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THE σ-COMPLETE MV-ALGEBRAS WHICH HAVE ENOUGH STATES

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Abstract. We characterize *Łukasiewicz tribes*, i.e., collections of fuzzy sets that are closed under the standard fuzzy complementation and the Łukasiewicz t-norm with countably many arguments. As a tool, we introduce *σ-McNaughton functions* as the closure of McNaughton functions under countable MV-algebraic operations. We give a measure-theoretical characterization of σ-complete MV-algebras which are isomorphic to Łukasiewicz tribes.

1. Introduction. The MV-algebra approach presents one of the most fruitful theoretical backgrounds of many-valued logics and a basis of successful applications in decision making, approximations, fuzzy control, etc. (see [3, 6]). In this paper we characterize the σ-complete MV-algebras represented by functions with values in $[0,1]$ with pointwise operations (*Łukasiewicz tribes*). Our result differs from a previous characterization which says that every σ-complete MV-algebra can be described as a collection of continuous $[0,1]$-valued functions on the space of *all* maximal ideals endowed with the spectral topology (see [3]). To specify the distinction, let us try to draw the analogy with Boolean algebras. Every Boolean algebra can be uniquely represented by two-valued (characteristic) functions on its Stone space (see [14]). Nevertheless, another set representation may be more useful on occasions. (For an analogy, the Borel σ-algebra on the real line is usually not studied via its Stone space.) Here we give a characterization of σ-complete MV-algebras by making use of *any* set representation. However, an additional condition is assumed that the operations coincide with the *pointwise* application of the operations of the standard MV-algebra $[0,1]$. This special case seems to be of considerable importance, in particular as a basis of many-valued probability theory [13]. Besides, every σ-complete

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MV-algebra can be obtained as a σ-homomorphic image of a Łukasiewicz tribe [5, 10].

It can happen that nonisomorphic σ-complete MV-algebras may be expressed as subdirect products of the same family of σ-complete MV-algebras even if they have the same Boolean skeletons. In contrast to this, we present a structure that represents a *unique* Łukasiewicz tribe. More exactly, we prove that there is an underlying σ-algebra and a sequence of σ-filters determining all elements of the tribe. We also give necessary and sufficient conditions for these structures to correspond to a σ-complete MV-algebra.

We prove that Łukasiewicz tribes are exactly those σ-complete MV-algebras which admit separating sets of pure states.

As an important tool, we study *σ-McNaughton functions*, i.e., the elements of the least Łukasiewicz tribe containing all McNaughton functions. Their role in the study of σ-complete MV-algebras is analogous to that of McNaughton functions in the theory of MV-algebras; every Łukasiewicz tribe is closed under pointwise application of all σ-McNaughton functions.

2. Basic notions. We refer to [3] for basic notions on MV-algebras. Unless stated otherwise, M is a σ-complete MV-algebra. By $\mathbb{N}$, resp. $\mathbb{Q}$, we denote the set of natural, resp. rational, numbers.

The *standard MV-algebra* is the real unit interval $S_\infty = [0,1]$ equipped with the Łukasiewicz operations $x \oplus y = \min(1, x+y)$ and $\neg x = 1-x$. The only σ-complete proper MV-subalgebras of S_∞ are of the form $S_n = \{i/n : i = 0,\dots,n\}$, $n \in \mathbb{N}$. All infinite MV-subalgebras of S_∞ are dense subsets of $[0,1]$.

DEFINITION 2.1 ([1]). Let X be a nonempty set. A collection $T \subseteq [0,1]^X$ is called a *Łukasiewicz clan* if it contains the constant zero function and is closed under the pointwise application of Łukasiewicz operations $\oplus, \neg$. If, moreover, T is closed under pointwise application of $\oplus$ to countably many arguments, then T is called a *Łukasiewicz tribe.*

As we shall work only with Łukasiewicz operations here, we shall speak briefly of a *clan* and a *tribe*. Every clan, resp. tribe, is an MV-algebra, resp. a σ-complete MV-algebra.

The *Boolean skeleton* of M is the Boolean algebra $\mathbf{B}(M) = \{a \in M : a \oplus a = a\}$ (of all *Boolean elements* of M). All operations of $\mathbf{B}(M)$ agree with the restrictions of the corresponding operations of M (see [3]). Boolean elements of a clan M are functions which attain only values $0, 1$, thus they coincide with those characteristic functions χ_A ($A \subseteq X$) which belong to M.

An n-ary function is a *McNaughton function* iff it belongs to the least clan of functions $[0,1]^n \to [0,1]$ containing all projections $\pi_{i,n}\colon (x_1,\dots,x_n) \mapsto x_i$, $i = 1,\dots,n$. Following [9], a function is a McNaughton function iff it is continuous, piecewise linear, and each piece is determined by a linear equation with integer coefficients.

COROLLARY 2.2. *Every Łukasiewicz clan*, T, *is closed under composition with all McNaughton functions*, *i.e.*, *if* $a_1, \ldots, a_n \in T$ *and* $f\colon [0,1]^n \to [0,1]$ *is a McNaughton function*, *then the function* $f(a_1, \ldots, a_n)\colon x \mapsto f(a_1(x), \ldots \ldots, a_n(x))$ *is in* T.

3. σ-completions of McNaughton functions. In this paper we deal with the class of functions which is obtained when we close the class of McNaughton functions under countable pointwise suprema and Łukasiewicz operations. Let us call an n-ary function a *σ-McNaughton function* iff it belongs to the least tribe of functions $[0,1]^n \to [0,1]$ containing all projections $\pi_{i,n}\colon (x_1, \ldots, x_n) \mapsto x_i$, $i = 1, \ldots, n$. (A similar notion for a different type of tribes was introduced in [11].) The tribe of unary σ-McNaughton functions was characterized already in [8] without reference to MV-algebras. Here we give an equivalent MV-algebraic characterization and its proof simplified by the use of the McNaughton theorem. Further, we extend it to n-ary functions.

The only σ-McNaughton constant functions are 0 and 1. We shall need functions which are "as close to constants as possible". For each $x \in [0,1]$, let $S_{(x)}$ denote the least σ-complete MV-subalgebra of $S_\infty = [0,1]$ such that $x \in S_{(x)}$. For each $r \in [0,1]$, we define a function $c_r\colon [0,1] \to [0,r]$ by

$$c_r(x) = \sup([0,r] \cap S_{(x)}). \tag{1}$$

PROPOSITION 3.1. *Let* $r \in [0,1]$. *Then the following cases may occur*:

- *If* $x \in [0,1] \setminus \mathbb{Q}$, *then* $c_r(x) = r$.
- *If* $x \in [0,1] \cap \mathbb{Q}$, *then* $c_r(x) \le r$, *and equality holds iff* $r \in S_{(x)}$.

For each $\varepsilon > 0$, *there are only finitely many points* $x \in [0,1]$ *such that* $c_r(x) \notin [r - \varepsilon, r]$.

THEOREM 3.2. *A function* $f\colon [0,1] \to [0,1]$ *is a σ-McNaughton function iff it satisfies the following conditions*:

(σMN0) f *is Borel measurable*,
(σMN1) $f(q) \in S_{(q)}$ *for all* $q \in [0,1] \cap \mathbb{Q}$.

Proof. The collection of all functions satisfying (σMN0), (σMN1) is a tribe, so it contains all σ-McNaughton functions. We have to prove that each function f satisfying (σMN0), (σMN1) is a σ-McNaughton function. We first prove it for special forms of f. For each $k \in \mathbb{N}$, $j \in \{0, \ldots, k-1\}$, we define the McNaughton function

$$s_{j,k}(x) = \begin{cases} 0 & \text{if } x \in [0, j/k], \\ -j + kx & \text{if } x \in (j/k, (j+1)/k), \\ 1 & \text{if } x \in [(j+1)/k, 1]. \end{cases}$$

For each $r = i/n \in [0,1) \cap \mathbb{Q}$, the characteristic function $\chi_{(r,1]}$ is a σ-McNaughton function, because

$$\chi_{(r,1]} = \bigvee_{p \in \mathbb{N}} s_{ip,np}.$$

By a standard Boolean construction, we deduce that also all characteristic functions of Borel subsets of $[0,1]$ are σ-McNaughton functions.

To prove that a function c_r for $r \in [0,1]$ is a σ-McNaughton function, we take the σ-McNaughton function $t_r = \mathrm{id} \wedge \chi_{[0,r]}$. We shall prove that $c_r = \bigvee_g (t_r \circ g)$, where the (countable) supremum is taken over all McNaughton functions g. Let $x \in [0,1]$. We shall use Proposition 3.1. If $x \in \mathbb{Q}$, it generates a finite MV-algebra $S_{(x)}$ and there is a McNaughton function whose value at x is $c_r(x)$. If $x \notin \mathbb{Q}$, then there is a sequence of McNaughton functions whose values at x converge to $r = c_r(x)$ and it is enough to compose them with t_r.

By a standard argument, all Borel measurable functions are obtained as pointwise suprema of simple functions (i.e., finitely-valued measurable functions). All simple functions $[0,1] \setminus \mathbb{Q} \to [0,1]$ are restrictions of σ-McNaughton functions. Taking their suprema, we obtain all Borel measurable functions on $[0,1] \setminus \mathbb{Q}$ as restrictions of σ-McNaughton functions. Thus, for each function f satisfying (σMN0), we may find a σ-McNaughton function f^* which coincides with f on $[0,1] \setminus \mathbb{Q}$. Using Proposition 3.1 and (σMN1), we see that the σ-McNaughton function

$$\bigvee_{q \in [0,1] \cap \mathbb{Q}} c_{f(q)} \wedge \chi_{\{q\}}$$

coincides with f on $[0,1] \cap \mathbb{Q}$ and vanishes on $[0,1] \setminus \mathbb{Q}$. Thus we obtain f as a σ-McNaughton function

$$f = (f^* \wedge \chi_{[0,1]\setminus\mathbb{Q}}) \vee \bigvee_{q \in [0,1] \cap \mathbb{Q}} c_{f(q)} \wedge \chi_{\{q\}}. \blacksquare$$

THEOREM 3.3. *An n-ary function $f\colon [0,1]^n \to [0,1]$ is a σ-McNaughton function iff it satisfies the following conditions*:

(σMN0) *f is Borel measurable.*

(σMNn) *Let $x_1, \ldots, x_n \in [0,1] \cap \mathbb{Q}$. Then $f(x_1, \ldots, x_n) \in S_k$, where $k \in \mathbb{N}$ is the least index such that $\{x_1, \ldots, x_n\} \subseteq S_k$.*

Proof. The proof follows the pattern of Theorem 3.2. The characteristic function of any n-dimensional subinterval of $[0,1]^n$ is σ-McNaughton, and this extends to all Borel subsets of $[0,1]$. We obtain all measurable functions $[0,1]^n \setminus \mathbb{Q}^n \to [0,1]$ as suprema of simple functions, and the values at the (countably many) remaining points are restricted only by (σMNn). ∎

In analogy to Corollary 2.2, σ-McNaughton functions play a similar role with respect to tribes as McNaughton functions do with respect to clans:

COROLLARY 3.4. *Every Łukasiewicz tribe T is closed under composition with all σ-McNaughton functions, i.e., if $a_1, \ldots, a_n \in T$ and $f\colon [0,1]^n \to [0,1]$ is a σ-McNaughton function, then $f(a_1, \ldots, a_n)\colon x \mapsto f(a_1(x), \ldots, a_n(x))$ is in T.*

We obtained an analytical characterization of σ-McNaughton functions. From the logical point of view, operations with countably many arguments should be avoided; if this is not possible, their use should be at least reduced to the very last step. Thus it is desirable to express any σ-McNaughton function as a supremum (or infimum) of McNaughton functions. However, $\chi_{\{r\}}$, $r \in [0,1]$, cannot be expressed as suprema of McNaughton functions. There are also σ-McNaughton functions which are neither suprema nor infima of McNaughton functions. Thus the σ-complete lattice generated by all n-ary McNaughton functions is a proper sublattice of the σ-complete lattice of all n-ary σ-McNaughton functions. (The case of MacNeille completions instead of σ-completions is clarified in [7, Th. 6.3, p. 91].)

4. Characterization of Łukasiewicz tribes and σ-complete MV-algebras. We shall refer to the lattice $(\mathbb{N}, |)$, where $|$ is the divisibility relation on $\mathbb{N}$. For a σ-algebra $\mathcal{B}$, we denote by $\mathcal{F}(\mathcal{B})$ the set of all its σ-filters. For a Łukasiewicz tribe T on X we define $C(T) = \{A \subseteq X : \chi_A \in T\}$ (which is a σ-algebra isomorphic to $\mathbf{B}(T)$).

THEOREM 4.1. *Let T be a Łukasiewicz tribe on X. Then there is a σ-algebra $\mathcal{B} \subseteq 2^X$ and an order-preserving mapping $\nabla\colon (\mathbb{N}, |) \to (\mathcal{F}(\mathcal{B}), \supseteq)$ such that*

$$T = \{a \in [0,1]^X : a \text{ is } \mathcal{B}\text{-measurable and } (\forall n \in \mathbb{N})\ a^{-1}[S_n] \in \nabla(n)\}. \tag{2}$$

Proof. We shall prove that T is of the above form for $\mathcal{B} = C(T)$ and $\nabla(n) = \{f^{-1}[S_n] : f \in T\}$, $n \in \mathbb{N}$.

For all $n \in \mathbb{N}$, $\nabla(n)$ is a σ-filter in $\mathcal{B}$. Indeed, the preimages $f^{-1}[S_n]$ under $f \in T$ belong to $\mathcal{B}$. If $A \in \nabla(n)$, $B \in \mathcal{B}$, and $A \subseteq B$, then there is an $f \in T$ such that $f^{-1}[S_n] = A$. We take $g = f \vee \chi_B \in T$ and obtain $g^{-1}[S_n] = A \cup B = B \in \nabla(n)$. For a sequence $(A_i)_{i\in\mathbb{N}} \in \nabla(n)^{\mathbb{N}}$, there are $f_i \in T$ such that $A_i = f_i^{-1}[S_n]$. We take an $r \in (1 - 1/2n, 1)$ and the σ-McNaughton function c_r from (1). The σ-McNaughton function $s_n = c_r \vee \chi_{S_n}$

equals 1 on S_n; at all other points it takes values from $[0,r] \setminus S_n$. We define the function $g = \bigwedge_{i\in\mathbb{N}}(s_n \circ f_i) \in T$. Then $\bigcap_{i\in\mathbb{N}} A_i = g^{-1}[S_n] \in \nabla(n)$.

Let $a \in T$. Each characteristic function χ_B of a Borel set $B \subseteq [0,1]$ is σ-McNaughton and hence $\chi_{a^{-1}[B]} = \chi_B \circ a \in T$. Thus a is $\mathcal{B}$-measurable and we are done.

On the other hand, let $a \in [0,1]^X$ be a $\mathcal{B}$-measurable function such that $a^{-1}[S_n] \in \nabla(n)$ for all $n \in \mathbb{N}$. Then $\chi_{a^{-1}[[r,1]]} \in T$ for all $r \in [0,1]$ and for each $n \in \mathbb{N}$ there exists a function $f_n \in T$ such that $a^{-1}[S_n] = f_n^{-1}[S_n]$. We shall reconstruct the function a from $\chi_{a^{-1}[[r,1]]}$, $r \in [0,1]$, and f_n, $n \in \mathbb{N}$, using σ-McNaughton functions.

Let C be the least clan containing all f_n, $n \in \mathbb{N}$. It is a countable subset of T. For each $q \in [0,1] \cap \mathbb{Q}$, we define a function g_q: $X \to [0,1]$ by

$$g_q = \chi_{a^{-1}[[q,1]]} \wedge \bigvee_{b\in C} (c_q \circ b) \in T. \tag{3}$$

Let

$$d = \bigvee_{q\in[0,1]\cap\mathbb{Q}} g_q \in T. \tag{4}$$

We shall prove that $a = d$. For each $b \in C$, the function $c_q \circ b$ attains only values from $[0,q]$; so does the supremum $\bigvee_{b\in C}(c_q \circ b)$. Therefore $g_q \le a$ and also $d \le a$. Suppose that there is an $x \in X$ such that $d(x) < a(x)$. Assume first that $a(x) \notin \mathbb{Q}$. Then, for each $n \in \mathbb{N}$, we have $x \notin a^{-1}[S_n]$, therefore $x \notin f_n^{-1}[S_n]$. We have values $f_n(x)$, $n \in \mathbb{N}$, satisfying $f_n(x) \notin S_n$. There is no $m \in \mathbb{N}$ such that S_m contains all these values $f_n(x)$. Therefore they generate an infinite MV-subalgebra which is dense in $[0,1]$. The restriction of the clan C to $\{x\}$ gives a dense set of values in $[0,1]$. For any rational number $q \in (d(x), a(x))$, the set $\{b(x) : b \in C\} \cap (d(x), q]$ is infinite. We apply c_q to this set. According to Proposition 3.1, there are only finitely many points at which c_q attains values outside the interval $(d(x), q]$, so there is a function $b \in C$ such that $(c_q \circ b)(x) \in (d(x), q]$. As $q < a(x)$, with the use of (3) we obtain

$$d(x) < (c_q \circ b)(x) = \underbrace{\chi_{a^{-1}[[q,1]]}(x)}_{1} \wedge (c_q \circ b)(x) \le g_q(x).$$

This contradicts (4), hence $a(x)$ cannot be irrational.

Suppose finally that $a(x) = i/m$, where $i, m \in \mathbb{N}$, $i \le m$, and i, m are relatively prime. Then $a(x) \in S_n$ iff $m \,|\, n$. Thus $f_n(x) \in S_n$ iff $m \,|\, n$. If the values $f_n(x)$, $n \in \mathbb{N}$, are not contained in S_k for any $k \in \mathbb{N}$, then they generate an MV-subalgebra dense in $[0,1]$ and we proceed as in the previous case. In the remaining case, there is the least $k \in \mathbb{N}$ such that $\{f_n(x) : n \in \mathbb{N}\} \subseteq S_k$. If k is not a multiple of m, then $f_k(x) \notin S_k$ which is impossible. Thus $m \,|\, k$. Hence $i/m \in S_k$. As S_k is the MV-subalgebra of

$[0,1]$ generated by $\{f_n(x) : n \in \mathbb{N}\}$, there is an element $b \in C$ such that $b(x) = i/m$. As $c_{i/m}(i/m) = i/m$, we obtain $g_{i/m}(x) = i/m = a(x) > d(x)$, a contradiction with (4). We proved that $d(x) = a(x)$ for all $x \in X$, therefore $a \in T$. ∎

We may ask which σ-algebras and sequences of σ-filters give rise to Łukasiewicz tribes. It appears that ∇ preserves not only the ordering, but also all meets.

THEOREM 4.2. *Let $\mathcal{B}$ be a σ-algebra of subsets of a set X and $(\nabla(n))_{n\in\mathbb{N}}$ be a sequence of σ-ideals in $\mathcal{B}$. Then the following are equivalent*:

(i) $\{a \in [0,1]^X : a$ *is $\mathcal{B}$-measurable and* $(\forall n \in \mathbb{N})\ a^{-1}[S_n] \in \nabla(n)\}$ *is a Łukasiewicz tribe*,

(ii) $\nabla\colon (\mathbb{N}, |) \to (\mathcal{F}(\mathcal{B}), \supseteq)$ *is a meet semilattice isomorphism.*

Proof. As (ii)⇒(i) follows from Theorem 4.1, we shall prove that (i)⇒(ii). For brevity, let $P = \big\{a \in [0,1]^X : a$ is $\mathcal{B}$-measurable and $(\forall n \in \mathbb{N})\ a^{-1}[S_n] \in \nabla(n)\big\}$. From the definition of P and the ordering of $(S_n)_{n\in\mathbb{N}}$ (by the set-theoretical inclusion), we see that ∇ preserves the ordering. It remains to prove that it also preserves all meets.

Let $p, q \in \mathbb{N}$, $n = p \wedge q$ (in the divisibility lattice $(\mathbb{N}, |)$, i.e., n is the greatest common divisor of p, q and $S_n = S_p \cap S_q$). We shall prove that $\nabla(n)$ coincides with the meet of $\nabla(p)$ and $\nabla(q)$ in $(\mathcal{F}(\mathcal{B}), \supseteq)$, i.e., with the σ-filter generated by $\nabla(p) \cup \nabla(q)$. One inclusion is obvious, because $\nabla(n) \supseteq \nabla(p) \cup \nabla(q)$. For the reverse inclusion, take $D \in \nabla(n)$. It is enough to find $g, h \in P$ such that $D \supseteq g^{-1}[S_p] \cap h^{-1}[S_q]$. Due to the definition of $\nabla(n)$, there is an $f \in P$ satisfying $D = f^{-1}[S_n]$. We define σ-McNaughton functions $d_p = \mathrm{id} \vee \chi_{S_p}$ and $d_q = \mathrm{id} \vee \chi_{S_q}$. The function $s = (d_p \circ f) \wedge (d_q \circ f)$ belongs to P and satisfies $s^{-1}[S_n] = f^{-1}[S_p] \cap f^{-1}[S_q]$. It suffices to take $g = d_p \circ f$ and $h = d_q \circ f$. ∎

The meet semilattice homomorphism ∇ need not be injective and need not preserve joins:

EXAMPLE 4.3. Let p, q be two different prime numbers, $n = p \cdot q = p \vee q$. Let $X = \{x, y\}$, $T = S_n^X$. Then $f^{-1}[S_p]$ may be empty for some $f \in T$ (e.g., for the constant function $1/q$ on X), so $\nabla(p) = C(T)$ and analogously $\nabla(q) = C(T)$. Nevertheless, $\nabla(n) = \{X\} \neq \nabla(p) \vee \nabla(q) = C(T)$.

REMARK 4.4. A related result has been proved in [2] (see also [12]): Let $n \in \mathbb{N}$ and let M be an MV-algebra from the variety generated by S_n. We denote by $(\mathbb{N}_n, |)$ the lattice of all divisors of n. Let X be the Stone space of $\mathbf{B}(M)$ and $\mathrm{Cl}(X)$ the lattice of all closed subsets of X. Then there is a meet-semilattice homomorphism $h\colon (\mathbb{N}_n, |) \to (\mathrm{Cl}(X), \supseteq)$ such that

$$M \cong \{f \in S_n^X : f \text{ is continuous and } f(h(j)) \subseteq S_j \text{ for all } j\}.$$

This theorem embraces also MV-algebras which are not isomorphic to tribes, but it does not cover all Łukasiewicz tribes.

5. Relation to states. Here we study a many-valued extension of probability according to [13]. A *state* (probability measure) on M is a mapping $s\colon M \to [0,1]$ such that

1. $s(1) = 1$,
2. if $a \odot b = 0$, then $s(a \oplus b) = s(a) + s(b)$,
3. if $(a_i)_{i\in\mathbb{N}}$ is an increasing sequence in M, then $s(\bigvee_{i\in\mathbb{N}} a_i) = \bigvee_{i\in\mathbb{N}} s(a_i)$.

As a consequence, $s(0) = 0$ and $s(\neg a) = 1 - s(a)$. Every countable convex combination of states is a state. A state s is called *pure* if it cannot be expressed as a nontrivial convex combination of different states, i.e., if s does not admit the equality

$$s = \lambda t + (1-\lambda)u \tag{5}$$

for different states t, u and $\lambda \in (0,1)$. As we required σ-additivity, the state space of M need not be a compact subset of $[0,1]^M$ and it need not have any pure states. Only the case when M admits many pure states is of importance in probability.

The restriction of a pure state s on M to $\mathbf{B}(M)$ attains only the values 0 and 1. Indeed, if $s\restriction\mathbf{B}(M)$ is not two-valued, then there is a $b \in \mathbf{B}(M)$ such that $s(b) = \lambda \in (0,1)$. Two different states t, u defined by

$$t(a) = \frac{s(a \wedge b)}{s(b)}, \qquad u(a) = \frac{s(a \wedge \neg b)}{s(\neg b)}$$

satisfy (5), so s is not pure.

We shall characterize all set representations of M. A set H of functionals on M is called *separating* if

$$(\forall a, b \in M)(a \neq b \Rightarrow (\exists h \in H)\ h(a) \neq h(b)).$$

PROPOSITION 5.1. *Let M be a σ-complete MV-algebra and let X be the space of all σ-homomorphisms from M to S_∞. Then M is isomorphic to a Łukasiewicz tribe iff X is separating. All set representations of M (as a Łukasiewicz tribe) are restrictions of the representation on X to separating subsets of X.*

Proof. If M is a tribe, then the restriction to a point of its domain is a σ-homomorphism into S_∞. The set of all such σ-homomorphisms is separating.

Conversely, suppose that X is separating. Then each $a \in M$ can be represented by a function $f(a)\colon X \to [0,1] = S_\infty$ defined by $f(a)(h) = h(a)$. Apparently, $f\colon M \to [0,1]^X = S_\infty^X$ is an injective σ-homomorphism and it is an isomorphism onto a σ-complete MV-subalgebra of the tribe $[0,1]^X$.

The final statement of the proposition follows directly. ∎

Every σ-homomorphism into S_∞ is a state, but not all states are σ-homomorphisms. This cannot happen with *pure* states:

PROPOSITION 5.2. *Each pure state s on a σ-complete MV-algebra M is a σ-homomorphism from M to S_∞.*

Proof. Let X be the Stone space of $\mathbf{B}(M)$. The restriction $s{\restriction}\mathbf{B}(M)$ is a two-valued state on a σ-algebra, hence there exists an ideal $\mathcal{J} \in X$ such that $s(b) = 0 \Leftrightarrow b \in \mathcal{J}$. It determines a unique maximal ideal $\mathcal{I}$ of M by $\mathcal{I} = \{a \in M : (\exists b \in \mathcal{J})\ a \leq b\}$ (see [3] or [4, Prop. 1.7]) which is the kernel of a σ-homomorphism, $h_\mathcal{I}$. We shall prove that $h_\mathcal{I} = s$. Then, due to monotonicity of s and maximality of $\mathcal{I}$, we infer that $\mathcal{I}$ equals the kernel of s and the quotient algebra $M/\mathcal{I}$ is a subalgebra of S_∞.

Suppose that $c, d \in M$ are such that $h_\mathcal{I}(c) = h_\mathcal{I}(d)$. Then there is $b \in \mathcal{I}$ satisfying $c \vee b = d \vee b$. Without loss of generality, we can assume that $b \in \mathcal{J}$. As $s(b) = 0$, we obtain $s(c) = s((c \wedge \neg b) \oplus (c \wedge b)) = s(c \wedge \neg b)$ and similarly $s(c \vee b) = s(c \wedge \neg b)$, $s(d) = s(d \wedge \neg b)$, and $s(d \vee b) = s(d \wedge \neg b)$. All these values are equal, hence $s(c) = s(d)$. We have proved that, for each $c \in M$, the value $s(c)$ depends only on $h_\mathcal{I}(c)$. Thus there exists a function $\varphi\colon M/\mathcal{I} \to [0, 1]$ such that $s = \varphi \circ h_\mathcal{I}$. Obviously, φ is a state on $M/\mathcal{I}$, i.e., the identity, hence $h_\mathcal{I} = s$. ∎

We obtained the following analogue of Proposition 5.1:

COROLLARY 5.3. *A σ-complete MV-algebra M is isomorphic to a Łukasiewicz tribe iff there exists a separating set of pure states on M.*

The latter condition cannot be replaced by the weaker requirement that M admits a separating set of states. (E.g., the Borel σ-algebra on the real line factorized over all sets of Lebesgue measure zero admits a separating set of states, but it has no pure states.)

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