ON pq-HYPERELLIPTIC RIEMANN SURFACES

by

EWA TYSZKOWSKA (Gdańsk)

Abstract. A compact Riemann surface $X$ of genus $g > 1$ is said to be $p$-hyperelliptic if $X$ admits a conformal involution $\varphi$, called a $p$-hyperelliptic involution, for which $X/\varphi$ is an orbifold of genus $p$. If in addition $X$ admits a $q$-hyperelliptic involution then we say that $X$ is $pq$-hyperelliptic. We give a necessary and sufficient condition on $p, q$ and $g$ for existence of a $pq$-hyperelliptic Riemann surface of genus $g$. Moreover we give some conditions under which $p$- and $q$-hyperelliptic involutions of a $pq$-hyperelliptic Riemann surface commute or are unique.

1. Introduction. A Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$ is said to be $p$-hyperelliptic if $X$ admits a conformal involution $\varphi$, called a $p$-hyperelliptic involution, such that $X/\varphi$ is an orbifold of genus $p$. In the particular cases $p = 0, 1$, $X$ is called a hyperelliptic and an elliptic-hyperelliptic Riemann surface respectively. The Hurwitz–Riemann formula asserts that a $p$-hyperelliptic involution has $2g + 2 - 4p$ fixed points. In [4] we proved that for $g$ in the range $3p + 2 < g \leq 4p + 1$, any two $p$-hyperelliptic involutions commute and $X$ can admit at most two such involutions. Thus their product is a central $q$-involution for some $q \neq p$. This leads to the study of surfaces $X$ admitting two involutions $\varphi$ and $\delta$ such that $X/\varphi$ and $X/\delta$ have genera $p$ and $q$ respectively. We shall call them $pq$-hyperelliptic and for simplicity we shall say that $\varphi$ and $\delta$ are their $p$- and $q$-involutions. We prove that the genus of a $pq$-hyperelliptic Riemann surface $X$ does not exceed $2p + 2q + 1$, which in particular gives the result of H. Farkas and I. Kra from [2] that a $p$-hyperelliptic involution is unique and central in the group of all automorphisms of $X$ if $g > 4p + 1$. On the other hand for $p \leq q$, the genus of such a surface cannot be smaller than $2q - 1$ since $2g + 2 - 4q$ is the number of fixed points of its $q$-involution. Consequently, for $q > 2p + 1$, the $p$-involution of $X$ is unique. Furthermore we argue that for any $g$ in the range $2q - 1 \leq g \leq 2p + 2q + 1$, there exists a $pq$-hyperelliptic Riemann surface of genus $g$ with commuting $p$- and $q$-involutions and for $g \geq 2p + 2q - 2$, their product is a $(g - p - q)$-involution. In particular we conclude that a Riemann surface which is simultaneously

2000 Mathematics Subject Classification: Primary 30F10; Secondary 20H10.

Key words and phrases: $p$-hyperelliptic surfaces, automorphisms of Riemann surfaces, fixed points of automorphisms.
hyperelliptic and elliptic-hyperelliptic has genus 2 or 3 and the product of its
0- and 1-involutions is a 1-involution or a 2-involution respectively. Finally,
we notice that $p$- and $q$- involutions of a Riemann surface commute if its genus
satisfies $3p + 3q + 2 < 2g \leq 4p + 4q + 2$. This allows us to prove that for
$2 \leq p < q < 2p$ and $g > 3q + 1$, the $p$- and $q$- involutions of a Riemann surface
$X$ of genus $g$ are central and unique in the full automorphism group of $X$.

In [1], Accola proved some similar results using detailed analysis of
branching coverings.

2. Preliminaries. We shall be using the Riemann uniformization the-
orem stating that each compact Riemann surface $X$ of genus $g \geq 2$ can be
represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of
some Fuchsian surface group $\Gamma$. Furthermore any group of automorphisms
of a surface $X = \mathcal{H}/\Gamma$ can be represented as $\Lambda/\Gamma$ for another Fuchsian
group $\Lambda$. The algebraic and geometric structure of a Fuchsian group $\Lambda$ is
determined by its signature

$$\sigma(\Lambda) = (g; m_1, \ldots, m_r),$$

where $g, m_i$ are integers satisfying $g \geq 0$, $m_i \geq 2$. The group with signature
(1) has a canonical presentation given by

$$\left\{ \begin{array}{l}
generators: \quad x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, \\
relations: \quad x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \cdots x_r[a_1, b_1] \cdots [a_g, b_g] = 1.
\end{array} \right.$$

Such a set of generators is called a canonical set of generators and often,
by abuse of language, the set of canonical generators. Geometrically $x_i$ are
elliptic elements which correspond to hyperbolic rotations and the remaining
generators are hyperbolic translations. The integers $m_1, \ldots, m_r$ are called the
periods of $\Lambda$ and $g$ is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with
signatures $(g; -)$ are called surface groups and they are characterized among
Fuchsian groups as those which are torsion free.

The group $\Lambda$ has a fundamental region whose area $\mu(\Lambda)$, called the area
of the group, is

$$\mu(\Lambda) = 2\pi \left( 2g - 2 + \sum_{i=1}^{r} \left( 1 - 1/m_i \right) \right).$$

An abstract group $\Lambda$ with presentation (2) is isomorphic to a Fuchsian
group with signature (1) if and only if the right hand side of (3) is greater
than 0; in that case (1) is called a Fuchsian signature.

If $\Gamma$ is a subgroup of finite index in $\Lambda$, then we have the Hurwitz–Riemann
formula

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.$$
We shall use the following result of Macbeath [3] concerning the number of fixed points of an automorphism of a Riemann surface.

**Theorem 2.1.** Let $G = \Gamma/\Lambda$ be an automorphism group of a Riemann surface $X = \mathcal{H}/\Gamma$ and let $x_1, \ldots, x_r$ be elliptic canonical generators of $\Lambda$ with periods $m_1, \ldots, m_r$ respectively. Let $\theta : \Lambda \rightarrow G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be 1 or 0 according as $g$ is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(g)$ of points of $X$ fixed by $g$ is given by the formula

$$F(g) = |N_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i.$$  

3. On $pq$-hyperelliptic Riemann surfaces. A Riemann surface $X$ of genus $g \geq 2$ is said to be $pq$-hyperelliptic if there exist two involutions $\varrho$ and $\delta$ of $X$ such that $X/\varrho$ and $X/\delta$ have genera $p$ and $q$ respectively. First we show that the genus of such a surface is bounded.

**Theorem 3.1.** For arbitrary integers $0 \leq p \leq q$ except $p = q = 0$, the genus $g$ of a $pq$-hyperelliptic Riemann surface satisfies $2q - 1 \leq g \leq 2p + 2q + 1$.

**Proof.** Suppose that a Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g$ admits a $p$-involution $\delta$ and a $q$-involution $\varrho$. Then $g \geq 2q - 1$ since $2g + 2 - 4q$ is the number of fixed points of $\varrho$. The involutions $\varrho$ and $\delta$ generate a dihedral group $G$, say of order $2n$, and there exist a Fuchsian group $\Lambda$ and an epimorphism $\theta : \Lambda \rightarrow G$ with kernel $\Gamma$. If $x_i$ is a canonical elliptic generator of $\Lambda$ corresponding to some period $m_i > 2$ then $\theta(x_i) \in \langle \varrho\delta \rangle$. But no conjugate of $\varrho$ nor of $\delta$ belongs to $\langle \varrho\delta \rangle$ and so in the notation of Macbeath’s theorem $\varepsilon_i(\varrho) = \varepsilon_i(\delta) = 0$. Now if $n$ is odd then $\varrho$ and $\delta$ are conjugate and so $p = q$. Moreover $|N_G(\langle \varrho \rangle)| = 2$ and $F(\varrho) = 2q + 2 - 4p$ imply that $\Lambda$ has $2g + 2 - 4p$ periods equal to 2. If $n$ is even then $|N_G(\langle \varrho \rangle)| = 4$ and so $g + 1 - 2p$ canonical elliptic generators are mapped by $\theta$ onto conjugates of $\varrho$. Similarly another $g + 1 - 2q$ canonical elliptic generators are mapped by $\theta$ onto conjugates of $\delta$. So in both cases $\sigma(\Lambda) = (\gamma; 2, \ldots, 2, m_{s+1}, \ldots, m_r)$ for $s = 2g + 2 - 2p - 2q$ and some integer $r \geq s$. Now applying the Hurwitz–Riemann formula for $(\Lambda, \Gamma)$, we obtain $2g - 2 = 2n(2\gamma - 2 + g + 1 - p - q + \sum_{i=s+1}^{r}(1 - 1/m_i))$, which implies

$$g - 1 \geq n(g - 1 - p - q).$$

Since $n \geq 2$, it follows that $g \leq 2p + 2q + 1$. Thus for $g > 2p + 2q + 1$, $X$ is not $pq$-hyperelliptic. 

The above theorem yields the following result of Farkas and Kra [2].
Corollary 3.2. A $p$-hyperelliptic involution of a Riemann surface $X$ of genus $g > 4p + 1$ is unique and central in the full automorphism group of $X$.

Proof. Let $G$ be the full automorphism group of a Riemann surface $X$ of genus $g > 4p + 1$ and let $\varrho \in G$ be a $p$-involution. By the previous theorem, $\varrho$ is unique in $G$. Moreover given $g \in G$, $g\varrho g^{-1}$ has the same number of fixed points as $\varrho$. So by the Hurwitz–Riemann formula it is also a $p$-involution, which implies that $g\varrho g^{-1} = \varrho$. ■

Furthermore using Theorem 3.1, it is easy to show that for appropriate parameters $g, p, q$, any $p$- and $q$-involutions of a $pq$-hyperelliptic Riemann surface of genus $g$ commute.

Corollary 3.3. Let $X$ be a $pq$-hyperelliptic Riemann surface of genus $g$. If $q > 2p + 1$ then a $p$-hyperelliptic involution is central and unique in the full automorphism group of $X$. Furthermore for $(p, q) \neq (0, 0)$, any $p$- and $q$-involutions of $X$ commute if the genus $g$ of $X$ satisfies $3p + 3q + 2 < 2g \leq 4p + 4q + 2$.

Proof. If $q > 2p + 1$ then by Theorem 3.1, $g \geq 2q - 1 > 4p + 1$ and so by Corollary 3.2, a $p$-involution of $X$ is central and unique in the full automorphism group of $X$.

Now, let $(p, q) \neq (0, 0).$ Then any $p$- and $q$-involutions of $X$ generate a dihedral group of order $2n$, for some $n$ satisfying (6). Since $n \geq 3$ implies $2g \leq 3p + 3q + 2$, it follows that $p$- and $q$-involutions commute if $2g > 3p + 3q + 2$. ■

Now we shall show that the necessary conditions from 3.1 on the genus of a $pq$-hyperelliptic Riemann surface are also sufficient for the existence of such a surface.

Theorem 3.4. Let $g \geq 2$ and $q \geq p \geq 0$ be integers such that $2q - 1 \leq g \leq 2p + 2q + 1$. Then there exists a Riemann surface of genus $g$ admitting commuting $p$- and $q$-involutions whose product is a $t$-involution if and only if $t$ is a nonnegative integer with $(g + 1)/2 - (p + 1) \leq t \leq (g + 1)/2$ such that $p + q + t - g$ is even and nonnegative.

Proof. Assume that a Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g$ admits a $p$-involution $\delta$ and a $q$-involution $\varrho$ whose product is a $t$-involution. Then $X$ is $pt$-hyperelliptic and so by Theorem 3.1, $2t - 1 \leq g \leq 2t + 2p + 1$. Thus $g/2 - p \leq t \leq g/2$ or $(g + 1)/2 - (p + 1) \leq t \leq (g + 1)/2$ according as $g$ is even or odd. Since $\varrho$ and $\delta$ generate a group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, there exists a Fuchsian group $\Delta$ with signature $(\gamma; 2, \ldots, 2)$ and an epimorphism $\theta : \Delta \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with kernel $\Gamma$. By Theorem 2.1, $r = 3g + 3 - 2p - 2q - 2t$ and so applying
the Hurwitz–Riemann formula for \((\Delta, \Gamma)\) we obtain \(\gamma = (p + q + t - g)/2\), which implies that \(p + q + t - g\) is even and nonnegative.

Conversely, let \(g\) be an integer with \(2q - 1 \leq g \leq 2p + 2q + 1\) and suppose that \(p + q + t - g\) is nonnegative and even for some nonnegative integer \(t\) with \((g + 1)/2 \leq t \leq (g + 1)/2\). Then for \(\gamma = (p + q + t - g)/2\) and \(r = 3g + 3 - 2p - 2q - 2t\), there exists a Fuchsian group \(\Delta\) with signature \((\gamma; 2, \ldots, 2)\). Define an epimorphism \(\theta : \Delta \to Z_2 \oplus Z_2 \cong \langle \phi \rangle \oplus \langle \delta \rangle\) by \(\theta(x_1) = \cdots = \theta(x_{s_1}) = \phi, \theta(x_{s_1+1}) = \cdots = \theta(x_{s_1+s_2}) = \delta, \theta(x_{s_1+s_2+1}) = \cdots = \theta(x_r) = g\delta\), where \(s_1 = g + 1 - 2q\) and \(s_2 = g + 1 - 2p\). By Theorem 2.1 and the Hurwitz–Riemann formula, \(g\) and \(\delta\) are commuting \(q\)- and \(p\)-involutions of a Riemann surface of genus \(g\) and their product is a \(t\)-involution.

**Corollary 3.5.** For any integers \(g \geq 2\) and \(q \geq p \geq 0\) such that \(2q - 1 \leq g \leq 2p + 2q + 1\), there exists a Riemann surface of genus \(g\) admitting commuting \(p\)- and \(q\)-involutions. Moreover if \(g \geq 2p + 2q - 2\) then the product of \(p\)- and \(q\)-involutions is a \((g - p - q)\)-involution.

**Proof.** We need to find an appropriate \(t\) satisfying the conditions of Lemma 3.4. If simultaneously \(g = 2p - 1\) and \(p = q\) then we can take \(t = 1\). In the remaining cases we can choose \(t = g - p - q\) and it is easy to check that for \(2p + 2q - 2 \leq g \leq 2p + 2q + 1\), such a \(t\) is unique and so the product of any \(p\)- and \(q\)-involutions of a Riemann surface of such genus is a \((g - p - q)\)-involution.

**Corollary 3.6.** There exists a Riemann surface which is simultaneously hyperelliptic and elliptic-hyperelliptic. It has genus 2 or 3 and the product of its 0- and 1-involutions is a 1- or a 2-involution respectively.

**Proof.** By Theorem 3.1, the genus of a 01-hyperelliptic Riemann surface is 2 or 3. Moreover by Corollaries 3.2 and 3.5, such a surface actually exists and the product of its 0- and 1-involutions is a 1- or a 2-involution according as \(g\) is 2 or 3.

The final theorem concerns the number of \(p\)- and \(q\)-involutions of a \(pq\)-hyperelliptic Riemann surface.

**Theorem 3.7.** If \(p < q < 2p\) and \(3q + 1 < g \leq 2p + 2q + 1\) then \(p\)- and \(q\)-involutions of a \(pq\)-hyperelliptic Riemann surface of genus \(g\) are central and unique in the full automorphism group. Moreover a Riemann surface of genus \(g\) with \(3q + 2 < g \leq 4q + 1\) can admit at most two \(q\)-involutions.

**Proof.** For \(p \leq q < 2p\), let \(X = \mathcal{H}/\Gamma\) be a \(pq\)-hyperelliptic Riemann surface of genus \(g\) with \(3q + 1 < g \leq 2p + 2q + 1\) and let \(T\) be the set of all \(p\)- and \(q\)-involutions of \(X\). By Corollary 3.3, any two elements of \(T\) commute. Moreover, by Theorem 3.4, the product of any two such elements can be neither a \(p\)- nor a \(q\)-involution. So if \(X\) admits a \(p\)-involution \(\varphi_p\) and
two $q$-involutions $\varrho_q$ and $\varrho_q'$ then they generate the group $G \cong Z_2 \oplus Z_2 \oplus Z_2$ which can be identified with $\Delta/\Gamma$ for some Fuchsian group $\Delta$, say with signature $(\gamma; 2, \ldots, 2)$. Let $\theta: \Delta \to G$ be the canonical epimorphism and for $1 \neq g \in G$, let $\varepsilon_i(g)$ be defined as in Theorem 2.1. Let $s_q = \sum_{i=1}^{r} \varepsilon_i(\varrho_q)$, $s'_q = \sum_{i=1}^{r} \varepsilon_i(\varrho_q')$ and $s_p = \sum_{i=1}^{r} \varepsilon_i(\varrho_p)$. By Theorem 2.1, $s_q = s'_q = (g+1-2q)/2$ and $s_p = (g+1-2p)/2$. Thus applying the Hurwitz–Riemann formula for $(\Delta, \Gamma)$, we obtain $2g-2 = 8(2\gamma - 2 + (3g + 3 - 4q - 2p)/4 + s/2)$, where $s = r - s_q - s'_q - s_p$. So $\gamma = (2 + 2q + p - g - s)/4 \geq 0$ if and only if $g \leq 2q + p + 2$. Repeating the argument we see that $X$ admits two $p$-involutions and a $q$-involution only if $g \leq 2p + q + 2$. Consequently, for $p < q$, the $p$- and $q$-involutions of a Riemann surface of genus $g > 3q + 1$ are unique and a Riemann surface of genus $g > 3q + 2$ can admit at most two $q$-involutions.

REFERENCES


Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: Ewa.Tyszkowska@math.univ.gda.pl

Received 23 October 2004;
revised 28 February 2005 (4517)