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CONTINUOUS DEPENDENCE ON FUNCTION PARAMETERS FOR SUPERLINEAR DIRICHLET PROBLEMS

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Abstract. We discuss the existence of solutions for a certain generalization of the membrane equation and their continuous dependence on function parameters. We apply variational methods and consider the PDE as the Euler–Lagrange equation for a certain integral functional, which is not necessarily convex and coercive. As a consequence of the duality theory we obtain variational principles for our problem and some numerical results concerning approximation of solutions.

1. Introduction. We are given two Carathéodory functions $\widetilde{G} : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying some growth conditions, Ω being a bounded domain in $\mathbb{R}^n$. We shall be dealing with the functional

$$J_u(x) = \int_\Omega \{H(y, \nabla x(y)) - \widetilde{G}(y, x(y), u(y))\}\, dy, \tag{1.1}$$

defined on a certain subset of $W_0^{1,2}(\Omega, \mathbb{R})$ for $u \in U \subset L^\infty(\Omega, \mathbb{R}^m)$. Investigation of the existence of a minimizer for (1.1) leads to the question whether there exists a solution of the boundary value problem

$$\begin{cases} -\operatorname{div} H_z(y, \nabla x(y)) = \widetilde{G}_x(y, x(y), u(y)) & \text{for a.e. } y \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $H_z(y,z) = \big[\frac{d}{dz_1} H(y,z), \ldots, \frac{d}{dz_n} H(y,z)\big]$ for $z = [z_1, \ldots, z_n] \in \mathbb{R}^n$, $\widetilde{G}_x(y,x,u)$ denotes the differential of the function $\widetilde{G}(y,\cdot,u)$ for $y \in \Omega$ and $u \in \mathbb{R}^m$, the function u in (1.2) is in $U := \{w \in L^\infty(\Omega, \mathbb{R}^m);\ w(y) \in \widetilde{U}$ for a.e. $y \in \Omega\}$ and $\widetilde{U}$ is a given subset of $\mathbb{R}^m$.

The aim of this paper is to study two issues: 1) the existence of solutions and 2) their continuous dependence on function parameters. We shall generalize the results presented in [14], where Ω is an interval, $\widetilde{G}(y,\cdot,u)$ is convex and satisfies some growth conditions. We show that weaker assumptions made on $\widetilde{G}$ ($\Omega \subset \mathbb{R}^n$, $n \geq 1$, $\mathbb{R} \ni x \mapsto \widetilde{G}(y,x,u)$ is not necessarily convex

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and coercive) are still sufficient to deduce the existence of a countable set of solutions for (1.2).

The problem of the continuous dependence on parameters for some systems of ODE of the second order with function parameters was considered, among others, in [8], [10], [17], [18]. The papers [8], [10] are based on direct methods and deal with scalar or two-dimensional systems. In [17], [18] some variational methods are applied in the case when $u \in L^\infty([0,\pi],\mathbb{R}^m)$. We shall investigate an analogous problem for a PDE of elliptic type which often appears in mathematical models of physical and technical phenomena.

Studying the existence problem for a given $u \in U$ leads to the generalization of the membrane equation

$$-\operatorname{div} H_z(y,\nabla x(y)) = G_x(y,x(y)) \quad \text{for a.e. } y \in \Omega, \tag{1.3}$$

$$x|_{\partial\Omega} = 0, \tag{1.4}$$

where $G(y,x) = \widetilde{G}(y,x,u(y))$ for $y \in \Omega$ and $x \in \mathbb{R}$. It is clear that (1.3) can be considered as the Euler–Lagrange equation for the action functional

$$J(x) = \int_\Omega \{H(y,\nabla x(y)) - G(y,x(y))\}\,dy. \tag{1.5}$$

By a solution of this problem we mean an element $x \in W_0^{1,2}(\Omega,R)$ such that for all $\varphi \in C_0^\infty(\Omega,\mathbb{R})$

$$\int_\Omega H_z(y,\nabla x(y))\nabla\varphi(y)\,dy = -\int_\Omega G_x(y,x(y))\varphi(y)\,dy.$$

Moreover we show that $H_z(\cdot,\nabla x(\cdot))$ has a distributional divergence that is an element of $L^2(\Omega,\mathbb{R})$.

Similar existence problems have been discussed by numerous authors. We mention [5] (for G of class C^1), [7], [9], [12], [13], [15], [16], [19] (in the case when H has the special form $H(y,z) = \frac{1}{2}|z|^2$ for $y \in \Omega$, $z \in \mathbb{R}^n$ and $G \in C^1(\Omega\times\mathbb{R},\mathbb{R})$). Most of these results are based on saddle point theorems or mountain pass theorems and require G to satisfy some smoothness conditions: $G_x(\cdot,\cdot) \in C(\Omega\times\mathbb{R},\mathbb{R})$, an estimate on G_x and the following relation between G and G_x: there exist $\mu > 0$ and $r \geq 0$ such that for $|x| \geq r$,

$$0 < \mu G(y,x) \leq xG_x(y,x). \tag{1.6}$$

Using these assumptions it is possible to obtain a classical solution of (1.3). In this paper we are looking for a nonzero solution of (1.3)–(1.4) in the sense mentioned above. A condition similar to (1.6) is also used in [2]. In [2], [11], [20] the right-hand side of the equation is continuous. Here we omit condition (1.6) and the continuity of $G_x(y,\cdot)$. There are many papers investigating PDEs of elliptic type in divergence form similar to our problem, among others D. Gilbarg and N. Trudinger [5] for $G \in C(\overline{\Omega}\times\mathbb{R})$, A. Benkirane and

A. Elmahi [1], and C. Ebmeyer and J. Frehse [3], where the right-hand side is independent of x and Ω is a bounded n-dimensional polyhedral domain. In [6] N. Grenon proves the existence of a solution $x \in W_0^{1,p}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$, $p > 1$, for the PDE

$$-\operatorname{div} A(y, x, Dx) = \mathbb{H}(y, x, Dx) \quad \text{in } \Omega, \tag{1.7}$$

with Ω being an open domain in $\mathbb{R}^n$, $n \geq 1$ and A, $\mathbb{H}$ satisfying additional estimates. He derives this result from the existence of solution of an associated symmetrized semilinear problem. Let us note that for $A(y, x, \xi) = H_z(y, \xi)$ and $\mathbb{H}(y, x, \xi) = G_x(y, x)$, (1.7) gives (1.3). In spite of this fact, we cannot use the results presented in [6]. In our paper we will assume that G satisfies the Carathéodory condition only, so that $G_x(y, \cdot)$ is not necessarily continuous. We also do not assume any additional estimate on G_x.

We shall assume throughout that the following holds:

Hypothesis (H). The set Ω is a bounded domain in $\mathbb{R}^n$ having a locally Lipschitz boundary. The functions $G : \Omega \times \mathbb{R} \to \mathbb{R}$ and $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy the Carathéodory condition, $H(y, \cdot)$ is Gateaux differentiable and convex for a.e. $y \in \Omega$, $G(y, \cdot)$ is differentiable for a.e. $y \in \Omega$ and $I \ni x \mapsto G(y, x)$ is convex, where I is a certain closed interval.

Moreover, there exist constants $b_1, b_2 > 0$ and functions $k_1, k_2 \in L^1(\Omega, \mathbb{R})$, $l_1 \in L^1(\Omega, \mathbb{R})$, $l_2 \in L^2(\Omega, \mathbb{R})$ such that

$$\frac{b_1}{2} |z|^2 + k_1(y) \leq H(y, z) \leq \frac{b_2}{2} |z|^2 + k_2(y)$$

for a.e. $y \in \Omega$ and all $z \in \mathbb{R}^n$ and

$$|G(y, x(y))| \leq l_1(y) \quad \text{for a.e. } y \in \Omega, \tag{1.8}$$

$$|G_x(y, x(y))| \leq l_2(y) \quad \text{for a.e. } y \in \Omega \tag{1.9}$$

for all $x \in W_0^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ such that $x(y) \in I$ for a.a. $y \in \Omega$.

We see that under hypothesis (H), J is not, in general, bounded on $W_0^{1,2}(\Omega, \mathbb{R})$, so that we must look for critical points of (1.5) of "minmax" type or find subsets on which the action functional J or its dual J_D is bounded. We shall apply another approach and choose special sets over which we will calculate the minimum of J and J_D. The usual methods applied to such problems include Morse theory and its generalizations, saddle points theorems and mountain pass theorems (see e.g. [12], [15], [19]). But none of these methods exhausts all critical points of J. Moreover, our assumptions are not strong enough to use, for example, the mountain pass theorem: G is not sufficiently smooth, we assume neither additional relations concerning G_x and G (see (1.6)) nor growth conditions on G, and in consequence, J is not necessarily of class C^1 and it does not satisfy, in general, the PS-condition. We shall develop a duality theory which permits us to omit deformation lemmas, the Ekeland variational principle and PS type conditions in our proof of the existence of critical points. Our approach also enables a numerical characterization of solutions of our problem and gives a measure of the duality gap between the primal and dual functionals.

2. The existence result. For given $M, K > 0$ and $z_1 \in W_0^{1,2}(\Omega,\mathbb{R}) \cap C(\Omega,\mathbb{R})$ define

$$\begin{aligned}\overline{X} := \{&x \in W_0^{1,2}(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R});\ \operatorname{div} H_z(\cdot,\nabla x(\cdot)) \in L^2(\Omega,\mathbb{R}),\\ &\|x - z_1\|_{L^\infty(\Omega,\mathbb{R})} \le M,\ \|\nabla x - \nabla z_1\|_{L^2(\Omega,\mathbb{R}^n)} \le K,\ x(y) \in I \text{ a.e. in } \Omega\}.\end{aligned}$$

Consider the function $\overline{G} : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\overline{G}(y,x) = \begin{cases} G(y,x) & \text{for } x \in I \text{ and } y \in \Omega,\\ +\infty & \text{for } x \in \mathbb{R}\setminus I \text{ and } y \in \Omega.\end{cases}$$

Now we define X as the largest subset of $\overline{X}$ having the property: for every $x \in X$, there exists $\widetilde{x} \in X$ such that

$$\int_\Omega \{\langle x(y), -\operatorname{div} H_z(y,\nabla\widetilde{x}(y))\rangle - G^*(y, -\operatorname{div} H_z(y,\nabla\widetilde{x}(y)))\}\,dy = \int_\Omega \overline{G}(y,x(y))\,dy, \tag{2.1}$$

where $G^*(y,z) = \sup_{x\in\mathbb{R}}\{zx - \overline{G}(y,x)\}$ for all $z \in \mathbb{R}$ and a.a. $y \in \Omega$, that is, for a.e. $y \in \Omega$, $\mathbb{R} \ni z \mapsto G^*(y,z)$ is the Fenchel conjugate of the function $\mathbb{R} \ni x \mapsto \overline{G}(y,x)$ ([4]).

Throughout the paper we shall assume hypotheses (H) and (H1) given below:

HYPOTHESIS (H1). $X \neq \emptyset$.

In Section 2.4 we shall consider (1.3) for $H(y,z) = \frac{1}{2}k(y)|z|^2$ for $y \in \Omega$, $z \in \mathbb{R}^n$ and $k \in C^1(\Omega,\mathbb{R})$ and formulate a sequence of assumptions concerning G which make the set X nonempty.

Let

$$X^d := \{p \in L^2(\Omega,\mathbb{R}^n);\ \text{there exists } x \in X \text{ such that } p(y) = H_z(y,\nabla x(y)) \text{ for a.e. } y \in \Omega\}. \tag{2.2}$$

Now we give the following auxiliary result:

REMARK 1. *For every $x \in X$, there exists $p \in X^d$ satisfying*

$$\int_\Omega \{\langle x(y), -\operatorname{div} p(y)\rangle - G^*(y, -\operatorname{div} p(y))\}\,dy = \int_\Omega G(y,x(y))\,dy. \tag{2.3}$$

Proof. Fix $x \in X$. Then there exists $\widetilde{x} \in X$ such that (2.1) holds, so $p(\cdot) = H_z(\cdot,\nabla\widetilde{x}(\cdot)) \in L^2(\Omega,\mathbb{R}^n)$ is in X^d and satisfies the required relation. ∎

2.1. *Duality.* The aim of this section is to develop a duality describing the connections between the critical values of J and the dual functional $J_D : X^d \to \mathbb{R}$ defined as follows:

$$J_D(p) = \int_\Omega \{-H^*(y, p(y)) + G^*(y, -\operatorname{div} p(y))\}\, dy, \tag{2.4}$$

where $H^*(y, v) = \sup_{u\in\mathbb{R}^n}\{\langle v, u\rangle - H(y, u)\}$ for all $v \in \mathbb{R}^n$ and a.a. $y \in \Omega$, that is, for a.e. $y \in \Omega$, $\mathbb{R}^n \ni v \mapsto H^*(y, v)$ is the Fenchel conjugate of $\mathbb{R}^n \ni u \mapsto H(y, u)$ ([4]).

For all $x \in X$, we consider the functional J_x defined in $L^2(\Omega,\mathbb{R})$ as

$$J_x(g) = \int_\Omega \{-H(y, \nabla x(y)) + \overline{G}(y, g(y) + x(y))\}\, dy.$$

It is clear that $J_x(0) = -J(x)$ for all $x \in X$. For every $x \in X$ define a conjugate $J_x^\# : X^d \to \mathbb{R}$ of J_x by

$$\begin{aligned} J_x^\#(p) \\ &= \sup_{g\in L^2(\Omega,\mathbb{R})} \int_\Omega \{\langle g(y), \operatorname{div} p(y)\rangle - \overline{G}(y, g(y) + x(y)) + H(y, \nabla x(y))\}\, dy. \end{aligned} \tag{2.5}$$

Now we will prove

Lemma 2.1.

$$J_x^\#(p) = \int_\Omega \{G^*(y, \operatorname{div} p(y)) + H(y, \nabla x(y)) - \langle x(y), \operatorname{div} p(y)\rangle\}\, dy.$$

Proof. Denote by F the functional defined in $L^2(\Omega,\mathbb{R})$ by

$$F(u) = \int_\Omega f(y, u(y))\, dy$$

and by F^* the conjugate function given by

$$F^*(u^*) = \sup_{u\in L^2(\Omega,\mathbb{R})} \Big\{\int_\Omega [u(y)u^*(y) - f(y, u(y))]\, dy\Big\} \tag{2.6}$$

for all $u^* \in L^2(\Omega,\mathbb{R})$. Since there exists $l_1 \in L^1(\Omega,\mathbb{R})$ such that $|G(y, x(y))| \le l_1(y)$ for a.e. $y \in \Omega$ and $x \in I$ and G is a Carathéodory function in $\Omega \times I$ we can apply Proposition 2.1 from [4, Chapter IX] to $f(y, x) = \overline{G}(y, x) + l_1(y)$, which gives

$$F^*(u^*) = \int_\Omega f^*(y, u^*(y))\, dy, \tag{2.7}$$

where f^* is the Fenchel transform of $\mathbb{R} \ni x \mapsto f(y, x)$ for a.a. $y \in \Omega$. We see that for a.a. $y \in \Omega$ and all $x \in \mathbb{R}$, $f^*(y, x^*) = \sup_{x\in\mathbb{R}}\{xx^* - \overline{G}(y, x) - l_1(y)\} = G^*(y, x^*) - l_1(y)$. So on the one hand (by (2.6))

$$F^*(u^*) = \sup_{u\in L^2(\Omega,\mathbb{R})} \Big\{\int_\Omega [u(y)u^*(y) - \overline{G}(y, u(y))]\, dy\Big\} - \int_\Omega l_1(y)\, dy$$

and on the other hand (by (2.7))

$$F^*(u^*) = \int_\Omega G^*(y, u^*(y))\,dy - \int_\Omega l_1(y)\,dy.$$

Finally,

$$\sup_{u\in L^2(\Omega,\mathbb{R})}\Big\{\int_\Omega [u(y)u^*(y) - \overline{G}(y,u(y))]\,dy\Big\} = \int_\Omega G^*(y,u^*(y))\,dy. \ \blacksquare$$

LEMMA 2.2. *For all $p \in X^d$,*

$$\sup_{x\in X}(-J_x^{\#}(-p)) = -J_D(p). \tag{2.8}$$

Proof. Fix $p \in X^d$. From (2.2) we obtain the existence of $\overline{x} \in X$ satisfying $p(\cdot) = H_z(\cdot, \nabla\overline{x}(\cdot))$ a.e. on Ω, and consequently,

$$\int_\Omega \{\langle \nabla\overline{x}(y), p(y)\rangle - H(y, \nabla\overline{x}(y))\}\,dy = \int_\Omega H^*(y,p(y))\,dy, \tag{2.9}$$

so that

$$\begin{aligned}\int_\Omega H^*(y,p(y))\,dy &\le \sup_{x\in X}\int_\Omega \{\langle \nabla x(y), p(y)\rangle - H(y,\nabla x(y))\}\,dy \\ &\le \int_\Omega H^*(y,p(y))\,dy.\end{aligned} \tag{2.10}$$

This implies

$$\begin{aligned}&\sup_{x\in X}(-J_x^{\#}(-p)) \\ &\quad = \sup_{x\in X}\int_\Omega \{\langle \nabla x(y), p(y)\rangle - H(y,\nabla x(y)) - G^*(y, -\operatorname{div} p(y))\}\,dy = -J_D(p). \ \blacksquare\end{aligned}$$

LEMMA 2.3. *For each $x \in X$,*

$$\sup_{p\in X^d}(-J_x^{\#}(-p)) = -J(x). \tag{2.11}$$

Proof. Taking into account Remark 1, we infer that for each $x \in X$ there exists $\overline{p} \in X^d$ such that

$$\int_\Omega \{\langle x(y), -\operatorname{div}\overline{p}(y)\rangle - G^*(y, -\operatorname{div}\overline{p}(y))\}\,dy = \int_\Omega G(y,x(y))\,dy. \tag{2.12}$$

By arguments similar to those in the proof of (2.10), we obtain

$$\begin{aligned}&\sup_{p\in X^d}\int_\Omega \{\langle x(y), -\operatorname{div} p(y)\rangle - G^*(y, -\operatorname{div} p(y))\}\,dy \\ &\qquad\qquad = \int_\Omega G^{**}(y,x(y))\,dy = \int_\Omega G(y,x(y))\,dy,\end{aligned} \tag{2.13}$$

where the last equality is due to the assumptions on G. By (2.13) and (2.5) we get

$$\begin{aligned}
&\sup_{p\in X^d}(-J_x^{\#}(-p))\\
&\quad=\sup_{p\in X^d}\int_\Omega\{\langle x(y),-\operatorname{div}p(y)\rangle-G^*(y,-\operatorname{div}p(y))-H(y,\nabla x(y))\}\,dy\\
&\quad=\int_\Omega\{-H(y,\nabla x(y))+G(y,x(y))\}\,dy=-J(x).\ \blacksquare
\end{aligned}$$

Combining the two lemmas leads to a duality principle:

THEOREM 2.4.

$$\inf_{x\in X}J(x)=\inf_{p\in X^d}J_D(p).$$

2.2. *Variational principles.* This section is devoted to conditions necessary for the existence of a minimizer for (1.5). We also present a variational principle for minimizing sequences of J and J_D. This result enables numerical approximation of solutions for (1.3).

THEOREM 2.5. *Let $\overline{x}\in X$ satisfy $J(\overline{x})=\inf_{x\in X}J(x)$. Then there exists $\overline{p}\in X^d$ which is a minimizer of J_D:*

$$J_D(\overline{p})=\inf_{p\in X^d}J_D(p) \tag{2.14}$$

and satisfies $-\operatorname{div}\overline{p}\in\partial J_{\overline{x}}(0)$ (where $\partial J_{\overline{x}}(0)$ denotes the subdifferential of $J_{\overline{x}}$ at 0). Moreover

$$J_{\overline{x}}^{\#}(-\overline{p})+J_{\overline{x}}(0)=0, \tag{2.15}$$

$$J_{\overline{x}}^{\#}(-\overline{p})-J_D(\overline{p})=0. \tag{2.16}$$

Proof. By Remark 1 there exists $\overline{p}\in X^d$ such that (2.3) holds. This gives (2.15). Let $J_{\overline{x}}^*$ denote the Fenchel conjugate of $J_{\overline{x}}$, that is, for all $g^*\in L^2(\Omega,\mathbb{R})$,

$$J_{\overline{x}}^*(g^*)=\sup_{g\in L^2(\Omega,\mathbb{R})}\Big\{\int_\Omega\langle g^*(y),g(y)\rangle\,dy-J_{\overline{x}}(g)\Big\}.$$

An easy computation shows that $J_{\overline{x}}^*(-\operatorname{div}\overline{p})=J_{\overline{x}}^{\#}(-\overline{p})$ and, as a consequence, by (2.15) and the properties of the subdifferential, we have the inclusion $-\operatorname{div}\overline{p}\in\partial J_{\overline{x}}(0)$.

Our task is now to prove that $\overline{p}$ is a minimizer of $J_D:X^d\to\mathbb{R}$. Combining the equalities $J_{\overline{x}}(0)=-J(\overline{x})$, (2.15) and Lemma 2.2 we deduce that

$$-J(\overline{x})=-J_{\overline{x}}^{\#}(-\overline{p})\le\sup_{x\in X}(-J_x^{\#}(-\overline{p}))=-J_D(\overline{p}) \tag{2.17}$$

and further Theorem 2.4 yields (2.14). Finally, (2.16) follows from (2.15) and the equalities $J_{\overline{x}}(0)=-J(\overline{x})=-J_D(\overline{p})$. $\blacksquare$

COROLLARY 2.6. *If $\overline{x} \in X$ is a minimizer of $J : X \to \mathbb{R}$, then $\overline{x}$ satisfies* (1.3).

Proof. By Theorem 2.5 we get the existence of $\overline{p} \in X$ for which (2.15) and (2.16) hold. Hence

$$\int_{\Omega} \{H^*(y, \overline{p}(y)) + H(y, \nabla \overline{x}(y)) - \langle \nabla \overline{x}(y), \overline{p}(y)\rangle\}\, dy = 0$$

and

$$\int_{\Omega} \{G^*(y, -\operatorname{div} \overline{p}(y)) + \overline{G}(y, \overline{x}(y)) - \langle \overline{x}(y), -\operatorname{div} \overline{p}(y)\rangle\}\, dy = 0.$$

Using the properties of the Fenchel conjugate, we obtain, for a.e. $y \in \Omega$,

$$\overline{p}(y) = H_z(y, \nabla \overline{x}(y)) \quad \text{and} \quad -\operatorname{div} \overline{p}(y) = G_x(y, \overline{x}(y))$$

for a.e. $y \in \Omega$. These equalities imply (1.3). ∎

Now we prove a numerical version of the above variational principle. We present a result for minimizing sequences that is analogous to the previous theorem.

THEOREM 2.7. *Let $\{x_m\}_{m\in\mathbb{N}} \subset X$ be a minimizing sequence of $J : X \to \mathbb{R}$. Then for each $m \in \mathbb{N}$ there exists $p_m \in X^d$ satisfying $-\operatorname{div} p_m \in \partial J_{x_m}(0)$ and*

$$\inf_{m\in\mathbb{N}} J_D(p_m) = \inf_{p\in X^d} J_D(p). \tag{2.18}$$

Moreover for all $m \in \mathbb{N}$,

$$J_{x_m}(0) + J^{\#}_{x_m}(-p_m) = 0, \tag{2.19}$$

and for each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$J^{\#}_{x_n}(-p_m) - J_D(p_m) \le \varepsilon, \tag{2.20}$$

$$|J_D(p_m) - J(x_m)| \le \varepsilon. \tag{2.21}$$

Proof. First observe that $J : X \to \mathbb{R}$ is bounded below. Indeed, from the definition of X and hypothesis (H) we infer that for all $x \in X$,

$$J(x) \ge \int_{\Omega} \left[\frac{b_1}{2} |\nabla x|^2 + k_1(y) - G(y, x(y))\right] dy \ge \int_{\Omega} [k_1(y) - l_1(y)]\, dy.$$

Hence $\inf_{m\in\mathbb{N}} J(x_m) =: c > -\infty$. As in the proof of Theorem 2.5, for each $m \in \mathbb{N}$ there exists $p_m \in X^d$ satisfying (2.19) and $-\operatorname{div} p_m \in \partial J_{x_m}(0)$. We proceed to show that $\{p_m\}_{m\in\mathbb{N}}$ is a minimizing sequence for $J_D : X^d \to \mathbb{R}$. To this end fix $\varepsilon > 0$. By the definition of c there exists $m_0 \in \mathbb{N}$ such that $c + \varepsilon > J(x_m)$ for all $m > m_0$. Therefore, from the equalities $J_{x_m}(0) = -J(x_m)$, (2.19) and (2.11) we deduce that for all $m > m_0$,

$$c + \varepsilon > J(x_m) = J^{\#}_{x_m}(-p_m) \ge \inf_{x\in X} (J^{\#}_x(-p_m)) = J_D(p_m).$$

Using the last assertion and Theorem 2.4 we can derive that $\inf_{p\in X^d} J_D(p) = \inf_{m\in\mathbb{N}} J_D(p_m) = c$. Now (2.20) and (2.21) follow from the assertion above and the fact that $J^{\#}_{x_m}(-p_m) \le c+\varepsilon$ for all $m > m_0$. ∎

As a consequence of the previous theorem we obtain the following

COROLLARY 2.8. *Suppose that $\{x_m\}_{m\in\mathbb{N}} \subset X$ is a minimizing sequence for J on X. Then there exists a minimizing sequence $\{p_m\}_{m\in\mathbb{N}} \subset X^d$ with*

$$-\operatorname{div} p_m(y) = G_x(y, x_m(y)) \tag{2.22}$$

for a.e. $y \in \Omega$ and every $m \in \mathbb{N}$. Moreover

$$\lim_{m\to\infty} \int_\Omega \{H^*(y, p_m(y)) + H(y, \nabla x_m(y)) - \langle p_m(y), \nabla x_m(y)\rangle\}\, dy = 0. \tag{2.23}$$

2.3. *The existence of solutions for the Dirichlet problem.* This section is devoted to the existence of a solution of (1.3).

THEOREM 2.9. *There exists a minimizer $x_0 \in X$ of the functional J on X satisfying* (1.3).

Proof. Consider a minimizing sequence $\{x_m\}_{m\in\mathbb{N}} \subset X$ of $J : X \to \mathbb{R}$. Since $X \subset \overline{X}$ we infer that $\{\nabla x_m\}_{m\in\mathbb{N}}$ is bounded in the norm $\|\cdot\|_{L^2(\Omega,\mathbb{R}^n)}$ and further $\{x_m\}_{m\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega,\mathbb{R})$. Thus, passing to a subsequence if necessary, we deduce that $\{x_m\}_{m\in\mathbb{N}}$ tends weakly to some $x_0 \in W_0^{1,2}(\Omega,\mathbb{R})$ and further $x_m \to x_0$ in $L^2(\Omega,\mathbb{R})$ as $m \to \infty$. As a consequence, we get pointwise convergence of a subsequence (still denoted by $\{x_m\}_{m\in\mathbb{N}}$): $\lim_{m\to\infty} x_m(y) = x_0(y)$ for a.e. $y \in \Omega$. Therefore

$$\begin{aligned} &\|x_0 - z_1\|_{L^\infty(\Omega,\mathbb{R})} \le M, \qquad \|\nabla x_0 - \nabla z_1\|_{L^2(\Omega,\mathbb{R}^n)} \le K,\\ &x_0(y) \in I \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.24}$$

By Corollary 2.8 there exists a minimizing sequence $\{p_m\}_{m\in\mathbb{N}} \subset X^d$ for J_D with

$$-\operatorname{div} p_m(y) = G_x(y, x_m(y)) \tag{2.25}$$

for a.e. $y \in \Omega$ and every $m \in \mathbb{N}$. Moreover

$$\lim_{m\to\infty} \int_\Omega \{H^*(y, p_m(y)) + H(y, \nabla x_m(y)) - \langle p_m(y), \nabla x_m(y)\rangle\}\, dy = 0. \tag{2.26}$$

By the assumptions concerning G_x and (2.25) the sequence $\{\operatorname{div} p_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega,\mathbb{R})$. Consequently, so is $\{\int_\Omega \langle \operatorname{div} p_m(y), x_m(y)\rangle\, dy\}_{m\in\mathbb{N}}$. Moreover, the assumptions on H imply $\{p_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega,\mathbb{R}^n)$. Finally, we conclude that there is a subsequence, still denoted by $\{p_m\}_{m\in\mathbb{N}}$, weakly convergent to some p_0 in $L^2(\Omega,\mathbb{R}^n)$ and such that $\operatorname{div} p_m \rightharpoonup z$ as $m \to \infty$, where $z \in L^2(\Omega,\mathbb{R})$.

Now we show that $\operatorname{div} p_0 = z$ in $L^2(\Omega,\mathbb{R})$. By the above,

$$\int_\Omega \langle p_0(y), \nabla h(y)\rangle\, dy = \lim_{m\to\infty} \int_\Omega \langle p_m(y), \nabla h(y)\rangle\, dy$$
$$= -\lim_{m\to\infty} \int_\Omega \langle \operatorname{div} p_m(y), h(y)\rangle\, dy = -\int_\Omega \langle z(y), h(y)\rangle\, dy$$

for any $h \in C_0^\infty(\Omega,\mathbb{R})$, hence, by the Euler–Lagrange lemma, $\operatorname{div} p_0(y) = z(y)$ for a.e. $y \in \Omega$. On account of the above reasoning we obtain

$$\int_\Omega \{G^*(y, -\operatorname{div} p_0(y)) + G(y, x_0(y)) + \langle \operatorname{div} p_0(y), x_0(y)\rangle\}\, dy \le 0. \tag{2.27}$$

Thus, by the properties of the Fenchel transform, we have equality in (2.27), and as a consequence,

$$-\operatorname{div} p_0(y) = G_x(y, x_0(y)) \quad \text{for a.e. } y \in \Omega. \tag{2.28}$$

Similarly, by (2.23),

$$0 = \lim_{m\to\infty} \int_\Omega \{H^*(y, p_m(y)) + H(y, \nabla x_m(y)) - \langle p_m(y), \nabla x_m(y)\rangle\}\, dy$$
$$\ge \int_\Omega [H^*(y, p_0(y)) + H(y, \nabla x_0(y)) - \langle p_0(y), \nabla x_0(y)\rangle]\, dy.$$

By the last relation, analysis similar to that in the proof of (2.28) shows that

$$p_0(y) = H_z(y, \nabla x_0(y)) \quad \text{for a.e. } y \in \Omega. \tag{2.29}$$

Combining (2.24), (2.29) and the relation $\operatorname{div} p_0 \in L^2(\Omega,\mathbb{R})$, we derive that $x_0 \in \overline{X}$. (2.28) and (2.29) imply

$$-\operatorname{div} H_z(y, \nabla x_0(y)) = G_x(y, x_0(y))$$

for a.e. $y \in \Omega$, so that $x_0 \in X$.

Now we show that

$$\inf_{x\in X} J(x) = \liminf_{m\to\infty} \int_\Omega \{H(y, \nabla x_m(y)) - G(y, x_m(y))\}\, dy \ge J(x_0). \tag{2.30}$$

To see this, note that

$$\liminf_{m\to\infty} \int_\Omega H(y, \nabla x_m(y))\, dy \ge \int_\Omega H(y, \nabla x_0(y))\, dy, \tag{2.31}$$

due to the weak lower semicontinuity of $L^2(\Omega,\mathbb{R}^n) \ni z \mapsto \int_\Omega H(y, z(y))\, dy$ and the fact that $\nabla x_m \rightharpoonup \nabla x_0$ in $L^2(\Omega,\mathbb{R}^n)$ as $m \to \infty$. Moreover, from the (strong) lower semicontinuity of $\overline{X} \ni x \mapsto \int_\Omega [-G(y, x(y))]\, dy$ and the strong convergence of $\{x_m\}_{m\in\mathbb{N}}$ to x_0 in $L^2(\Omega,\mathbb{R})$, we infer that

$$\liminf_{m\to\infty} \int_\Omega [-G(y, x_m(y))]\, dy \ge \int_\Omega [-G(y, x_0(y))]\, dy. \tag{2.32}$$

Assertions (2.31) and (2.32) imply (2.30). ∎

An immediate consequence of this theorem and the definition of X is the following

COROLLARY 2.10. *If* $\|z_1\|_{L^\infty(\Omega,\mathbb{R})} > M$ *or* $\|\nabla z_1\|_{L^2(\Omega,\mathbb{R}^n)} > K$, *then there exists a nonzero solution of* (1.3).

2.3.1. *Applications.* We shall apply our theory to derive an existence result for the Dirichlet problem for a certain class of partial differential equations. It is fairly easy to find an example of functions G and H satisfying (H); checking (H1) for a given G is more difficult. So we shall consider the case when H has a special form and assume some additional conditions on G, which make X nonempty. To this and we need the relevant theorems from [5]:

THEOREM 2.11 ([5, Theorem 9.15 for the PDE given below]). *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. If $f \in L^p(\Omega,\mathbb{R})$ with $1 < p < \infty$, then the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(k(y)\nabla u(y)) = f(y) & \text{for a.e. } y \in \Omega, \\ u \in W_0^{1,p}(\Omega,\mathbb{R}), \end{cases} \tag{2.33}$$

where $k \in C^1(\overline{\Omega},\mathbb{R})$ with $\overline{k}_0 \geq k(y) \geq k_0 > 0$ for all $y \in \Omega$, has a unique solution $u \in W^{2,p}(\Omega,\mathbb{R})$.

THEOREM 2.12 ([5, Theorem 9.17]). *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. Then there exists a constant $\widetilde{c}$ (independent of u) such that*

$$\|u\|_{W^{2,p}(\Omega,\mathbb{R})} \leq \widetilde{c}\|\operatorname{div}(k\nabla u)\|_{L^p(\Omega,\mathbb{R})} \tag{2.34}$$

for all $u \in W^{2,p}(\Omega,\mathbb{R}) \cap W_0^{1,p}(\Omega,\mathbb{R})$ with $1 < p < \infty$ and $k \in C^1(\overline{\Omega},\mathbb{R})$ with $\overline{k}_0 \geq k(y) \geq k_0 > 0$ for any $y \in \Omega$.

REMARK 2. Now we shall apply Theorem 2.9 to show that there exists at least one solution of (1.3) for $H(y,z) = \frac{1}{2}k(y)|z|^2$ and G satisfying hypothesis (H). To this end we have to make some additional assumptions on G, which guarantee that hypothesis (H1) holds. In the proof we will use Theorems 2.11 and 2.12.

THEOREM 2.13. *Assume that*: $k \in C^1(\overline{\Omega},\mathbb{R})$, $\overline{k}_0 \geq k(y) \geq k_0$ *for all* $y \in \Omega$, *where Ω is a $C^{1,1}$ bounded domain in $\mathbb{R}^n$, G is differentiable with respect to the second variable on $\mathbb{R}$ for a.e. $y \in \Omega$ and measurable in y for all $x \in \mathbb{R}$, and $I \ni x \mapsto G(y,x)$ is convex, where $I := [-b,b]$ and b is a certain positive number. Let $\widetilde{c}$ denote the constant of* (2.34) *for $p = q$ and let s be the Sobolev constant (see the remark below). Suppose additionally that there exists $z \in L^q(\Omega,\mathbb{R})$ such that $G_x(\cdot,z(\cdot)) \in L^q(\Omega,\mathbb{R})$ and $\|G_x(\cdot,z(\cdot))\|_{L^q(\Omega,\mathbb{R})} \leq b/2s\widetilde{c}$, and there exist constants $q > n/2$, $0 < S_1 \leq b/2s\widetilde{c}$ and a function $l_1 \in L^1(\Omega,\mathbb{R})$ such that*

$$|G(y,x(y))| \leq l_1(y) \quad \text{for a.e. } y \in \Omega,$$

and

$$\|G_x(\cdot,x(\cdot))-G_x(\cdot,z(\cdot))\|_{L^q(\Omega,\mathbb{R})}<S_1 \tag{2.35}$$

for all $x\in W_0^{1,2}(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$ *with* $x(y)\in I$ *for a.a.* $y\in\Omega$. *Then there exists a solution* $x_0\in\overline{X}$ *of the Dirichlet problem for the PDE*

$$-\operatorname{div}(k(y)\nabla x_0(y))=G_x(y,x_0(y)) \quad \text{for a.e. } y\in\Omega. \tag{2.36}$$

REMARK 3. Since $q>n/2$ and $\Omega\subset\mathbb{R}^n$ is bounded, the Sobolev inequality implies that $\|u\|_{C(\Omega,\mathbb{R})}\le s\|u\|_{W_0^{2,q}(\Omega,\mathbb{R})}$ for each $u\in W_0^{2,q}(\Omega,\mathbb{R})$.

Proof of Theorem 2.13. Let $\widetilde{\widetilde{c}}$ denote the constant of (2.34) for $p=2$, $M:=s\widetilde{c}S_1$, $K:=\widetilde{\widetilde{c}}S_1|\Omega|^{1/2-1/q}$, and let $\overline{z}\in W_0^{2,q}(\Omega,\mathbb{R})$ be a solution of the equation

$$-\operatorname{div}(k(y)\nabla\overline{z}(y))=G_x(y,z(y)) \quad \text{for a.e. } y\in\Omega$$

(the existence of $\overline{z}$ follows from Theorem 2.11). Then, by Theorem 2.12, we have

$$\begin{aligned}\|\overline{z}\|_{L^\infty(\Omega,\mathbb{R})}\le s\|\overline{z}\|_{W_0^{2,q}(\Omega,\mathbb{R})}&\le s\widetilde{c}\|\operatorname{div}(k\nabla\overline{z})\|_{L^q(\Omega,\mathbb{R})}\\&=s\widetilde{c}\|G_x(\cdot,z(\cdot))\|_{L^q(\Omega,\mathbb{R})}<b.\end{aligned} \tag{2.37}$$

First we recall that

$$\begin{aligned}\overline{X}:=\{&x\in W_0^{1,2}(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R});\operatorname{div}(k\nabla x)\in L^2(\Omega,\mathbb{R}),\\&\|u-\overline{z}\|_{L^\infty(\Omega,\mathbb{R})}\le M,\ \|\nabla u-\nabla\overline{z}\|_{L^2(\Omega,\mathbb{R}^n)}\le K,\ x(y)\in I \text{ for a.a. } y\in\Omega\}.\end{aligned}$$

Let

$$\begin{aligned}X_0:=\{x\in W_0^{2,2}(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R});\|\operatorname{div}(k\nabla x)+G_x(\cdot,z(\cdot))\|_{L^q(\Omega,\mathbb{R})}\le S_1\\ \text{and } x(y)\in I \text{ for a.a. } y\in\Omega\}.\end{aligned}$$

Taking into account the definition of $\overline{z}$ and (2.37), we see that $\overline{z}\in X_0$. Now we show that $X_0\subset\overline{X}$. Indeed, fix $x\in X_0$; using Remark 3 and Theorem 2.12 for $p=q$ we get

$$\begin{aligned}\|x-\overline{z}\|_{L^\infty(\Omega,\mathbb{R})}\le s\|x-\overline{z}\|_{W_0^{2,q}(\Omega,\mathbb{R})}&\le s\widetilde{c}\|\operatorname{div}(k\nabla x-k\nabla\overline{z})\|_{L^q(\Omega,\mathbb{R})}\\&=s\widetilde{c}\|\operatorname{div}(k\nabla x)+G_x(\cdot,z(\cdot))\|_{L^q(\Omega,\mathbb{R})}<s\widetilde{c}S_1.\end{aligned} \tag{2.38}$$

Applying again Theorem 2.12 for $p=2$ we have

$$\begin{aligned}\|\nabla x-\nabla\overline{z}\|_{L^2(\Omega,\mathbb{R}^n)}\le\|x-\overline{z}\|_{W_0^{2,2}(\Omega,\mathbb{R})}&\le\widetilde{\widetilde{c}}\|\operatorname{div}(k\nabla x-k\nabla\overline{z})\|_{L^2(\Omega,\mathbb{R})}\\&\le\widetilde{\widetilde{c}}|\Omega|^{1/2-1/q}\|\operatorname{div}(k\nabla x-k\nabla\overline{z})\|_{L^q(\Omega,\mathbb{R})}\le\widetilde{\widetilde{c}}|\Omega|^{1/2-1/q}S_1.\end{aligned}$$

It is clear that $\operatorname{div}(k\nabla x)\in L^2(\Omega,\mathbb{R})$. Finally, $x\in\overline{X}$.

Now we prove that X_0 has the following property: for every $x \in X_0$, there exists $\widetilde{x} \in X_0$ such that

$$\int_\Omega \{\langle x(y), -\operatorname{div}(k(y)\nabla\widetilde{x}(y))\rangle - G^*(y, -\operatorname{div}(k(y)\nabla\widetilde{x}(y))\}\, dy = \int_\Omega G(y, x(y))\, dy.$$

To this end fix $\overline{x} \in X_0$. Since $G_x(\cdot, \overline{x}(\cdot)) \in L^q(\Omega, \mathbb{R})$, by Theorem 2.11 there exists a unique solution $x_0 \in W_0^{1,q}(\Omega, \mathbb{R}) \cap W^{2,q}(\Omega, \mathbb{R})$ of the Dirichlet problem for the equation

$$-\operatorname{div}(k(y)\nabla x(y)) = G_x(y, \overline{x}(y)) \quad \text{a.e. on } \Omega, \tag{2.39}$$

so that

$$-\operatorname{div}(k(y)\nabla x_0(y)) \in \partial_x G(y, \overline{x}(y)).$$

The properties of the subdifferential now yield the required relation.

Moreover, by (2.35), we obtain

$$\begin{aligned}\|\operatorname{div}(k(y)\nabla x_0(y)) + G_x(\cdot, z(\cdot))\|_{L^q(\Omega,\mathbb{R})}& \\ = \|G_x(\cdot, \overline{x}(\cdot)) - G_x(\cdot, z(\cdot))\|_{L^q(\Omega,\mathbb{R})} &< S_1.\end{aligned}$$

Taking into account this estimate, we see that

$$\begin{aligned}\|x_0\|_{L^\infty(\Omega,\mathbb{R})} \le s\|x_0\|_{W_0^{2,q}(\Omega,\mathbb{R})} &\le s\widetilde{c}\|\operatorname{div}(k\nabla x_0 z)\|_{L^q(\Omega,\mathbb{R})} \\ &= s\widetilde{c}\|G_x(\cdot, \overline{x}(\cdot))\|_{L^q(\Omega,\mathbb{R})} \le b.\end{aligned}$$

Finally, we conclude that $x_0 \in X_0$. Summarizing, $X_0 \subset \overline{X}$ and X_0 has the required property, so that $X_0 \subset X$. The relation $X_0 \neq \emptyset$ leads to $X \neq \emptyset$. Now Theorem 2.9 yields the existence of a solution $x_0 \in \overline{X}$ of (2.36). ∎

Now we give an explicit example of (2.36) with G satisfying the assumption of the previous theorem.

Example 1. We consider the special case of (2.36), when $n = 4$, $q = 9$, Ω is a $C^{1,1}$ bounded domain in $\mathbb{R}^4$, $k(y) = \|y\|^2_{\mathbb{R}^4} + 1$ for all $y \in \Omega$, $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$G(y, x) = a(y)(x^9 - 5x^8 - 2x^7 + 24x^6 + x),$$

where $a \in L^\infty(\Omega, \mathbb{R})$ with

$$0 < \|a\|_{L^\infty(\Omega,\mathbb{R})} \le \frac{1}{\widetilde{\widetilde{c}}s} \min\left\{\frac{1}{60\sqrt[9]{|\Omega|}}, \frac{1}{12}\right\}$$

and the constants $\widetilde{\widetilde{c}}$, s, $\widetilde{c}$ are defined as in Theorem 2.13. Then there exists at least one solution $x \in W_0^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ of the problem

$$\begin{aligned} &-\operatorname{div}((\|y\|^2_{\mathbb{R}^4} + 1)\nabla x(y)) \\ &= a(y)[9(x(y))^8 - 40(x(y))^7 - 14(x(y))^6 + 144(x(y))^5 + 1] \quad \text{for a.e. } y \in \Omega \end{aligned} \tag{2.40}$$

such that $\Omega \ni y \mapsto \operatorname{div}((\|y\|^2_{\mathbb{R}^4} + 1)\nabla x(y))$ belongs to $L^2(\Omega, \mathbb{R})$.

Indeed, G satisfies all the assumptions of Theorem 2.13. Of course $k \in C^1(\overline{\Omega}, \mathbb{R})$, $1 \leq k(y) \leq 1+\sup_{y\in\overline{\Omega}} \|y\|^2_{\mathbb{R}^4} < \infty$ for all $y \in \Omega$, G is differentiable with respect to the second variable in $\mathbb{R}$ for a.e. $y \in \Omega$ and measurable in y for all $x \in \mathbb{R}$.

Let $S_1 = 1/2\widetilde{c}s$, $z(y) = \frac{1}{2}\sin\|y\|_{\mathbb{R}^4} \in L^\infty(\Omega, \mathbb{R})$ and let $\overline{z} \in W^{2,9}(\Omega, \mathbb{R})$ be a solution of the linear Dirichlet problem

$$\begin{cases} -\operatorname{div}((\|y\|^2_{\mathbb{R}^4} + 1)\nabla\overline{z}(y)) \\ \qquad = a(y)[9(z(y))^8 - 40(z(y))^7 - 14(z(y))^6 + 144(z(y))^5 + 1] \\ \hfill \text{for a.e. } y \in \Omega, \\ \overline{z} \in W^{1,2}_0(\Omega, \mathbb{R}). \end{cases}$$

(The existence of $\overline{z}$ follows from Theorem 2.11.) It is worth noting that G is convex with respect to the second variable in the interval $[-1, 1]$, but it is not convex in $\mathbb{R}$. By the estimate on a, we have $\|G_x(\cdot, z(\cdot))\|_{L^q(\Omega,\mathbb{R})} \leq 1/2s\widetilde{c}$.

It is obvious that if $x(y) \in [-1, 1]$ for a.a. $y \in \Omega$, then there exists a constant $l_1 > 0$ such that $|G(y, x(y))| \leq l_1$ for a.e. $y \in \Omega$.

To end the proof it is sufficient to show that

$$\|G_x(\cdot, x(\cdot)) - G_x(\cdot, z(\cdot))\|_{L^9(\Omega,\mathbb{R})} < S_1.$$

Indeed,

$$\begin{aligned} &\|G_x(\cdot, x(\cdot)) - G_x(\cdot, z(\cdot))\|_{L^9(\Omega,\mathbb{R})} \\ &\quad \leq \|a\|_{L^\infty(\Omega,\mathbb{R})}[\|9(x(y))^8 - 40(x(y))^7 - 14(x(y))^6 + 144(x(y))^5\|_{L^9(\Omega,\mathbb{R})} \\ &\qquad + \|9(z(y))^8 - 40(z(y))^7 - 14(z(y))^6 + 144(z(y))^5\|_{L^9(\Omega,\mathbb{R})}] \\ &\quad \leq 25\sqrt[9]{|\Omega|}\,\|a\|_{L^\infty(\Omega,\mathbb{R})} < S_1. \end{aligned}$$

We have proved that G satisfies the assumptions of Theorem 2.13, which yields the existence of a solution $x_0 \in \overline{X}$ to (2.40).

2.3.2. *Multiple solutions*

Hypothesis (H2). Ω is a bounded domain of $\mathbb{R}^n$ having a locally Lipschitz boundary. The functions $G : \Omega \times \mathbb{R} \to \mathbb{R}$ and $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy the Carathéodory condition, $H(y, \cdot)$ is Gateaux differentiable and convex for a.e. $y \in \Omega$, $G(y, \cdot)$ is differentiable for a.e. $y \in \Omega$ and there exist constants $b_3, b_4 > 0$ and functions k_3, $k_4 \in L^1(\Omega, \mathbb{R})$ such that for a.e. $y \in \Omega$ and all $z \in \mathbb{R}^n$,

$$\frac{b_3}{2}|z|^2 + k_3(y) \leq H(y, z) \leq \frac{b_4}{2}|z|^2 + k_4(y).$$

Moreover for each i from a certain subset $N_0 \subset \mathbb{N}$ there exist functions $l^1_i \in L^1(\Omega, \mathbb{R})$, $l^2_i \in L^2(\Omega, \mathbb{R})$ and an interval I_i such that $I_i \ni x \mapsto G(y, x)$

is convex and

$$|G(y,x(y))| \le l_i^1(y) \quad \text{and} \quad |G_x(y,x(y))| \le l_i^2(y)$$

for all $x \in W_0^{1,2}(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$ such that $x(y) \in I_i$ for a.a. $y \in \Omega$.

For given $M_i, K_i > 0$ and $z_{1i} \in W_0^{1,2}(\Omega,\mathbb{R}) \cap C(\Omega,\mathbb{R})$, $i \in N_0$, define

$$\begin{aligned}\overline{X}_i := \{x \in W_0^{1,2}(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R});\ &\operatorname{div} H_z(\cdot,\nabla x(\cdot)) \in L^2(\Omega,\mathbb{R}),\\ \|x - z_{1i}\|_{L^\infty(\Omega,\mathbb{R})} \le M_i,\ &\|\nabla x - \nabla z_{1i}\|_{L^2(\Omega,\mathbb{R}^n)} \le K_i,\ x(y) \in I_i \text{ a.e. in } \Omega\}.\end{aligned}$$

Hypothesis (H2′). $X_i \ne \emptyset$ for all $i \in I$, where X_i is the largest subset of $\overline{X}_i$ with property (2.1).

Using Theorem 2.9 for each X_i we get the existence of a countable set of solutions for our problem.

Theorem 2.14. *Assume hypotheses* (H2) *and* (H2′)*. Then for all $i \in N_0$ there exists a solution $\overline{x}_i$ of the Dirichlet problem for* (1.3) *such that* $\operatorname{div}(H_z(y,\nabla\overline{x}_i(y)) \in L^2(\Omega,\mathbb{R})$ *and $\overline{x}_i \in \overline{X}_i$. If $\overline{X}_i \cap \overline{X}_j = \emptyset$ for all $i,j \in N_0$, $i \ne j$, we obtain $\#S \ge \#N_0$, where S denotes the set of solutions for* (1.3).

3. Dependence of solutions on function parameters. In this section we shall consider the continuous dependence of solutions on parameters for the elliptic PDE with function parameters and boundary conditions of Dirichlet type described by (1.2).

Hypothesis (H3). $\widetilde{G}(y,x,u) = F(y,x) + xg(y,u)$, where $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function, $\Omega \ni y \mapsto g(y,u(y))$ belongs to $L^2(\Omega,\mathbb{R})$ for all $u \in U$, and F satisfies hypothesis (H) (with $G = F$).

For each $u \in U$ we define X_u to be the largest subset of $\overline{X}$ having the property: for every $x \in X_u$, there exists $\widetilde{x} \in X_u$ such that

$$\begin{aligned}\int_\Omega \{F(y,x(y)) + F^*(y, -\operatorname{div} H_z(y,\nabla\widetilde{x}(y)) - g(y,u(y)))\\ + x(y)[\operatorname{div} H_z(y,\nabla\widetilde{x}(y)) + g(y,u(y))]\}\,dy = 0.\end{aligned} \tag{3.1}$$

Hypothesis (H4). For all $u \in U$, $X_u \ne \emptyset$.

Let

$$\begin{aligned}X_u^d := \{p \in L^2(\Omega,\mathbb{R}^n);\ \text{there exists } x \in X_u \text{ such that}\\ p(y) = H_z(y,\nabla x(y)) \text{ for a.e. } y \in \Omega\}.\end{aligned} \tag{3.2}$$

Without loss of generality we assume that $0 \in U$ and $g(y,0) = 0$ a.e. on Ω.

Theorem 3.1. *Assume hypotheses* (H3) *and* (H4)*. Let $\{u_m\}_{m\in\mathbb{N}} \subset U$ be a sequence such that $\{g(\cdot,u_m(\cdot))\}_{m\in\mathbb{N}}$ converges weakly to $g(\cdot,0)$ in $L^2(\Omega,\mathbb{R})$.*

For each $m \in \mathbb{N}$ let $x_m \in X_{u_m}$ satisfy

$$(3.3)\quad -\operatorname{div} H_z(y, \nabla x_m(y)) - g(y, u_m(y)) = F_x(y, x_m(y)) \quad \text{for a.e. } y \in \Omega.$$

Then there exists $x_0 \in X_0$ which is a solution of the equation

$$(3.4)\qquad -\operatorname{div} H_z(y, \nabla x(y)) = F_x(y, x(y)) \quad \text{for a.e. } y \in \Omega$$

such that $\{x_m\}_{m\in\mathbb{N}}$ converges weakly to x_0 in $W_0^{1,2}(\Omega, \mathbb{R})$.

Proof. Using the definition of $\overline{X}$ we derive that $x_m \rightharpoonup x_0$ in $W_0^{1,2}(\Omega, \mathbb{R})$ as $m \to \infty$, and consequently, passing to a subsequence if necessary, $x_m \to x_0$ in $L^2(\Omega, \mathbb{R})$ as $m \to \infty$. Consider the sequence $\{p_m\}_{m\in\mathbb{N}}$ given by

$$(3.5)\qquad p_m(y) = H_z(y, \nabla x_m(y)) \quad \text{a.e. on } \Omega.$$

An analysis similar to that in the proof of Theorem 2.9 shows that $p_m \rightharpoonup p_0$, where $p_0 \in L^2(\Omega, \mathbb{R}^n)$, $\operatorname{div} p_0 \in L^2(\Omega, \mathbb{R})$ and $\operatorname{div} p_m \rightharpoonup \operatorname{div} p_0$. Combining (3.5) and (3.3) we have

$$\begin{aligned}(3.6)\quad 0 &= \lim_{m\to\infty} \int_\Omega \{H^*(y, p_m(y)) + H(y, \nabla x_m(y)) - \langle p_m(y), \nabla x_m(y)\rangle\}\, dy \\ &\geq \int_\Omega \{H^*(y, p_0(y)) + H(y, \nabla x_0(y)) - \langle p_0(y), \nabla x_0(y)\rangle\}\, dy,\end{aligned}$$

and

$$\begin{aligned}(3.7)\quad 0 &= \lim_{m\to\infty} \int_\Omega \{F(y, x_m(y)) + F^*(y, -\operatorname{div} p_m(y) - g(y, u_m(y))) \\ &\qquad + x_m(y) \operatorname{div} p_m(y) + x_m(y) g(y, u_m(y))\}\, dy \\ &\geq \int_\Omega [F(y, x_0(y)) + F^*(y, -\operatorname{div} p_0(y))\, dy + x_0(y) \operatorname{div} p_0(y)]\, dy.\end{aligned}$$

Thus, by the properties of the Fenchel transform and the subdifferential, we obtain for a.e. $y \in \Omega$,

$$p_0(y) = H_z(y, \nabla x_0(y)) \quad \text{and} \quad -\operatorname{div} p_0(y) = F_x(y, x_0(y)).$$

This gives (3.4) and the relation $x_0 \in X_0$. ∎

Example 2. Suppose that the assumptions of Example 1 are satisfied. Let us consider the following sequence of problems: for each $m \in \mathbb{N}$,

$$\begin{aligned}(3.8)\quad -\operatorname{div}((\|y\|_{\mathbb{R}^4}^2 + 1)\nabla x(y)) = a(y)\Big[&9(x(y))^8 - 40(x(y))^7 \\ &- 14(x(y))^6 + 144(x(y))^5 + 1 + \frac{\|y\|_{\mathbb{R}^4}}{[\|y\|_{\mathbb{R}^4}^2 + 1]m}\Big]\end{aligned}$$

for a.e. $y \in \Omega$ with Dirichlet boundary condition $x|_{\partial\Omega} = 0$. An analysis similar to that in Example 1 shows that for each $m \in \mathbb{N}$, (3.8) has a solution $x_m \in W_0^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ with $\Omega \ni y \mapsto \operatorname{div}((\|y\|_{\mathbb{R}^4}^2 + 1)\nabla x_m(y))$ belonging to $L^2(\Omega, \mathbb{R})$. Hence, by the uniform convergence of $\{\|y\|_{\mathbb{R}^4}/[(\|y\|_{\mathbb{R}^4}^2 + 1)m]\}_{m\in\mathbb{N}}$

to 0 in Ω, Theorem 3.1 leads to the conclusion that $\{x_m\}_{m\in\mathbb{N}}$ converges weakly to x_0 in $W_0^{1,2}(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$, where x_0 is a solution of the Dirichlet problem for the PDE

$$\begin{aligned}&-\operatorname{div}((\|y\|^2_{\mathbb{R}^4}+1)\nabla x(y))\\&\qquad= a(y)[9(x(y))^8-40(x(y))^7-14(x(y))^6+144(x(y))^5+1].\end{aligned}$$

for a.e. $y\in\Omega$ and $\Omega\ni y\mapsto \operatorname{div}((\|y\|^2_{\mathbb{R}^4}+1)\nabla x_m(y))$ belongs to $L^2(\Omega,\mathbb{R})$.

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