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ON pq-HYPERELLIPTIC RIEMANN SURFACES

BY

EWA TYSZKOWSKA (Gdańsk)

Abstract. A compact Riemann surface X of genus $g > 1$ is said to be *p-hyperelliptic* if X admits a conformal involution ϱ, called a *p-hyperelliptic involution*, for which X/ϱ is an orbifold of genus p. If in addition X admits a q-hypereliptic involution then we say that X is *pq-hyperelliptic*. We give a necessary and sufficient condition on p, q and g for existence of a pq-hyperelliptic Riemann surface of genus g. Moreover we give some conditions under which p- and q-hyperelliptic involutions of a pq-hyperelliptic Riemann surface commute or are unique.

1. Introduction. A Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ϱ, called a *p-hyperelliptic involution*, such that X/ϱ is an orbifold of genus p. In the particular cases $p = 0, 1$, X is called a *hyperelliptic* and an *elliptic-hyperelliptic* Riemann suface respectively. The Hurwitz–Riemann formula asserts that a p-hyperelliptic involution has $2g + 2 - 4p$ fixed points. In [4] we proved that for g in the range $3p + 2 < g \leq 4p + 1$, any two p-hyperelliptic involutions commute and X can admit at most two such involutions. Thus their product is a central q-involution for some $q \neq p$. This leads to the study of surfaces X admitting two involutions ϱ and δ such that X/ϱ and X/δ have genera p and q respectively. We shall call them *pq-hyperelliptic* and for simplicity we shall say that ϱ and δ are their p- and *q-involutions*. We prove that the genus of a pq-hyperelliptic Riemann surface X does not exceed $2p+2q+1$, which in particular gives the result of H. Farkas and I. Kra from [2] that a p-hyperelliptic involution is unique and central in the group of all automorphisms of X if $g > 4p + 1$. On the other hand for $p \leq q$, the genus of such a surface cannot be smaller than $2q - 1$ since $2g + 2 - 4q$ is the number of fixed points of its q-involution. Consequently, for $q > 2p+1$, the p-involution of X is unique. Furthermore we argue that for any g in the range $2q - 1 \leq g \leq 2p+2q+1$, there exists a pq-hyperelliptic Riemann surface of genus g with commuting p- and q-involutions and for $g \geq 2p+2q-2$, their product is a $(g-p-q)$-involution. In particular we conclude that a Riemann surface which is simultaneously

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hyperelliptic and elliptic-hyperelliptic has genus 2 or 3 and the product of its 0- and 1-involutions is a 1-involution or a 2-involution respectively. Finally, we notice that p- and q-involutions of a Riemann surface commute if its genus satisfies $3p + 3q + 2 < 2g \leq 4p + 4q + 2$. This allows us to prove that for $2 \leq p < q < 2p$ and $g > 3q+1$, the p- and q-involutions of a Riemann surface X of genus g are central and unique in the full automorphism group of X.

In [1], Accola proved some similar results using detailed analysis of branching coverings.

2. Preliminaries. We shall be using the Riemann uniformization theorem stating that each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore any group of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as Λ/Γ for another Fuchsian group Λ. The algebraic and geometric structure of a Fuchsian group Λ is determined by its signature

$$\sigma(\Lambda) = (g; m_1, \ldots, m_r), \tag{1}$$

where g, m_i are integers satisfying $g \geq 0$, $m_i \geq 2$. The group with signature (1) has a canonical presentation given by

$$\begin{cases} \text{generators:} & x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, \\ \text{relations:} & x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \cdots x_r [a_1, b_1] \cdots [a_g, b_g] = 1. \end{cases} \tag{2}$$

Such a set of generators is called a *canonical set of generators* and often, by abuse of language, the set of *canonical generators*. Geometrically x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1, \ldots, m_r$ are called the *periods* of Λ and g is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with signatures $(g; -)$ are called *surface groups* and they are characterized among Fuchsian groups as those which are torsion free.

The group Λ has a fundamental region whose area $\mu(\Lambda)$, called the *area of the group*, is

$$\mu(\Lambda) = 2\pi\Big(2g - 2 + \sum_{i=1}^{r} (1 - 1/m_i)\Big). \tag{3}$$

An abstract group Λ with presentation (2) is isomorphic to a Fuchsian group with signature (1) if and only if the right hand side of (3) is greater than 0; in that case (1) is called a *Fuchsian signature*.

If Γ is a subgroup of finite index in Λ, then we have the *Hurwitz–Riemann formula*

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{4}$$

We shall use the following result of Macbeath [3] concerning the number of fixed points of an automorphism of a Riemann surface.

THEOREM 2.1. *Let $G = \Lambda/\Gamma$ be an automorphism group of a Riemann surface $X = H/\Gamma$ and let $x_1, \ldots, x_r$ be elliptic canonical generators of Λ with periods $m_1, \ldots, m_r$ respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be 1 or 0 according as g is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(g)$ of points of X fixed by g is given by the formula*

$$F(g) = |N_G(\langle g\rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i. \tag{5}$$

3. On pq-hyperelliptic Riemann surfaces. A Riemann surface X of genus $g \geq 2$ is said to be *pq-hyperelliptic* if there exist two involutions ϱ and δ of X such that X/ϱ and X/δ have genera p and q respectively. First we show that the genus of such a surface is bounded.

THEOREM 3.1. *For arbitrary integers $0 \leq p \leq q$ except $p = q = 0$, the genus g of a pq-hyperelliptic Riemann surface satisfies $2q - 1 \leq g \leq 2p + 2q + 1$.*

Proof. Suppose that a Riemann surface $X = \mathcal{H}/\Gamma$ of genus g admits a p-involution δ and a q-involution ϱ. Then $g \geq 2q - 1$ since $2g + 2 - 4q$ is the number of fixed points of ϱ. The involutions ϱ and δ generate a dihedral group G, say of order $2n$, and there exist a Fuchsian group Λ and an epimorphism $\theta : \Lambda \to G$ with kernel Γ. If x_i is a canonical elliptic generator of Λ corresponding to some period $m_i > 2$ then $\theta(x_i) \in \langle\varrho\delta\rangle$. But no conjugate of ϱ nor of δ belongs to $\langle\varrho\delta\rangle$ and so in the notation of Macbeath's theorem $\varepsilon_i(\varrho) = \varepsilon_i(\delta) = 0$. Now if n is odd then ϱ and δ are conjugate and so $p = q$. Moreover $|N_G(\langle\varrho\rangle)| = 2$ and $F(\varrho) = 2g+2-4p$ imply that Λ has $2g+2-4p$ periods equal to 2. If n is even then $|N_G(\langle\varrho\rangle)| = 4$ and so $g+1-2p$ canonical elliptic generators are mapped by θ onto conjugates of ϱ. Similarly another $g+1-2q$ canonical elliptic generators are mapped by θ onto conjugates of δ. So in both cases $\sigma(\Lambda) = (\gamma; 2, \overset{s}{\ldots}, 2, m_{s+1}, \ldots, m_r)$ for $s = 2g + 2 - 2p - 2q$ and some integer $r \geq s$. Now applying the Hurwitz–Riemann formula for (Λ, Γ), we obtain $2g - 2 = 2n(2\gamma - 2 + g + 1 - p - q + \sum_{i=s+1}^{r}(1 - 1/m_i))$, which implies

$$g - 1 \geq n(g - 1 - p - q). \tag{6}$$

Since $n \geq 2$, it follows that $g \leq 2p + 2q + 1$. Thus for $g > 2p + 2q + 1$, X is not pq-hyperelliptic. ∎

The above theorem yields the following result of Farkas and Kra [2].

COROLLARY 3.2. *A p-hyperelliptic involution of a Riemann surface X of genus $g > 4p+1$ is unique and central in the full automorphism group of X.*

Proof. Let G be the full automorphism group of a Riemann surface X of genus $g > 4p+1$ and let $\varrho \in G$ be a p-involution. By the previous theorem, ϱ is unique in G. Moreover given $g \in G$, $g\varrho g^{-1}$ has the same number of fixed points as ϱ. So by the Hurwitz–Riemann formula it is also a p-involution, which implies that $g\varrho g^{-1} = \varrho$. ■

Furthermore using Theorem 3.1, it is easy to show that for appropriate parameters g, p, q, any p- and q-involutions of a pq-hyperelliptic Riemann surface of genus g commute.

COROLLARY 3.3. *Let X be a pq-hyperelliptic Riemann surface of genus g. If $q > 2p+1$ then a p-hyperelliptic involution is central and unique in the full automorphism group of X. Furthermore for $(p,q) \neq (0,0)$, any p- and q-involutions of X commute if the genus g of X satisfies $3p+3q+2 < 2g \leq 4p+4q+2$.*

Proof. If $q > 2p+1$ then by Theorem 3.1, $g \geq 2q-1 > 4p+1$ and so by Corollary 3.2, a p-involution of X is central and unique in the full automorphism group of X.

Now, let $(p,q) \neq (0,0)$. Then any p- and q-involutions of X generate a dihedral group of order $2n$, for some n satisfying (6). Since $n \geq 3$ implies $2g \leq 3p+3q+2$, it follows that p- and q-involutions commute if $2g > 3p+3q+2$. ■

Now we shall show that the necessary conditions from 3.1 on the genus of a pq-hyperelliptic Riemann surface are also sufficient for the existence of such a surface.

THEOREM 3.4. *Let $g \geq 2$ and $q \geq p \geq 0$ be integers such that $2q-1 \leq g \leq 2p+2q+1$. Then there exists a Riemann surface of genus g admitting commuting p- and q-involutions whose product is a t-involution if and only if t is a nonnegative integer with $(g+1)/2-(p+1) \leq t \leq (g+1)/2$ such that $p+q+t-g$ is even and nonnegative.*

Proof. Assume that a Riemann surface $X = \mathcal{H}/\Gamma$ of genus g admits a p-involution δ and a q-involution ϱ whose product is a t-involution. Then X is pt-hyperelliptic and so by Theorem 3.1, $2t-1 \leq g \leq 2t+2p+1$. Thus $g/2-p \leq t \leq g/2$ or $(g+1)/2-(p+1) \leq t \leq (g+1)/2$ according as g is even or odd. Since ϱ and δ generate a group $Z_2 \oplus Z_2$, there exists a Fuchsian group Δ with signature $(\gamma; 2, \overset{r}{\ldots}, 2)$ and an epimorphism $\theta : \Delta \to Z_2 \oplus Z_2$ with kernel Γ. By Theorem 2.1, $r = 3g+3-2p-2q-2t$ and so applying the Hurwitz–Riemann formula for (Δ, Γ) we obtain $\gamma = (p+q+t-g)/2$, which implies that $p+q+t-g$ is even and nonnegative.

Conversely, let g be an integer with $2q-1 \le g \le 2p+2q+1$ and suppose that $p+q+t-g$ is nonnegative and even for some nonnegative integer t with $(g+1)/2-(p+1) \le t \le (g+1)/2$. Then for $\gamma = (p+q+t-g)/2$ and $r = 3g+3-2p-2q-2t$, there exists a Fuchsian group Δ with signature $(\gamma; 2, \overset{r}{\ldots}, 2)$. Define an epimorphism $\theta : \Delta \to Z_2 \oplus Z_2 \cong \langle \varrho \rangle \oplus \langle \delta \rangle$ by $\theta(x_1) = \cdots = \theta(x_{s_1}) = \varrho$, $\theta(x_{s_1+1}) = \cdots = \theta(x_{s_1+s_2}) = \delta$, $\theta(x_{s_1+s_2+1}) = \cdots = \theta(x_r) = \varrho\delta$, where $s_1 = g+1-2q$ and $s_2 = g+1-2p$. By Theorem 2.1 and the Hurwitz–Riemann formula, ϱ and δ are commuting q- and p-involutions of a Riemann surface of genus g and their product is a t-involution. ∎

Corollary 3.5. *For any integers $g \ge 2$ and $q \ge p \ge 0$ such that $2q-1 \le g \le 2p+2q+1$, there exists a Riemann surface of genus g admitting commuting p- and q-involutions. Moreover if $g \ge 2p+2q-2$ then the product of p- and q-involutions is a $(g-p-q)$-involution.*

Proof. We need to find an appropriate t satisfying the conditions of Lemma 3.4. If simultaneously $g = 2p-1$ and $p = q$ then we can take $t = 1$. In the remaining cases we can choose $t = g-p-q$ and it is easy to check that for $2p+2q-2 \le g \le 2p+2q+1$, such a t is unique and so the product of any p- and q-involutions of a Riemann surface of such genus is a $(g-p-q)$-involution. ∎

Corollary 3.6. *There exists a Riemann surface which is simultaneously hyperelliptic and elliptic-hyperelliptic. It has genus* 2 *or* 3 *and the product of its* 0- *and* 1-*involutions is a* 1- *or a* 2-*involution respectively.*

Proof. By Theorem 3.1, the genus of a 01-hyperelliptic Riemann surface is 2 or 3. Moreover by Corollaries 3.2 and 3.5, such a surface actually exists and the product of its 0- and 1-involutions is a 1- or a 2-involution according as g is 2 or 3. ∎

The final theorem concerns the number of p- and q-involutions of a pq-hyperelliptic Riemann surface.

Theorem 3.7. *If $p < q < 2p$ and $3q+1 < g \le 2p+2q+1$ then p- and q-involutions of a pq-hyperelliptic Riemann surface of genus g are central and unique in the full automorphism group. Moreover a Riemann surface of genus g with $3q+2 < g \le 4q+1$ can admit at most two q-involutions.*

Proof. For $p \le q < 2p$, let $X = \mathcal{H}/\Gamma$ be a pq-hyperelliptic Riemann surface of genus g with $3q+1 < g \le 2p+2q+1$ and let T be the set of all p- and q-involutions of X. By Corollary 3.3, any two elements of T commute. Moreover, by Theorem 3.4, the product of any two such elements can be neither a p- nor a q-involution. So if X admits a p-involution ϱ_p and two q-involutions ϱ_q and ϱ'_q then they generate the group $G \cong Z_2 \oplus Z_2 \oplus Z_2$ which can be identified with Δ/Γ for some Fuchsian group Δ, say with signature $(\gamma; 2, \overset{r}{\ldots}, 2)$. Let $\theta : \Delta \to G$ be the canonical epimorphism and for $1 \neq g \in G$, let $\varepsilon_i(g)$ be defined as in Theorem 2.1. Let $s_q = \sum_{i=1}^r \varepsilon_i(\varrho_q)$, $s'_q = \sum_{i=1}^r \varepsilon_i(\varrho'_q)$ and $s_p = \sum_{i=1}^r \varepsilon_i(\varrho_p)$. By Theorem 2.1, $s_q = s'_q = (g+1-2q)/2$ and $s_p = (g+1-2p)/2$. Thus applying the Hurwitz–Riemann formula for (Δ, Γ), we obtain $2g-2 = 8(2\gamma - 2 + (3g+3-4q-2p)/4 + s/2)$, where $s = r - s_q - s'_q - s_p$. So $\gamma = (2+2q+p-g-s)/4 \geq 0$ if and only if $g \leq 2q+p+2$. Repeating the argument we see that X admits two p-involutions and a q-involution only if $g \leq 2p+q+2$. Consequently, for $p < q$, the p- and q-involutions of a Riemann surface of genus $g > 3q+1$ are unique and a Riemann surface of genus $g > 3q+2$ can admit at most two q-involutions. ∎

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Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: Ewa.Tyszkowska@math.univ.gda.pl

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