GROUPS SATISFYING THE MAXIMAL CONDITION ON SUBNORMAL NON-NORMAL SUBGROUPS

BY

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Abstract. The structure of (generalized) soluble groups for which the set of all subnormal non-normal subgroups satisfies the maximal condition is described, taking as a model the known theory of groups in which normality is a transitive relation.

1. Introduction. A group $G$ is called a $T$-group if all its subnormal subgroups are normal, i.e. if normality is a transitive relation in $G$. The structure of soluble $T$-groups has been described by W. Gaschütz [12] in the finite case and by D. J. S. Robinson [13] for arbitrary groups. It turns out that soluble groups with the property $T$ are metabelian and hypercyclic, and that finitely generated soluble $T$-groups are either finite or abelian; moreover, Sylow properties of periodic soluble $T$-groups have been studied.

In recent years, many papers deal with the structure of (generalized) soluble groups in which normality is imposed only on certain systems of subnormal subgroups (see [8], [9], [11]). Other classes of generalized $T$-groups can be introduced by imposing that the set of all subnormal non-normal subgroups of the group is small in some sense; this point of view was for instance adopted in [1], [6] and [10]. Groups satisfying the minimal condition on subnormal non-normal subgroups have recently been investigated (see [7]), and the aim of this paper is to study soluble groups satisfying the maximal condition on subnormal non-normal subgroups.

We shall say that a group $G$ is a $\hat{T}$-group (or that $G$ has the property $\hat{T}$) if the set of all subnormal non-normal subgroups of $G$ satisfies the maximal condition, i.e. if there does not exist in $G$ an infinite properly ascending chain

$$X_1 < X_2 < \cdots < X_n < \cdots$$

of subnormal non-normal subgroups. Any group satisfying the maximal condition on subnormal subgroups (in particular, any polycyclic-by-finite group) is obviously a $\hat{T}$-group; notice also that groups with finitely many subnormal non-normal subgroups, as well as groups in which every sub-

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normal subgroup of infinite index is normal, have the property $\hat{T}$. Observe finally that nilpotent $\hat{T}$-groups must satisfy the maximal condition on non-normal subgroups. Locally soluble groups with this latter property have been completely described by G. Cutolo [5]; in particular, they either satisfy the maximal condition or are nilpotent of class at most 2. For instance, it turns out that the direct product $H = X \times Y$ of a non-abelian group $X$ of order $p^3$ and exponent $p$ and a group $Y$ of type $p^\infty$ is not a $\hat{T}$-group, while the property $\hat{T}$ holds for the factor group $H/\langle xy \rangle$, where $Z(X) = \langle x \rangle$ and $y$ is an element of order $p$ of $Y$.

We take the known theory of $T$-groups as our model, and show that a number of properties of such groups have an analogue in the class of groups satisfying $\hat{T}$.

Most of our notation is standard and can be found in [14].

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2. General properties of $\hat{T}$-groups. In the first part of this section we describe the behaviour of the Fitting subgroup of a $\hat{T}$-group. Recall that the *Baer radical* of a group $G$ is the subgroup generated by all abelian subnormal subgroups of $G$, and that $G$ is a *Baer group* if it coincides with its Baer radical. It is easy to prove that $G$ is a Baer group if and only if all its cyclic subgroups are subnormal.

**Lemma 2.1.** Let $G$ be a $\hat{T}$-group, and let $X$ be a subnormal non-normal subgroup of $G$. If $X$ is a Baer group, then the normal closure $X^G$ of $X$ satisfies the maximal condition on subgroups.

**Proof.** Since $X$ is contained in the Baer radical of $G$, also its normal closure $X^G$ is a Baer group. Assume by contradiction that $X$ is not finitely generated. Since all finitely generated subgroups of $X$ are subnormal in $G$, we may consider a maximal element $M$ of the set of all finitely generated subgroups of $X$ which are not normal in $G$. Thus $\langle M, x \rangle$ is a normal subgroup of $G$ for each $x \in X \setminus M$, and hence also

$$X = \langle \langle M, x \rangle \mid x \in X \setminus M \rangle$$

is normal in $G$. This contradiction shows that $X$ is finitely generated, and so it satisfies the maximal condition on subgroups. Clearly, the set of all subnormal subgroups $H$ of $G$ such that $X \leq H \leq X^G$ satisfies the maximal condition. It follows that for each positive integer $i$ the locally nilpotent group $X^{G,i}/X^{G,i+1}$ has the maximal condition on subnormal subgroups, and so also the maximal condition on subgroups (see [14, Part 1, Theorem 5.37]). Therefore $X^G$ satisfies the maximal condition on subgroups. \[\blacksquare\]

**Lemma 2.2.** Let $G$ be a locally nilpotent group whose derived subgroup $G'$ is finitely generated. Then $G$ is nilpotent.
Proof. As $G'$ is nilpotent, it is enough to prove that $G/G''$ is nilpotent, so that without loss of generality it can be assumed that $G'$ is abelian. Moreover, since the subgroup $T$ consisting of all elements of finite order of $G'$ is finite, we may also replace $G$ by the factor group $G/T$ and suppose that $G'$ is a finitely generated torsion-free abelian group. If $G'$ has rank $r$, it follows that $[G',rG]$, is contained in $(G')^p$ for each prime number $p$, so that

$$[G',rG] \leq \bigcap_p (G')^p = \{1\}$$

and the group $G$ is nilpotent. ■

Lemma 2.3. Let $G$ be a Baer group with the property $\hat{T}$. Then the derived subgroup $G'$ of $G$ satisfies the maximal condition on subgroups.

Proof. We may obviously suppose that $G$ is not a Dedekind group. Let $X$ be a maximal subnormal non-normal subgroup of $G$; then $G/X^G$ is a Dedekind group, and so $G'X^G/X^G$ is finite. On the other hand, $X^G$ satisfies the maximal condition on subgroups by Lemma 2.1, so that in particular $G' \cap X^G$ has this property, and hence $G'$ itself satisfies the maximal condition on subgroups. ■

Corollary 2.4. If $G$ is a $\hat{T}$-group, then the Baer radical of $G$ is nilpotent. In particular, the Baer radical and the Fitting subgroup of $G$ coincide.

Proof. Let $B$ be the Baer radical of $G$. By Lemma 2.3 the derived subgroup $B'$ of $B$ satisfies the maximal condition on subgroups, and hence $B$ is nilpotent by Lemma 2.2. ■

Corollary 2.5. Let $G$ be a $\hat{T}$-group, and let $F$ be the Fitting subgroup of $G$. If $X$ is any subgroup of $F$ which is not finitely generated, then $X$ is normal in $G$. In particular, all infinite periodic subgroups of $F$ are normal in $G$.

Proof. As $F$ is nilpotent by Corollary 2.4, the subgroup $X$ is subnormal in $G$ and does not satisfy the maximal condition on subgroups. Thus it follows from Lemma 2.1 that $X$ is normal in $G$. ■

A group $G$ is said to be subsoluble if it has an ascending series with abelian factors consisting of subnormal subgroups. Clearly, all hyperabelian groups are subsoluble, and for groups satisfying $\hat{T}$ subsolubility is equivalent to solubility, as the following result shows.

Theorem 2.6. Let $G$ be a subsoluble $\hat{T}$-group. Then $G$ is soluble.

Proof. Suppose first that the group $G$ is hyperabelian, and let

$$\{1\} = G_0 < G_1 < \cdots < G_\alpha < G_{\alpha+1} < \cdots < G_\tau = G$$

be an ascending normal series of $G$ with abelian factors. Assume that $G$ is not soluble, and consider the least ordinal $\mu \leq \tau$ such that $G_\mu$ is not
soluble. Clearly, $\mu$ is a limit ordinal and $G_\alpha$ is soluble for each $\alpha < \mu$. Let $\Lambda$ be the set of all ordinals $\alpha < \mu$ for which there exists a subnormal non-normal subgroup $X$ of $G$ such that $G_\alpha < X < G_\beta$ for some $\beta < \mu$; since $G$ satisfies $\widehat{T}$, the set $\Lambda$ must be finite, and so there is an ordinal $\delta < \mu$ such that if $X$ is any subnormal subgroup of $G$ with $G_\delta \leq X \leq G_\beta$ and $\delta < \beta < \mu$, then $X$ is normal in $G$. In particular, $G_\beta/G_\delta$ is a $T$-group for $\delta < \beta < \mu$ and so it is metabelian. As

$$G_\mu = \bigcup_{\delta < \beta < \mu} G_\beta,$$

it follows that $G_\mu/G_\delta$ is likewise metabelian, contradicting the assumption that $G_\mu$ is insoluble.

In the general case, let

$$\{1\} = X_0 < X_1 < \cdots < X_\alpha < X_{\alpha+1} < \cdots < X_\tau = G$$

be an ascending subnormal series of $G$ with abelian factors. The set of all ordinals $\alpha$ such that the subgroup $X_\alpha$ is not normal in $G$ is obviously finite, of order $k$ say. If $k = 0$, then $G$ is hyperabelian, and so even soluble by the first part of the proof. Assume that $k > 0$, and let $\varrho < \tau$ be the largest ordinal such that $X_\varrho$ is not normal in $G$. Then $X_{\varrho+1}$ is normal in $G$, and $G/X_{\varrho+1}$ is a hyperabelian $\widehat{T}$-group, so that it is soluble. Moreover, $X_{\varrho+1}$ has an ascending subnormal series with abelian factors in which there are at most $k - 1$ non-normal terms, so that by induction on $k$ the subgroup $X_{\varrho+1}$ is soluble, and hence $G$ itself is soluble.

**Lemma 2.7.** Let $G$ be a soluble $\widehat{T}$-group which is not polycyclic, and let $F$ be the Fitting subgroup of $G$. Then $F/Z(F)$ is a finite abelian group. In particular, if $F$ is torsion-free, then it is abelian.

**Proof.** Since soluble groups of automorphisms of polycyclic groups are likewise polycyclic (see [14, Part 1, Theorem 3.27]), the subgroup $F$ is not polycyclic. On the other hand, the nilpotent group $F$ satisfies the maximal condition on non-normal subgroups, and so $F/Z(F)$ is finite and abelian (see [5, Corollary 2.5]). In particular, $F'$ is finite by Schur’s theorem and hence $F$ is abelian, provided that it is torsion-free.

A power automorphism of a group $G$ is an automorphism mapping every subgroup of $G$ onto itself, and the set $\text{PAut} G$ of all power automorphisms of $G$ is an abelian residually finite normal subgroup of the full automorphism group $\text{Aut} G$ of $G$. Recall also that the set $\text{IAut} G$ of all automorphisms of $G$ fixing every infinite subgroup is a subgroup of $\text{Aut} G$ containing $\text{PAut} G$. The behaviour of the groups $\text{PAut} G$ and $\text{IAut} G$ has been investigated in [3] and [4], respectively. Power automorphisms play a central role in the study of $T$-groups, and they will also be important in our considerations.
Lemma 2.8. Let $G$ be a soluble group, and let $F$ be the Fitting subgroup of $G$. If all subgroups of $F$ are normal in $G$, then $C_G(G') = F$. In particular, $G$ is metabelian.

Proof. The group $G/C_G(F)$ is abelian, since it is isomorphic to a group of power automorphisms of $F$, and so $G' \leq C_G(F) \leq F$. It follows that $F$ is contained in $C_G(G')$, so that $C_G(G') = F$ and $G'$ is abelian.

Lemma 2.9. Let $G$ be an infinite soluble $\hat{T}$-group with periodic Fitting subgroup $F$. Then either all subgroups of $F$ are normal in $G$, or $F$ is a finite extension of a Prüfer group and $G^{(3)} = \{1\}$.

Proof. By Corollary 2.5 all infinite subgroups of $F$ are normal in $G$, so that $G/C_G(F)$ is isomorphic to a subgroup of $IAut F$. In particular, if $IAut F = PAut F$, then all subgroups of $F$ are normal in $G$. On the other hand, if $IAut F \neq PAut F$, it follows that $F$ is a finite extension of a Prüfer group and $G/C_G(F)$ is metabelian (see [4, Corollary 2.4 and Proposition 2.5]), so that $G'' \leq C_G(F) = Z(F)$ and $G^{(3)} = \{1\}$.

Corollary 2.10. Let $G$ be a periodic soluble $\hat{T}$-group, and let $F$ be the Fitting subgroup of $G$. Then either all subgroups of $F$ are normal in $G$, or $G$ is a finite extension of a Prüfer group and $G^{(3)} = \{1\}$.

Proof. The statement follows directly from Lemma 2.9, because if $F$ is a finite extension of a Prüfer group, then $G/C_G(F)$ is finite (see [14, Part 1, Corollary to Theorem 3.29.2]) and so $G$ itself is a finite extension of a Prüfer group.

Corollary 2.11. Let $G$ be a periodic soluble $\hat{T}$-group which is not a finite extension of a Prüfer group. Then $G$ is metabelian and hypercyclic.

Proof. Let $F$ be the Fitting subgroup of $G$. All subgroups of $F$ are normal in $G$ by Corollary 2.10, so that $G$ is metabelian by Lemma 2.8 and $G'$ is hypercyclically embedded in $G$. Therefore $G$ is hypercyclic.

Lemma 2.12. Let $G$ be a $\hat{T}$-group, and let $A$ be a torsion-free abelian subnormal subgroup of $G$. If $A$ is not finitely generated, then all subgroups of $A$ are normal in $G$.

Proof. The subgroup $A$ is normal in $G$ by Corollary 2.5. Assume by contradiction that the statement is false, and let $X$ be a maximal element of the set of all subgroups of $A$ which are not normal in $G$. As $X$ is finitely generated, the group $A/X$ must be infinite. Moreover, in $A/X$ the identity subgroup cannot be obtained as intersection of a collection of non-trivial subgroups, and hence $A/X$ is a group of type $p^\infty$ for some prime number $p$. Write

$$X = \langle x_1 \rangle \times \cdots \times \langle x_t \rangle,$$
where each $\langle x_i \rangle$ is an infinite cyclic subgroup. For each prime number $q \neq p$ and for all $i = 1, \ldots, t$ put
\[ X_{i,q} = \langle x_i^q \rangle \times \cdots \times \langle x_{i-1}^q \rangle \times \langle x_i \rangle \times \langle x_{i+1}^q \rangle \times \cdots \times \langle x_t^q \rangle. \]

Then
\[ A/X_{i,q} = X/X_{i,q} \times B_{i,q}/X_{i,q}, \]
where $B_{i,q}/X_{i,q}$ is a group of type $p^\infty$. Clearly, each $B_{i,q}$ is a normal subgroup of $G$ by Corollary 2.5, so that also the intersection
\[ B_i = \bigcap_{p \neq q} B_{i,q} \]
is normal in $G$. Since
\[ B_i \cap X = \bigcap_{q \neq p} (B_{i,q} \cap X) = \bigcap_{q \neq p} X_{i,q} = \langle x_i \rangle, \]
it follows that $B_i$ has rank 1 and hence the normal closure $\langle x_i \rangle^G$ is cyclic by Lemma 2.1. Therefore the subgroup $\langle x_i \rangle$ is normal in $G$ for all $i = 1, \ldots, t$, so that $X$ itself is normal in $G$, and this contradiction proves the lemma.

**Lemma 2.13.** Let $G$ be a soluble $\hat{T}$-group which is not polycyclic, and let $F$ be the Fitting subgroup of $G$. If the largest periodic subgroup $T$ of $F$ is neither finite nor a finite extension of a Prüfer group, then all subgroups of $F$ are normal in $G$.

**Proof.** If $F$ is a periodic group, the statement follows from Lemma 2.9. Thus suppose that $T$ is properly contained in $F$, and that $T$ is neither finite nor a finite extension of a Prüfer group. Let $A$ be a maximal abelian normal subgroup of $T$; then $C_T(A) = A$ and $A$ contains a subgroup $B$ such that $B = B_1 \times B_2$, where both $B_1$ and $B_2$ are infinite. Let $a$ be any element of infinite order of $F$. The subgroups $\langle a, B_1 \rangle$ and $\langle a, B_2 \rangle$ are normal in $G$ by Corollary 2.5, so that also $\langle a \rangle = \langle a, B_1 \rangle \cap \langle a, B_2 \rangle$ is a normal subgroup of $G$. If $x$ is any element of $T$, then $\langle x \rangle$ is characteristic in $\langle a, x \rangle = \langle a \rangle \langle x \rangle$ and so normal in $G$. Therefore all subgroups of $F$ are normal in $G$.

**Lemma 2.14.** Let $G$ be a soluble $\hat{T}$-group which is not polycyclic, and let $F$ be the Fitting subgroup of $G$. If the largest periodic subgroup $T$ of $F$ is finite, then $C_G(G') = F$ and in particular $G$ is metabelian.

**Proof.** By Corollary 2.4 the group $F$ is nilpotent, so that it satisfies the maximal condition on non-normal subgroups. As $F$ is not polycyclic, it follows that either $F$ is abelian or it is isomorphic to $\mathbb{Q}_2 \times E$, where $\mathbb{Q}_2$ is the additive group of rational numbers whose denominators are powers of 2 and $E$ is finite (see [5]). In any case, $F$ contains a torsion-free abelian subgroup $A$ such that $F = T \times A$, and all subgroups of $A$ are normal in $G$ by Lemma 2.12. Let $x$ be any element of $T$. Since $A$ is not finitely generated,
there exist elements $a_1, a_2, \ldots, a_n, \ldots$ of $A$ such that
\[ \langle x, a_1 \rangle < \langle x, a_1, a_2 \rangle < \cdots < \langle x, a_1, a_2, \ldots, a_n \rangle < \cdots, \]
and so the subgroup $\langle x, a_1, \ldots, a_m \rangle$ is normal in $G$ for some positive integer $m$. Clearly, $\langle x \rangle$ is the subgroup of all elements of finite order of $\langle x, a_1, \ldots, a_m \rangle$ and hence it is likewise normal in $G$. Therefore $G$ induces groups of power automorphisms on both $T$ and $A$, so that in particular
\[ G' \leq C_G(T) \cap C_G(A) = C_G(F) \]
and hence $C_G(G') = F$. ■

Our next result shows in particular that, with the obvious exception of polycyclic groups, soluble $\hat{T}$-groups have derived length at most 3.

**Theorem 2.15.** If $G$ is a soluble $\hat{T}$-group which is not polycyclic, then $G''$ is abelian. Moreover, if $G$ is not an extension of a Prüfer group by a polycyclic group, then $G'$ is nilpotent of class at most 2 and $G''$ is cyclic with prime power order.

**Proof.** Let $F$ be the Fitting subgroup of $G$. If $F$ has no Prüfer subgroups, it follows from Lemmas 2.13 and 2.14 that $G$ is metabelian. Suppose now that $F$ contains a subgroup $P$ of type $p^\infty$ for some prime number $p$, and let $X$ be any subgroup of $G$ such that $P \leq X \leq F$. Then $X$ is not finitely generated and so it is normal in $G$ by Corollary 2.5. Therefore $G'$ acts trivially on both $P$ and $F/P$, so that $G'' \leq C_G(F) \leq F$ and $G''$ is abelian.

Assume that $G$ is not an extension of a Prüfer group by a polycyclic group. We may obviously suppose that $G'' \neq \{1\}$, so that it follows again from Lemmas 2.13 and 2.14 that the largest periodic subgroup $T$ of $F$ contains a subgroup $P$ of type $p^\infty$ with $T/P$ finite. Put $\overline{G} = G/P$ and let $\overline{K}$ be the Fitting subgroup of $\overline{G}$. If $\overline{Q} = Q/P$ is a Prüfer subgroup of $\overline{K}$, then $P \leq Z(\overline{Q})$ and $Q$ lies in $F$. This contradiction shows that $\overline{K}$ cannot contain Prüfer subgroups, and hence a further application of Lemmas 2.13 and 2.14 shows that $\overline{G}$ is metabelian and so $G''$ is contained in $P$. In particular, $G' \leq C_G(G'')$ and $G'$ is nilpotent of class 2. Finally, $G''$ satisfies the maximal condition on subgroups by Lemma 2.3 and hence it is cyclic with prime power order. ■

As finitely generated soluble $T$-groups are either finite or abelian, the last result of this section shows that finitely generated soluble groups behave similarly with respect to the properties $T$ and $\hat{T}$.

**Theorem 2.16.** Let $G$ be a finitely generated soluble $\hat{T}$-group. Then $G$ is polycyclic.

**Proof.** Let $A$ be the smallest non-trivial term of the derived series of $G$. By induction on the derived length of $G$ it can be assumed that the factor
group $G/A$ is polycyclic, so that $A$ contains a finitely generated subgroup $E$ such that $A = E^G$. Since $E$ is subnormal in $G$, it follows from Lemma 2.1 that $A$ is finitely generated. Therefore the group $G$ is polycyclic. ■

3. Periodic $\hat{T}$-groups. Recall that a group $G$ is called an $IT$-group if all its infinite subnormal subgroups are normal. The structure of $IT$-groups has been described in [8], where it is proved in particular that a periodic soluble group $G$ has the property $IT$ if and only if $G$ is either a $T$-group or an extension of a Prüfer group by a finite $T$-group. Clearly, every $IT$-group satisfies the minimal condition on subnormal non-normal subgroups; our next result shows that in the periodic soluble case the property $IT$ forces the group to satisfy also the maximal condition on subnormal non-normal subgroups.

**Lemma 3.1.** Let $G$ be a periodic soluble $IT$-group. Then $G$ is a $\hat{T}$-group.

**Proof.** We may obviously suppose that $G$ is infinite and it is not a $T$-group. Thus $G$ contains a normal subgroup $P$ of type $p^\infty$ for some prime number $p$ such that $G/P$ is a finite $T$-group. Assume by contradiction that $G$ is not a $\hat{T}$-group, so that there exist infinitely many subnormal non-normal subgroups $X_1, X_2, \ldots, X_n, \ldots$ of $G$ such that

$$X_1 < X_2 < \cdots < X_n < \cdots,$$

and each $X_n$ must be finite since $G$ is an $IT$-group. Then $[P, X_n] = \{1\}$ and so $P$ normalizes all $X_n$. The subgroup

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

is infinite and so it contains $P$. As $G/P$ is finite, there is a positive integer $m$ such that $X_KP = X_mP$ for every $k \geq m$, so that in particular $X = X_mP$ is subnormal and hence also normal in $G$. Let $\{g_1, \ldots, g_t\}$ be a set of representatives of the cosets of $P$ in $G$. For each $i = 1, \ldots, t$ the subgroup $X_{g_i}^m$ is contained in $X$ and so $X_{g_i}^m \leq X_{s_i}$ for a suitable $s_i \geq m$. Moreover,

$$X_r = X_mP \cap X_r = X_m(P \cap X_r)$$

for all $r \geq s_i$ and then

$$X_r^{g_i} = X_m^{g_i}(P \cap X_r) \leq X_r,$$

so that $X_r^{g_i} = X_r$. In particular, if $s = \max\{s_1, \ldots, s_t\}$, then $X_s^{g_i} = X_s$ for each $i = 1, \ldots, t$, and hence $X_s$ is normal in $G$. This contradiction proves the lemma. ■

It can be observed that there exist soluble $\hat{T}$-groups which are finite extensions of a Prüfer group but do not have the property $IT$. In fact, let $K$ be a group of type $p^\infty$ (where $p$ is a prime number), $H = \langle x, y \rangle$ a dihedral group of order 8 with $x^2 = y^2 = 1$, and consider the semidirect product
$G = H \ltimes K$, where $a^x = a^y = a^{-1}$ for each $a \in K$. Then $C_G(K) = \langle xy \rangle \times K$ is abelian and all its subgroups are normal in $G$. Since each finite subnormal subgroup of $G$ is contained in $C_G(K)$, it follows that any subnormal non-normal subgroup of $G$ is infinite; thus $G$ has finitely many subnormal non-normal subgroups and so it is a $\hat{T}$-group. On the other hand, the infinite subnormal subgroup $\langle x, K \rangle$ is not normal and hence $G$ is not an $IT$-group.

The next results of this section show that periodic soluble $\hat{T}$-groups either have the property $T$ or contain an abelian subgroup of finite index; moreover, any periodic soluble $\hat{T}$-group has finite conjugacy classes of subnormal subgroups.

**Theorem 3.2.** Let $G$ be a periodic soluble $\hat{T}$-group which is not a $T$-group. Then $G$ is abelian-by-finite and $G/G'$ is either finite or a finite extension of a Prüfer group.

**Proof.** It can obviously be assumed that $G$ is neither finite nor a finite extension of a Prüfer group. If $F$ is the Fitting subgroup of $G$, it follows from Corollary 2.10 that all subgroups of $F$ are normal in $G$, and so $G'$ is abelian by Lemma 2.8. Let $X$ be a subnormal non-normal subgroup of $G$. Then $X'$ is normal in $G$ and $X/X'$ is an abelian subnormal non-normal subgroup of $G/X'$. Another application of Corollary 2.10 shows that $G/X'$ is either finite or a finite extension of a Prüfer group, so that $G/G'$ has the same structure. If $G/G'$ is finite, then $G$ is abelian-by-finite. Suppose that $G/G'$ contains a subgroup $P/G'$ of type $p^\infty$ such that $G/P$ is finite. As $G/C_G(G')$ is isomorphic to a group of power automorphisms of $G'$, it is residually finite and hence $P \leq C_G(G')$. Thus $P$ is nilpotent, so that it is contained in $F$ and $G/F$ finite. Since $F$ is a Dedekind group, it follows that the group $G$ is abelian-by-finite. 

**Corollary 3.3.** Let $G$ be a periodic soluble $\hat{T}$-group, and let $X$ be a subnormal non-normal subgroup of $G$. Then $G/X_G$ is either finite or a finite extension of a Prüfer group. Moreover, if $G$ is neither finite nor a finite extension of a Prüfer group, then $X/X_G$ is abelian and $X^G/X_G$ is finite.

**Proof.** It can obviously be assumed that $G$ is neither finite nor a finite extension of a Prüfer group. The argument used in the proof of Theorem 3.2 shows that $X'$ is normal in $G$ and $G/X'$ is either finite or a finite extension of a Prüfer group, so that in particular $G/X_G$ has the same structure and $X/X_G$ is abelian. Moreover, $X/X_G$ has finitely many conjugates in $G/X_G$ (see [14, Part 1, Theorem 5.49]). Since the Fitting subgroup $F$ of $G$ has finite index and $X \cap F \leq X_G$, it follows that $X/X_G$ is finite, and hence its normal closure $X^G/X_G$ is likewise finite.

If $G$ is any $T$-group, then $\gamma_3(G) = \gamma_4(G)$ and so the factor group $G/\gamma(G)$ is nilpotent (here $\gamma(G)$ denotes the last term of the lower central series of $G$).
Our next lemma shows in particular that also in the case of periodic soluble $\hat{T}$-groups the lower central series stops after finitely many steps.

**Lemma 3.4.** Let $G$ be a periodic soluble $\hat{T}$-group, and let $L$ be the last term of the lower central series of $G$. Then the factor group $G/L$ is nilpotent.

**Proof.** The statement is obvious for Černikov groups. Suppose that $G$ is not a Černikov group, and assume by contradiction that $G/L$ is not nilpotent; then neither is $G/\gamma(G)$, and of course the latter group does not satisfy the minimal condition on subgroups. Replacing $G$ by $G/\gamma(G)$, we may also suppose that the group $G$ is residually nilpotent. Thus $G$ is the direct product of its Sylow subgroups (see [14, Part 2, p. 8]). Let $D$ be the largest divisible abelian normal subgroup of $G$. As $[D,G] \leq \gamma_n(G)$ for each positive integer $n$, also $[D,G] \leq \omega(G) = \{1\}$, and hence $D$ is contained in $Z(G)$. On the other hand, each subnormal subgroup of $G$ has finite index in its normal closure by Corollary 3.3, and it follows that all Sylow subgroups of $G$ are nilpotent (see [2, Theorem 3.2]). Therefore the group $G$ itself is nilpotent, and this contradiction proves that $G/L$ is nilpotent.

**Lemma 3.5.** Let $G$ be an infinite periodic soluble $\hat{T}$-group, and let $L$ be the smallest term of the lower central series of $G$. If $G$ is not a finite extension of a Prüfer group, then $L^2 = L$.

**Proof.** Clearly, it can be assumed that $G$ is not a $T$-group, so that $G/G'$ is either finite or a finite extension of a Prüfer group by Theorem 3.2. As the factor group $G/L$ is nilpotent by Lemma 3.4, it satisfies the maximal condition on non-normal subgroups and hence is a finite extension of a Prüfer group (see [5]). Thus $L$ must be infinite. Let $H$ be a subgroup of $L$ such that $[L:H] \leq 2$. Then $H$ is normal in $G$ by Theorem 2.15 and Lemma 2.10, and clearly $G/H$ is nilpotent, so that $H = L$ and $L^2 = L$.

It was proved in [13] that a soluble $p$-group $G$ with the property $T$ is abelian if $p > 2$, while if $p = 2$ then either $G$ is a Dedekind group or it has a very restricted structure. For primary soluble $\hat{T}$-groups we have the following result.

**Theorem 3.6.** Let $G$ be an infinite primary soluble $\hat{T}$-group which is not a finite extension of a Prüfer group.

(a) If $G$ is a $p$-group for some odd prime $p$, then $G$ is abelian.

(b) If $G$ is a 2-group, then it has finitely many subnormal non-normal subgroups.

**Proof.** By Corollary 3.3 every subnormal subgroup of $G$ has finite index in its normal closure. If $G$ is a $p$-group with $p$ odd, then $G$ is abelian (see [2, Theorem 3.2]). Suppose now that $G$ is a 2-group. The same result of Casolo shows that the Fitting subgroup $F$ of $G$ has index at most 2. Moreover, all
subgroups of $F$ are normal in $G$ by Corollary 2.10, so that it can be assumed that $|G : F| = 2$ and hence $G = \langle F, z \rangle$ for any element $z$ of $G \setminus F$. As the last term $L$ of the lower central series of $G$ is a non-trivial divisible subgroup by Lemmas 3.4 and 3.5, it follows that $F$ is abelian and $a^2 = a^{-1}$ for all $a \in F$. Thus $G' = [F, z] = F^2$ and so $L = \gamma_{2n+1}(G) = F^{2^n}$ for some non-negative integer $n$. Therefore $G/L$ has finite exponent and hence it is either finite or a Dedekind group (see [5]). Let $X$ be any subnormal non-normal subgroup of $G$. Then $X$ cannot be contained in $F$, so that $[L, X] = L^2 = L$ and $L \leq X$. Therefore the group $G$ has only finitely many subnormal non-normal subgroups.

In the last part of this section we consider the Sylow structure of periodic soluble groups with the property $\hat{T}$. If $G$ is a periodic soluble $T$-group, then

Theorem 3.7. Let $G$ be a periodic soluble $\hat{T}$-group which is not a finite extension of a Prüfer group, and let $L$ be the last term of the lower central series of $G$. If for some prime number $p$ the $p$-component $L_p$ of $L$ is infinite and $q > p$ is a prime in $\pi(L)$, then $q \notin \pi(G/L)$.

Proof. Let $\pi_p$ be the set of all prime numbers $q > p$. The group $G$ is locally supersoluble by Corollary 2.11, and hence the set $N$ consisting of all $\pi_p$-elements of $G$ is a subgroup. Let $X$ be any subnormal subgroup of $N$. As $L_p$ is infinite and all its subgroups are normal in $G$, there exists a finite subgroup $E$ of $L_p$ such that $XE$ is normal in $G$; but $X$ is a characteristic subgroup of $XE$, and so it is normal in $G$. Therefore all subnormal subgroups of $N$ are normal in $G$, and in particular $N$ is a $T$-group; it follows that the last term $K$ of the lower central series of $N$ is a Hall subgroup of $N$ (see [13, Theorem 4.2.2]). In particular, $\pi(K) \cap \pi(G/L) = \emptyset$. Moreover, $N/K$ is nilpotent, so that all its subgroups are normal in $G/K$ and $N/K$ must be abelian. Let $q > p$ be an element of $\pi(L) \setminus \pi(K)$ and let $M/K$ be the $q$-component of $N/K$. Then $L_q$ is contained in $M$ and $C_G(L_q K/K) = C_G(L_q)$ is a proper subgroup of $G$. Since $G/C_G(M/K)$ is isomorphic to a periodic $q'$-group of power automorphisms of $M/K$, it follows that $G$ acts fixed-point-freely on $M/L_q K$. On the other hand, $ML/L$ lies in the centre of $G/L$, so that

$$[M, G] \leq M \cap L = L_q K$$

and hence $M = L_q K \leq L$. Therefore $q \notin \pi(G/L)$. ■
Corollary 3.8. Let $G$ be a periodic soluble $\hat{T}$-group, and let $L$ be the last term of the lower central series of $G$. If $p$ is a prime number such that the $p$-component $L_p$ of $L$ is neither finite nor a finite extension of a Prüfer group and $q \geq p$ is an odd prime in $\pi(L)$, then $q \notin \pi(G/L)$.

Proof. By Theorem 3.7 it is enough to prove that if the prime $p$ is odd, then $p \notin \pi(L)$. Let $L_p'$ be the $p'$-component of $L$. Replacing $G$ by the factor group $G/L_p'$, it can be assumed without loss of generality that $L$ is a $p$-group. As $G/L$ is nilpotent by Lemma 3.4, it follows that $G$ contains a unique Sylow $p$-subgroup $M$. Then $M$ is abelian by Theorem 3.6 and so all its subgroups are normal in $G$ by Corollary 2.10. In particular, $G/C_G(L)$ is isomorphic to a non-trivial $p'$-group of power automorphisms of $L$. Thus $G$ acts fixed-point-freely on $L$ and so also on $M$ and on $M/L$ (see for instance [13, Lemma 4.1.2]). On the other hand, $M/L$ is contained in the centre of $G/L$, so that $L = M$ and $G/L$ is a $p'$-group.

Corollary 3.9. Let $G$ be a periodic soluble $\hat{T}$-group, and let $L$ be the last term of the lower central series of $G$. If $L$ has no elements of order 2, then there exists a finite set $\pi$ of prime numbers such that the $\pi$-component $L_\pi$ of $L$ is either finite or a finite extension of a Prüfer group and $L_{\pi'}$ is a Hall subgroup of $G$.

Proof. We may obviously suppose that $L$ is not a Hall subgroup of $G$, so that in particular $G$ is not a $T$-group. The nilpotent group $G/L$ satisfies the maximal condition on non-normal subgroups and so its derived subgroup $G'/L$ is finite (see [5]). Then it follows from Theorem 3.2 that the set $\pi = \pi(G/L)$ is finite. Clearly, $L_{\pi'}$ is a Hall subgroup of $G$ and $L_{\pi'} < L$. Assume that $L_\pi$ is infinite, and let $p$ be the smallest prime in $\pi$ such that $L_p$ is infinite. As $p > 2$, Corollary 3.8 implies that $L_p$ must be a finite extension of a Prüfer group. Moreover, by Theorem 3.7 the set $\pi(L) \cap \pi(G/L)$ cannot contain primes greater than $p$. Therefore $L_\pi$ is a finite extension of a Prüfer group.

We finally show that the primary structure of a periodic soluble $\hat{T}$-group is quite similar to that of periodic soluble groups with the property $T$, provided that the last term of the lower central series contains elements of order 2.

Theorem 3.10. Let $G$ be an infinite periodic soluble $\hat{T}$-group which is not a finite extension of a Prüfer group, and let $L$ be the last term of the lower central series of $G$. If $L$ has elements of order 2, then the odd component $L_{2'}$ of $L$ is a Hall subgroup of $G$, $G'/L$ is a 2-group, $2 \in \pi(G/G')$ and each element of $G$ acts on $L_2$ either as the identity or as the inversion.

Proof. Since $L_2 = L$ by Lemma 3.5, the 2-component $L_2$ of $L$ must be infinite and so it follows from Theorem 3.7 that $L_{2'}$ is a Hall subgroup.
of $G$. Moreover, all subgroups of $L$ are normal in $G$ by Corollary 2.10, and in particular each element of $G \setminus C_G(L_2)$ induces on $L_2$ the inversion map. On the other hand, $L_2$ is not contained in $Z(G)$, and so $2 \in \pi(G/G')$. Let $K/L_2'$ be the unique Sylow 2-subgroup of $G/L_2$. As $G$ is hypercyclic by Corollary 2.11, the elements of odd order of $G$ form a characteristic subgroup $M$ and $M/L_2'$ is nilpotent. Clearly, 

$$G/L_2' = K/L_2' \times M/L_2'$$

and all subgroups of $M/L_2'$ are normal in $G/L_2'$ because $L_2$ is infinite. It follows that $M/L_2'$ is abelian, so that $G'$ lies in $K$ and $G'/L$ is a 2-group. ■

4. Non-periodic $\hat{T}$-groups. A group $G$ is called an LT-group if each subnormal non-normal subgroup of $G$ has finite index. Clearly, groups with the property $LT$ can be considered as duals of IT-groups, and the structure of soluble LT-groups has been studied in [11]; in particular, it turns out that infinite soluble LT-groups are metabelian. It is also clear that all LT-groups have the property $\hat{T}$.

**Theorem 4.1.** Let $G$ be a soluble $\hat{T}$-group which is not polycyclic, and let $F$ be the Fitting subgroup of $G$. If either $F$ is torsion-free, or the largest periodic subgroup $T$ of $F$ is infinite but it is not a finite extension of a Prüfer group, then $G$ is an LT-group.

**Proof.** We may obviously suppose that $G$ is not a T-group, so that it contains a subnormal non-normal subgroup $X$. It follows from Lemmas 2.7, 2.12 and 2.13 that all subgroups of $F$ are normal in $G$, so that in particular $F$ is abelian and $C_G(F) = F$. Then $G/F$ is isomorphic to a non-trivial group of power automorphisms of $F$, so that $|G:F| = 2$ and $G = \langle F,z \rangle$ where $a^z = a^{-1}$ for all $a \in F$. It follows that $\gamma_{n+1}(G) = F^{2n}$ for each positive integer $n$, and so $G/\gamma_{n+1}(G)$ has finite exponent. Assume by contradiction that the nilpotent group $G/\gamma_4(G)$ is infinite. As $G/\gamma_4(G)$ satisfies the maximal condition on non-normal subgroups, it is a Dedekind group (see [5]). In particular, $L = \gamma_3(G)$ is the last term of the lower central series of $G$ and $L^2 = L$. As $X$ is not contained in $F$, there is an element $x$ of $X$ such that $a^x = a^{-1}$ for all $a \in F$; then $[L,x] = L^2 = L$, so that $L \leq X$ and $X$ is normal in $G$. This contradiction shows that $G/\gamma_4(G)$ is finite. In particular, the group $G/F^2$ is finite and hence $G$ is an LT-group (see [11, Theorem 3.3]). ■

**Corollary 4.2.** Let $G$ be a torsion-free soluble $\hat{T}$-group. If $G$ is not polycyclic, then it is abelian.

**Proof.** The group $G$ has the property LT by Theorem 4.1 and hence it is abelian (see [11, Corollary 3.4]). ■
Observe finally that there exist soluble non-polycyclic $\hat{T}$-groups with torsion-free Fitting subgroup for which the set of subnormal non-normal subgroups is infinite. In fact, let $p$ be an odd prime number and consider the semidirect product $G = \langle x \rangle \rtimes A$, where $A$ is isomorphic to the additive group of rational numbers whose denominators are powers of $p$ and $x$ is an element of order 2 such that $a^x = a^{-1}$ for all $a \in A$. Then $A$ is the Fitting subgroup of $G$ and $G$ is an LT-group (see [11, Theorem 3.3]), so that in particular $G$ has the property $\hat{T}$. On the other hand, $\langle x, A^{2^n} \rangle$ is a subnormal non-normal subgroup of $G$ for each integer $n \geq 2$, so that $G$ contains infinitely many subnormal non-normal subgroups.

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