group $\mathrm{Aut}\, G$ of G. Recall also that the set $\mathrm{IAut}\, G$ of all automorphisms of G fixing every infinite subgroup is a subgroup of $\mathrm{Aut}\, G$ containing $\mathrm{PAut}\, G$. The behaviour of the groups $\mathrm{PAut}\, G$ and $\mathrm{IAut}\, G$ has been investigated in [3] and [4], respectively. Power automorphisms play a central role in the study of T-groups, and they will also be important in our considerations.

Lemma 2.8. *Let G be a soluble group, and let F be the Fitting subgroup of G. If all subgroups of F are normal in G, then $C_G(G') = F$. In particular, G is metabelian.*

Proof. The group $G/C_G(F)$ is abelian, since it is isomorphic to a group of power automorphisms of F, and so $G' \le C_G(F) \le F$. It follows that F is contained in $C_G(G')$, so that $C_G(G') = F$ and G' is abelian. ∎

Lemma 2.9. *Let G be an infinite soluble $\widehat{T}$-group with periodic Fitting subgroup F. Then either all subgroups of F are normal in G, or F is a finite extension of a Prüfer group and $G^{(3)} = \{1\}$.*

Proof. By Corollary 2.5 all infinite subgroups of F are normal in G, so that $G/C_G(F)$ is isomorphic to a subgroup of $\mathrm{IAut}\, F$. In particular, if $\mathrm{IAut}\, F = \mathrm{PAut}\, F$, then all subgroups of F are normal in G. On the other hand, if $\mathrm{IAut}\, F \ne \mathrm{PAut}\, F$, it follows that F is a finite extension of a Prüfer group and $G/C_G(F)$ is metabelian (see [4, Corollary 2.4 and Proposition 2.5]), so that $G'' \le C_G(F) = Z(F)$ and $G^{(3)} = \{1\}$. ∎

Corollary 2.10. *Let G be a periodic soluble $\widehat{T}$-group, and let F be the Fitting subgroup of G. Then either all subgroups of F are normal in G, or G is a finite extension of a Prüfer group and $G^{(3)} = \{1\}$.*

Proof. The statement follows directly from Lemma 2.9, because if F is a finite extension of a Prüfer group, then $G/C_G(F)$ is finite (see [14, Part 1, Corollary to Theorem 3.29.2]) and so G itself is a finite extension of a Prüfer group. ∎

Corollary 2.11. *Let G be a periodic soluble $\widehat{T}$-group which is not a finite extension of a Prüfer group. Then G is metabelian and hypercyclic.*

Proof. Let F be the Fitting subgroup of G. All subgroups of F are normal in G by Corollary 2.10, so that G is metabelian by Lemma 2.8 and G' is hypercyclically embedded in G. Therefore G is hypercyclic. ∎

Lemma 2.12. *Let G be a $\widehat{T}$-group, and let A be a torsion-free abelian subnormal subgroup of G. If A is not finitely generated, then all subgroups of A are normal in G.*

Proof. The subgroup A is normal in G by Corollary 2.5. Assume by contradiction that the statement is false, and let X be a maximal element of the set of all subgroups of A which are not normal in G. As X is finitely generated, the group A/X must be infinite. Moreover, in A/X the identity subgroup cannot be obtained as intersection of a collection of non-trivial subgroups, and hence A/X is a group of type p^∞ for some prime number p. Write

$$X = \langle x_1 \rangle \times \cdots \times \langle x_t \rangle,$$

where each $\langle x_i\rangle$ is an infinite cyclic subgroup. For each prime number $q \neq p$ and for all $i = 1, \dots, t$ put

$$X_{i,q} = \langle x_1^q\rangle \times \cdots \times \langle x_{i-1}^q\rangle \times \langle x_i\rangle \times \langle x_{i+1}^q\rangle \times \cdots \times \langle x_t^q\rangle.$$

Then

$$A/X_{i,q} = X/X_{i,q} \times B_{i,q}/X_{i,q},$$

where $B_{i,q}/X_{i,q}$ is a group of type p^∞. Clearly, each $B_{i,q}$ is a normal subgroup of G by Corollary 2.5, so that also the intersection

$$B_i = \bigcap_{p\neq q} B_{i,q}$$

is normal in G. Since

$$B_i \cap X = \bigcap_{q\neq p}(B_{i,q} \cap X) = \bigcap_{q\neq p} X_{i,q} = \langle x_i\rangle,$$

it follows that B_i has rank 1 and hence the normal closure $\langle x_i\rangle^G$ is cyclic by Lemma 2.1. Therefore the subgroup $\langle x_i\rangle$ is normal in G for all $i = 1, \dots, t$, so that X itself is normal in G, and this contradiction proves the lemma. ∎

LEMMA 2.13. *Let G be a soluble $\widehat{T}$-group which is not polycyclic, and let F be the Fitting subgroup of G. If the largest periodic subgroup T of F is neither finite nor a finite extension of a Prüfer group, then all subgroups of F are normal in G.*

Proof. If F is a periodic group, the statement follows from Lemma 2.9. Thus suppose that T is properly contained in F, and that T is neither finite nor a finite extension of a Prüfer group. Let A be a maximal abelian normal subgroup of T; then $C_T(A) = A$ and A contains a subgroup B such that $B = B_1 \times B_2$, where both B_1 and B_2 are infinite. Let a be any element of infinite order of F. The subgroups $\langle a, B_1\rangle$ and $\langle a, B_2\rangle$ are normal in G by Corollary 2.5, so that also $\langle a\rangle = \langle a, B_1\rangle \cap \langle a, B_2\rangle$ is a normal subgroup of G. If x is any element of T, then $\langle x\rangle$ is characteristic in $\langle a, x\rangle = \langle a\rangle\langle x\rangle$ and so normal in G. Therefore all subgroups of F are normal in G. ∎

LEMMA 2.14. *Let G be a soluble $\widehat{T}$-group which is not polycyclic, and let F be the Fitting subgroup of G. If the largest periodic subgroup T of F is finite, then $C_G(G') = F$ and in particular G is metabelian.*

Proof. By Corollary 2.4 the group F is nilpotent, so that it satisfies the maximal condition on non-normal subgroups. As F is not polycyclic, it follows that either F is abelian or it is isomorphic to $\mathbb{Q}_2 \times E$, where $\mathbb{Q}_2$ is the additive group of rational numbers whose denominators are powers of 2 and E is finite (see [5]). In any case, F contains a torsion-free abelian subgroup A such that $F = T \times A$, and all subgroups of A are normal in G by Lemma 2.12. Let x be any element of T. Since A is not finitely generated, there exist elements $a_1, a_2, \dots, a_n, \dots$ of A such that

$$\langle x, a_1 \rangle < \langle x, a_1, a_2 \rangle < \cdots < \langle x, a_1, a_2, \dots, a_n \rangle < \cdots,$$

and so the subgroup $\langle x, a_1, \dots, a_m \rangle$ is normal in G for some positive integer m. Clearly, $\langle x \rangle$ is the subgroup of all elements of finite order of $\langle x, a_1, \dots, a_m \rangle$ and hence it is likewise normal in G. Therefore G induces groups of power automorphisms on both T and A, so that in particular

$$G' \leq C_G(T) \cap C_G(A) = C_G(F)$$

and hence $C_G(G') = F$. ∎

Our next result shows in particular that, with the obvious exception of polycyclic groups, soluble $\widehat{T}$-groups have derived length at most 3.

THEOREM 2.15. *If G is a soluble $\widehat{T}$-group which is not polycyclic, then G'' is abelian. Moreover, if G is not an extension of a Prüfer group by a polycyclic group, then G' is nilpotent of class at most 2 and G'' is cyclic with prime power order.*

Proof. Let F be the Fitting subgroup of G. If F has no Prüfer subgroups, it follows from Lemmas 2.13 and 2.14 that G is metabelian. Suppose now that F contains a subgroup P of type p^∞ for some prime number p, and let X be any subgroup of G such that $P \leq X \leq F$. Then X is not finitely generated and so it is normal in G by Corollary 2.5. Therefore G' acts trivially on both P and F/P, so that $G'' \leq C_G(F) \leq F$ and G'' is abelian.

Assume that G is not an extension of a Prüfer group by a polycyclic group. We may obviously suppose that $G'' \neq \{1\}$, so that it follows again from Lemmas 2.13 and 2.14 that the largest periodic subgroup T of F contains a subgroup P of type p^∞ with T/P finite. Put $\overline{G} = G/P$ and let $\overline{K}$ be the Fitting subgroup of $\overline{G}$. If $\overline{Q} = Q/P$ is a Prüfer subgroup of $\overline{K}$, then $P \leq Z(Q)$ and Q lies in F. This contradiction shows that $\overline{K}$ cannot contain Prüfer subgroups, and hence a further application of Lemmas 2.13 and 2.14 shows that $\overline{G}$ is metabelian and so G'' is contained in P. In particular, $G' \leq C_G(G'')$ and G' is nilpotent of class 2. Finally, G'' satisfies the maximal condition on subgroups by Lemma 2.3 and hence it is cyclic with prime power order. ∎

As finitely generated soluble T-groups are either finite or abelian, the last result of this section shows that finitely generated soluble groups behave similarly with respect to the properties T and $\widehat{T}$.

THEOREM 2.16. *Let G be a finitely generated soluble $\widehat{T}$-group. Then G is polycyclic.*

Proof. Let A be the smallest non-trivial term of the derived series of G. By induction on the derived length of G it can be assumed that the factor group G/A is polycyclic, so that A contains a finitely generated subgroup E such that $A = E^G$. Since E is subnormal in G, it follows from Lemma 2.1 that A is finitely generated. Therefore the group G is polycyclic. ■

3. Periodic $\widehat{T}$-groups. Recall that a group G is called an *IT-group* if all its infinite subnormal subgroups are normal. The structure of IT-groups has been described in [8], where it is proved in particular that a periodic soluble group G has the property IT if and only if G is either a T-group or an extension of a Prüfer group by a finite T-group. Clearly, every IT-group satisfies the minimal condition on subnormal non-normal subgroups; our next result shows that in the periodic soluble case the property IT forces the group to satisfy also the maximal condition on subnormal non-normal subgroups.

LEMMA 3.1. *Let G be a periodic soluble IT-group. Then G is a $\widehat{T}$-group.*

Proof. We may obviously suppose that G is infinite and it is not a T-group. Thus G contains a normal subgroup P of type p^∞ for some prime number p such that G/P is a finite T-group. Assume by contradiction that G is not a $\widehat{T}$-group, so that there exist infinitely many subnormal non-normal subgroups $X_1, X_2, \dots, X_n, \dots$ of G such that

$$X_1 < X_2 < \dots < X_n < \cdots,$$

and each X_n must be finite since G is an IT-group. Then $[P, X_n] = \{1\}$ and so P normalizes all X_n. The subgroup

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

is infinite and so it contains P. As G/P is finite, there is a positive integer m such that $X_kP = X_mP$ for every $k \geq m$, so that in particular $X = X_mP$ is subnormal and hence also normal in G. Let $\{g_1, \dots, g_t\}$ be a set of representatives of the cosets of P in G. For each $i = 1, \dots, t$ the subgroup $X_m^{g_i}$ is contained in X and so $X_m^{g_i} \leq X_{s_i}$ for a suitable $s_i \geq m$. Moreover,

$$X_r = X_mP \cap X_r = X_m(P \cap X_r)$$

for all $r \geq s_i$ and then

$$X_r^{g_i} = X_m^{g_i}(P \cap X_r) \leq X_r,$$

so that $X_r^{g_i} = X_r$. In particular, if $s = \max\{s_1, \dots, s_t\}$, then $X_s^{g_i} = X_s$ for each $i = 1, \dots, t$, and hence X_s is normal in G. This contradiction proves the lemma. ■

It can be observed that there exist soluble $\widehat{T}$-groups which are finite extensions of a Prüfer group but do not have the property IT. In fact, let K be a group of type p^∞ (where p is a prime number), $H = \langle x, y\rangle$ a dihedral group of order 8 with $x^2 = y^2 = 1$, and consider the semidirect product

$G = H \ltimes K$, where $a^x = a^y = a^{-1}$ for each $a \in K$. Then $C_G(K) = \langle xy \rangle \times K$ is abelian and all its subgroups are normal in G. Since each finite subnormal subgroup of G is contained in $C_G(K)$, it follows that any subnormal non-normal subgroup of G is infinite; thus G has finitely many subnormal non-normal subgroups and so it is a $\widehat{T}$-group. On the other hand, the infinite subnormal subgroup $\langle x, K \rangle$ is not normal and hence G is not an IT-group.

The next results of this section show that periodic soluble $\widehat{T}$-groups either have the property T or contain an abelian subgroup of finite index; moreover, any periodic soluble $\widehat{T}$-group has finite conjugacy classes of subnormal subgroups.

THEOREM 3.2. *Let G be a periodic soluble $\widehat{T}$-group which is not a T-group. Then G is abelian-by-finite and G/G' is either finite or a finite extension of a Prüfer group.*

Proof. It can obviously be assumed that G is neither finite nor a finite extension of a Prüfer group. If F is the Fitting subgroup of G, it follows from Corollary 2.10 that all subgroups of F are normal in G, and so G' is abelian by Lemma 2.8. Let X be a subnormal non-normal subgroup of G. Then X' is normal in G and X/X' is an abelian subnormal non-normal subgroup of G/X'. Another application of Corollary 2.10 shows that G/X' is either finite or a finite extension of a Prüfer group, so that G/G' has the same structure. If G/G' is finite, then G is abelian-by-finite. Suppose that G/G' contains a subgroup P/G' of type p^∞ such that G/P is finite. As $G/C_G(G')$ is isomorphic to a group of power automorphisms of G', it is residually finite and hence $P \leq C_G(G')$. Thus P is nilpotent, so that it is contained in F and G/F finite. Since F is a Dedekind group, it follows that the group G is abelian-by-finite. ■

COROLLARY 3.3. *Let G be a periodic soluble $\widehat{T}$-group, and let X be a subnormal non-normal subgroup of G. Then G/X_G is either finite or a finite extension of a Prüfer group. Moreover, if G is neither finite nor a finite extension of a Prüfer group, then X/X_G is abelian and X^G/X_G is finite.*

Proof. It can obviously be assumed that G is neither finite nor a finite extension of a Prüfer group. The argument used in the proof of Theorem 3.2 shows that X' is normal in G and G/X' is either finite or a finite extension of a Prüfer group, so that in particular G/X_G has the same structure and X/X_G is abelian. Moreover, X/X_G has finitely many conjugates in G/X_G (see [14, Part 1, Theorem 5.49]). Since the Fitting subgroup F of G has finite index and $X \cap F \leq X_G$, it follows that X/X_G is finite, and hence its normal closure X^G/X_G is likewise finite. ■

If G is any T-group, then $\gamma_3(G) = \gamma_4(G)$ and so the factor group $G/\overline{\gamma}(G)$ is nilpotent (here $\overline{\gamma}(G)$ denotes the last term of the lower central series of G).

Our next lemma shows in particular that also in the case of periodic soluble $\widehat{T}$-groups the lower central series stops after finitely many steps.

Lemma 3.4. *Let G be a periodic soluble $\widehat{T}$-group, and let L be the last term of the lower central series of G. Then the factor group G/L is nilpotent.*

Proof. The statement is obvious for Černikov groups. Suppose that G is not a Černikov group, and assume by contradiction that G/L is not nilpotent; then neither is $G/\gamma_\omega(G)$, and of course the latter group does not satisfy the minimal condition on subgroups. Replacing G by $G/\gamma_\omega(G)$, we may also suppose that the group G is residually nilpotent. Thus G is the direct product of its Sylow subgroups (see [14, Part 2, p. 8]). Let D be the largest divisible abelian normal subgroup of G. As $[D,G] \leq \gamma_n(G)$ for each positive integer n, also $[D,G] \leq \gamma_\omega(G) = \{1\}$, and hence D is contained in $Z(G)$. On the other hand, each subnormal subgroup of G has finite index in its normal closure by Corollary 3.3, and it follows that all Sylow subgroups of G are nilpotent (see [2, Theorem 3.2]). Therefore the group G itself is nilpotent, and this contradiction proves that G/L is nilpotent. ∎

Lemma 3.5. *Let G be an infinite periodic soluble $\widehat{T}$-group, and let L be the smallest term of the lower central series of G. If G is not a finite extension of a Prüfer group, then $L^2 = L$.*

Proof. Clearly, it can be assumed that G is not a T-group, so that G/G' is either finite or a finite extension of a Prüfer group by Theorem 3.2. As the factor group G/L is nilpotent by Lemma 3.4, it satisfies the maximal condition on non-normal subgroups and hence is a finite extension of a Prüfer group (see [5]). Thus L must be infinite. Let H be a subgroup of L such that $|L : H| \leq 2$. Then H is normal in G by Theorem 2.15 and Lemma 2.10, and clearly G/H is nilpotent, so that $H = L$ and $L^2 = L$. ∎

It was proved in [13] that a soluble p-group G with the property T is abelian if $p > 2$, while if $p = 2$ then either G is a Dedekind group or it has a very restricted structure. For primary soluble $\widehat{T}$-groups we have the following result.

Theorem 3.6. *Let G be an infinite primary soluble $\widehat{T}$-group which is not a finite extension of a Prüfer group.*

(a) *If G is a p-group for some odd prime p, then G is abelian.*
(b) *If G is a 2-group, then it has finitely many subnormal non-normal subgroups.*

Proof. By Corollary 3.3 every subnormal subgroup of G has finite index in its normal closure. If G is a p-group with p odd, then G is abelian (see [2, Theorem 3.2]). Suppose now that G is a 2-group. The same result of Casolo shows that the Fitting subgroup F of G has index at most 2. Moreover, all subgroups of F are normal in G by Corollary 2.10, so that it can be assumed that $|G : F| = 2$ and hence $G = \langle F, z\rangle$ for any element z of $G \setminus F$. As the last term L of the lower central series of G is a non-trivial divisible subgroup by Lemmas 3.4 and 3.5, it follows that F is abelian and $a^z = a^{-1}$ for all $a \in F$. Thus $G' = [F, z] = F^2$ and so $L = \gamma_{n+1}(G) = F^{2^n}$ for some non-negative integer n. Therefore G/L has finite exponent and hence it is either finite or a Dedekind group (see [5]). Let X be any subnormal non-normal subgroup of G. Then X cannot be contained in F, so that $[L, X] = L^2 = L$ and $L \leq X$. Therefore the group G has only finitely many subnormal non-normal subgroups. ∎

In the last part of this section we consider the Sylow structure of periodic soluble groups with the property $\widehat{T}$. If G is a periodic soluble T-group, then the intersection $\pi([G', G]) \cap \pi(G/[G', G])$ contains no odd primes; moreover, if $2 \in \pi([G', G])$, it is known that the Sylow 2-subgroups of G satisfy certain strong restrictions (see [13, Theorem 4.2.2]). Our previous results can be applied to obtain a corresponding information for periodic soluble $\widehat{T}$-groups. It is not surprising that the only exceptions are produced by Sylow subgroups of small size.

THEOREM 3.7. *Let G be a periodic soluble $\widehat{T}$-group which is not a finite extension of a Prüfer group, and let L be the last term of the lower central series of G. If for some prime number p the p-component L_p of L is infinite and $q > p$ is a prime in $\pi(L)$, then $q \notin \pi(G/L)$.*

Proof. Let π_p be the set of all prime numbers $q > p$. The group G is locally supersoluble by Corollary 2.11, and hence the set N consisting of all π_p-elements of G is a subgroup. Let X be any subnormal subgroup of N. As L_p is infinite and all its subgroups are normal in G, there exists a finite subgroup E of L_p such that XE is normal in G; but X is a characteristic subgroup of XE, and so it is normal in G. Therefore all subnormal subgroups of N are normal in G, and in particular N is a T-group; it follows that the last term K of the lower central series of N is a Hall subgroup of N (see [13, Theorem 4.2.2]). In particular, $\pi(K) \cap \pi(G/L) = \emptyset$. Moreover, N/K is nilpotent, so that all its subgroups are normal in G/K and N/K must be abelian. Let $q > p$ be an element of $\pi(L) \setminus \pi(K)$ and let M/K be the q-component of N/K. Then L_q is contained in M and $C_G(L_qK/K) = C_G(L_q)$ is a proper subgroup of G. Since $G/C_G(M/K)$ is isomorphic to a periodic q'-group of power automorphisms of M/K, it follows that G acts fixed-point-freely on M/L_qK. On the other hand, ML/L lies in the centre of G/L, so that

$$[M, G] \leq M \cap L = L_qK$$

and hence $M = L_qK \leq L$. Therefore $q \notin \pi(G/L)$. ∎

Corollary 3.8. *Let G be a periodic soluble $\widehat{T}$-group, and let L be the last term of the lower central series of G. If p is a prime number such that the p-component L_p of L is neither finite nor a finite extension of a Prüfer group and $q \geq p$ is an odd prime in $\pi(L)$, then $q \notin \pi(G/L)$.*

Proof. By Theorem 3.7 it is enough to prove that if the prime p is odd, then $p \notin \pi(L)$. Let $L_{p'}$ be the p'-component of L. Replacing G by the factor group $G/L_{p'}$, it can be assumed without loss of generality that L is a p-group. As G/L is nilpotent by Lemma 3.4, it follows that G contains a unique Sylow p-subgroup M. Then M is abelian by Theorem 3.6 and so all its subgroups are normal in G by Corollary 2.10. In particular, $G/C_G(L)$ is isomorphic to a non-trivial p'-group of power automorphisms of L. Thus G acts fixed-point-freely on L and so also on M and on M/L (see for instance [13, Lemma 4.1.2]). On the other hand, M/L is contained in the centre of G/L, so that $L = M$ and G/L is a p'-group. ∎

Corollary 3.9. *Let G be a periodic soluble $\widehat{T}$-group, and let L be the last term of the lower central series of G. If L has no elements of order 2, then there exists a finite set π of prime numbers such that the π-component L_π of L is either finite or a finite extension of a Prüfer group and $L_{\pi'}$ is a Hall subgroup of G.*

Proof. We may obviously suppose that L is not a Hall subgroup of G, so that in particular G is not a T-group. The nilpotent group G/L satisfies the maximal condition on non-normal subgroups and so its derived subgroup G'/L is finite (see [5]). Then it follows from Theorem 3.2 that the set $\pi = \pi(G/L)$ is finite. Clearly, $L_{\pi'}$ is a Hall subgroup of G and $L_{\pi'} < L$. Assume that L_π is infinite, and let p be the smallest prime in π such that L_p is infinite. As $p > 2$, Corollary 3.8 implies that L_p must be a finite extension of a Prüfer group. Moreover, by Theorem 3.7 the set $\pi(L) \cap \pi(G/L)$ cannot contain primes greater than p. Therefore L_π is a finite extension of a Prüfer group. ∎

We finally show that the primary structure of a periodic soluble $\widehat{T}$-group is quite similar to that of periodic soluble groups with the property T, provided that the last term of the lower central series contains elements of order 2.

Theorem 3.10. *Let G be an infinite periodic soluble $\widehat{T}$-group which is not a finite extension of a Prüfer group, and let L be the last term of the lower central series of G. If L has elements of order 2, then the odd component $L_{2'}$ of L is a Hall subgroup of G, G'/L is a 2-group, $2 \in \pi(G/G')$ and each element of G acts on L_2 either as the identity or as the inversion.*

Proof. Since $L^2 = L$ by Lemma 3.5, the 2-component L_2 of L must be infinite and so it follows from Theorem 3.7 that $L_{2'}$ is a Hall subgroup of G. Moreover, all subgroups of L are normal in G by Corollary 2.10, and in particular each element of $G \setminus C_G(L_2)$ induces on L_2 the inversion map. On the other hand, L_2 is not contained in $Z(G)$, and so $2 \in \pi(G/G')$. Let $K/L_{2'}$ be the unique Sylow 2-subgroup of $G/L_{2'}$. As G is hypercyclic by Corollary 2.11, the elements of odd order of G form a characteristic subgroup M and $M/L_{2'}$ is nilpotent. Clearly,

$$G/L_{2'} = K/L_{2'} \times M/L_{2'}$$

and all subgroups of $M/L_{2'}$ are normal in $G/L_{2'}$ because L_2 is infinite. It follows that $M/L_{2'}$ is abelian, so that G' lies in K and G'/L is a 2-group. ∎

4. Non-periodic $\widehat{T}$-groups. A group G is called an *LT-group* if each subnormal non-normal subgroup of G has finite index. Clearly, groups with the property LT can be considered as duals of IT-groups, and the structure of soluble LT-groups has been studied in [11]; in particular, it turns out that infinite soluble LT-groups are metabelian. It is also clear that all LT-groups have the property $\widehat{T}$.

THEOREM 4.1. *Let G be a soluble $\widehat{T}$-group which is not polycyclic, and let F be the Fitting subgroup of G. If either F is torsion-free, or the largest periodic subgroup T of F is infinite but it is not a finite extension of a Prüfer group, then G is an LT-group.*

Proof. We may obviously suppose that G is not a T-group, so that it contains a subnormal non-normal subgroup X. It follows from Lemmas 2.7, 2.12 and 2.13 that all subgroups of F are normal in G, so that in particular F is abelian and $C_G(F) = F$. Then G/F is isomorphic to a non-trivial group of power automorphisms of F, so that $|G : F| = 2$ and $G = \langle F, z\rangle$ where $a^z = a^{-1}$ for all $a \in F$. It follows that $\gamma_{n+1}(G) = F^{2^n}$ for each positive integer n, and so $G/\gamma_{n+1}(G)$ has finite exponent. Assume by contradiction that the nilpotent group $G/\gamma_4(G)$ is infinite. As $G/\gamma_4(G)$ satisfies the maximal condition on non-normal subgroups, it is a Dedekind group (see [5]). In particular, $L = \gamma_3(G)$ is the last term of the lower central series of G and $L^2 = L$. As X is not contained in F, there is an element x of X such that $a^x = a^{-1}$ for all $a \in F$; then $[L, x] = L^2 = L$, so that $L \le X$ and X is normal in G. This contradiction shows that $G/\gamma_4(G)$ is finite. In particular, the group G/F^2 is finite and hence G is an LT-group (see [11, Theorem 3.3]). ∎

COROLLARY 4.2. *Let G be a torsion-free soluble $\widehat{T}$-group. If G is not polycyclic, then it is abelian.*

Proof. The group G has the property LT by Theorem 4.1 and hence it is abelian (see [11, Corollary 3.4]). ∎

Observe finally that there exist soluble non-polycyclic $\widehat{T}$-groups with torsion-free Fitting subgroup for which the set of subnormal non-normal subgroups is infinite. In fact, let p be an odd prime number and consider the semidirect product $G = \langle x \rangle \ltimes A$, where A is isomorphic to the additive group of rational numbers whose denominators are powers of p and x is an element of order 2 such that $a^x = a^{-1}$ for all $a \in A$. Then A is the Fitting subgroup of G and G is an LT-group (see [11, Theorem 3.3]), so that in particular G has the property $\widehat{T}$. On the other hand, $\langle x, A^{2^n} \rangle$ is a subnormal non-normal subgroup of G for each integer $n \geq 2$, so that G contains infinitely many subnormal non-normal subgroups.

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