COLLOQUIUM MATHEMATICUM

VOL. 96 2003 NO. 2

ON SOME GENERALIZED EINSTEIN METRIC CONDITIONS ON HYPERSURFACES IN SEMI-RIEMANNIAN SPACE FORMS

BY

RYSZARD DESZCZ (Wrocław), MAŁGORZATA GŁOGOWSKA (Wrocław), MARIAN HOTLOŚ (Wrocław) and LEOPOLD VERSTRAELEN (Leuven)

Dedicated to the memory of Professor Stanisław Gołąb

Abstract. Solutions of the P. J. Ryan problem as well as investigations of curvature properties of Cartan hypersurfaces and Ricci-pseudosymmetric hypersurfaces lead to curvature identities holding on every hypersurface M isometrically immersed in a semi-Riemannian space form. These identities, under some assumptions, give rises to new generalized Einstein metric conditions on M. We investigate hypersurfaces satisfying such curvature conditions.

1. Some generalized Einstein metric conditions. In [14, Theorem 3.1] a curvature property of pseudosymmetry type of Einstein manifolds was found. It was shown that on any semi-Riemannian Einstein manifold (M,g), $n \geq 4$, the following identity holds:

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}\, Q(g,R) = \frac{\kappa}{(n-1)n}\, Q(g,C).$$

For precise definitions of the symbols used we refer to Sections 2 and 3 of the present paper. The above theorem gives rise to a family of curvature conditions of pseudosymmetry type ([14]). In particular, curvature properties of non-Einstein and non-conformally flat semi-Riemannian manifolds of dimension ≥ 4 satisfying at every point the condition: the tensors $R\cdot C - C\cdot R$ and $Q(g,C)$ are linearly dependent, were investigated in [14]. This condition is equivalent on $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$ to

$$R \cdot C - C \cdot R = L_1\, Q(g,C), \tag{1}$$

2000 *Mathematics Subject Classification*: 53B20, 53B25, 53B30, 53B50, 53C25, 53C40.

Key words and phrases: Einstein manifold, quasi-Einstein manifold, pseudosymmetry type manifold, warped product, hypersurface, the Ryan problem.

The first and third named authors were supported by the grant 2 P03A 006 17 from the Polish State Committee of Scientific Research (KBN). The second named author was supported by a grant from the Ministerie van de Vlaamse Gemeenschap (Belgium) and the fourth named author was supported by the Katholieke Universiteit Leuven (Belgium).

where L_1 is some function on $\mathcal{U}_C$. In [14, Theorem 4.1] it was shown that if (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying (1) then on $\mathcal{U}_S \cap \mathcal{U}_C$ we have $R \cdot R = L_1 Q(g, R)$ and $C \cdot R = 0$, where $\mathcal{U}_S = \{x \in M \,|\, S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$.

Curvature properties of semi-Riemannian manifolds satisfying at every point the condition: the tensors $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent, were investigated in [12]. This condition is equivalent on $\mathcal{U}_R = \{x \in M \,|\, R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$ to

$$R \cdot C - C \cdot R = L_2 \, Q(g, R), \tag{2}$$

where L_2 is some function on $\mathcal{U}_R$. In [12, Theorem 4.2] it was shown that if (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying (2) then $R \cdot R = 0$ on $\mathcal{U}_S \cap \mathcal{U}_C$.

The study of semi-Riemannian manifolds satisfying at every point the condition: the tensors $R \cdot C - C \cdot R$ and $Q(S, R)$ are linearly dependent, was initiated in [22]. This condition is equivalent on $\mathcal{U}_3 = \{x \in M \mid Q(S, R) \neq 0 \text{ at } x\}$ to

$$R \cdot C - C \cdot R = L_3 Q(S, R), \tag{3}$$

where L_3 is some function on $\mathcal{U}_3$. In [22] it was shown that if (M, g), $n \geq 4$, is a Ricci-semisymmetric ($R \cdot S = 0$) semi-Riemannian manifold satisfying (3) then at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ at which L_3 does not vanish we have

$$R \cdot C - C \cdot R = \frac{1}{n-2} \, Q(S, R). \tag{4}$$

In Section 5 we consider hypersurfaces of semi-Euclidean spaces $\mathbb{E}^{n+1}_s$ with signature $(s, n + 1 - s)$, $n \geq 4$, satisfying (4).

We can also investigate semi-Riemannian manifolds satisfying at every point the condition: the tensors $R \cdot C - C \cdot R$ and $Q(S, C)$ are linearly dependent. This condition is equivalent on $\mathcal{U}_4 = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$ to

$$R \cdot C - C \cdot R = L_4 Q(S, C), \tag{5}$$

where L_4 is some function on $\mathcal{U}_4$. In this paper we present results on hypersurfaces of $\mathbb{E}^{n+1}_s$, $n \geq 4$, satisfying (5). Semi-Riemannian manifolds satisfying (5) will be investigated in subsequent papers.

(1)–(5) as well as other conditions of this kind are called *generalized Einstein metric conditions* ([12], [14]) and also *curvature conditions of pseudosymmetry type*. Recently, a review of results on semi-Riemannian manifolds satisfying such conditions was given in [3] (see also [6] and [24]).

Let M be a hypersurface in a semi-Riemannian space of constant curvature $N^{n+1}_s(c)$ with signature $(s, n + 1 - s)$, $n \geq 4$. We denote by $\mathcal{U}_H$ the set of all points of M at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H of M. It is known that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C$.

Let now M be a hypersurface in a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 4$. The following results pertain to (4).

THEOREM 1.1. *Let M be a Ricci-semisymmetric hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$.*

(i) ([13, Lemma 3.1]) *On $\mathcal{U}_H \subset M$ we have $H^3 = \mathrm{tr}(H)H^2 + \lambda H$ and*

$$R \cdot C - C \cdot R = \frac{1}{n-2}\, Q(S,R) - \frac{1}{n-2}\left(\varepsilon\lambda + \frac{\kappa}{n-1}\right) Q(g,R), \tag{6}$$

where λ is some function on $\mathcal{U}_H$.

(ii) ([15, Theorem 5.1]) *In addition, if M is a quasi-Einstein hypersurface then on $\mathcal{U}_H$,* (6) *reduces to* (4).

Curvature properties of Ricci-pseudosymmetric hypersurfaces in semi-Riemannian spaces of constant curvature $N^{n+1}_s(c)$, $n \geq 4$, were investigated in [4], [8], [9], [18] and [19], among others. From Proposition 3.2 and Theorem 3.1 of [4] it follows that for every Ricci-pseudosymmetric hypersurface M in $N^{n+1}_s(c)$, $n \geq 4$, on the set $\mathcal{U}_H \subset M$ we have

$$R \cdot S = \frac{\tau}{n(n+1)}\, Q(g,S), \tag{7}$$

where τ is the scalar curvature of the ambient space. In [21] a curvature characterization of pseudosymmetry type of Ricci-pseudosymmetric hypersurfaces M in $N^{n+1}_s(c)$, $n \geq 4$, was found. Namely, we have

THEOREM 1.2 ([21, Proposition 5.1(iii) and Theorem 6.1]). *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$. On $\mathcal{U}_H \subset M$,* (7) *is equivalent to*

$$R \cdot C = Q(S,R) - \frac{(n-2)\tau}{n(n+1)}\, Q(g,R) - \frac{(n-3)\tau}{(n-2)n(n+1)}\, Q(S,G). \tag{8}$$

Cartan hypersurfaces are Ricci-pseudosymmetric ([18], [19]). In [8] further curvature properties of pseudosymmetry type for Cartan hypersurfaces of dimension ≥ 6 were found.

THEOREM 1.3 ([8, Theorem 4.3]). *On every Cartan hypersurface M in $S^{n+1}(c)$, $n = 6, 12$ or 24, we have*: (7), (8),

$$\begin{aligned} C \cdot R = \frac{n-3}{n-2}\, Q(S,R) &- \frac{(n-3)\tau}{(n-1)(n+1)}\, Q(g,R) \\ &- \frac{(n-3)\tau}{(n-2)n(n+1)}\, Q(S,G), \end{aligned}$$

$$R \cdot C - C \cdot R = \frac{1}{n-2}\, Q(S,R) - \frac{2\tau}{(n-1)n(n+1)}\, Q(g,R). \tag{9}$$

In Section 3 we consider an extension of the standard Kulkarni–Nomizu product $E \wedge F$ of two $(0,2)$-tensors E and F. Namely, we define the Kulkarni–Nomizu product $Q(E,T)$ of a $(0,2)$-tensor E and a $(0,k)$-tensor T, $k \geq 2$ (see [8]). We present some properties of this product. We use these properties to prove (see Theorem 3.1) that on any hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, the following identities hold:

$$R \cdot C = Q(S,R) - \frac{(n-2)\tau}{n(n+1)}\, Q(g,R) - \frac{(n-3)\tau}{(n-2)n(n+1)}\, Q(S,G) + \frac{1}{n-2}\, g \wedge Q(H,A), \tag{10}$$

$$C \cdot R = \frac{n-3}{n-2}\, Q(S,R) - \frac{(n^2-3n+3)\tau}{(n-2)n(n+1)}\, Q(g,R) - \frac{(n-3)\tau}{(n-2)n(n+1)}\, Q(S,G) + \frac{1}{n-2}\, H \wedge Q(g,A)\,, \tag{11}$$

$$R \cdot C - C \cdot R = \frac{1}{n-2}\, Q(S,R) + \frac{(n-1)\tau}{(n-2)n(n+1)}\, Q(g,R) + \frac{1}{n-2}\, (g \wedge Q(H,A) - H \wedge Q(g,A))\,, \tag{12}$$

where τ, g and H are the scalar curvature of $N_s^{n+1}(c)$, the metric tensor of M and the second fundamental tensor of M, respectively. The $(0,2)$-tensor A is defined by

$$A = H^3 - \operatorname{tr}(H)\, H^2 + \frac{\varepsilon\kappa}{n-1}\, H. \tag{13}$$

We mention that from Theorem 5.1 of [15] it follows that A vanishes on the subset $\mathcal{U}_H$ of any quasi-Einstein Ricci-semisymmetric hypersurface M in $\mathbb{E}_s^{n+1}$, $n \geq 4$. In Section 5 we prove that (4) holds on the subset $\mathcal{U}_H$ of a hypersurface M in $\mathbb{E}_s^{n+1}$, $n \geq 4$, if and only if $A = 0$ on $\mathcal{U}_H$. We also present examples of hypersurfaces with nonzero A.

From Proposition 5.2 of [21] it follows that if on the subset $\mathcal{U}_H$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have

$$\sum_{(X_1,X_2),\,(X_3,X_4),\,(X,Y)} (R \cdot C)(X_1,X_2,X_3,X_4;X,Y) = 0, \tag{14}$$

then

$$A = \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right) H + \varrho g,$$
$$\varrho = \frac{1}{n}\left(\operatorname{tr}(A) - \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right)\operatorname{tr}(H)\right), \tag{15}$$

on $\mathcal{U}_H$, where λ is some function on $\mathcal{U}_H$. In Section 4 we prove (see Proposition 4.1) that the following conditions: (14),

$$\sum_{(X_1,X_2),\,(X_3,X_4),\,(X,Y)} (R\cdot C - C\cdot R)(X_1,X_2,X_3,X_4;X,Y) = 0, \tag{16}$$

$$\sum_{(X_1,X_2),\,(X_3,X_4),\,(X,Y)} (C\cdot R)(X_1,X_2,X_3,X_4;X,Y) = 0 \tag{17}$$

are equivalent on any semi-Riemannian manifold of dimension ≥ 4. Thus on the subset $\mathcal{U}_H$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, each of the condition (14), (16), (17) implies (15) on $\mathcal{U}_H$ (see Theorem 4.1).

2. Preliminaries. Throughout this paper all manifolds are assumed to be connected paracompact of class C^∞. Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where $X,Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. We define the endomorphisms $X\wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ of $\Xi(M)$ by $(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y$, $\mathcal{R}(X,Y)Z = [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z$ and

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X\wedge_g \mathcal{S}Y + \mathcal{S}X\wedge_g Y - \frac{\kappa}{n-1}X\wedge_g Y\right)Z,$$

where $X,Y,Z \in \Xi(M)$ and A is a symmetric $(0,2)$-tensor. Now the Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0,4)$-tensor G of (M,g) are defined by

$$\begin{aligned} R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4),\\ G(X_1,X_2,X_3,X_4) &= g((X_1\wedge_g X_2)X_3,X_4), \end{aligned}$$

where $X,Y,Z,X_1,X_2,\ldots \in \Xi(M)$. Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4). \tag{18}$$

B is said to be a *generalized curvature tensor* if

$$\begin{aligned} &B(X_1,X_2,X_3,X_4) + B(X_2,X_3,X_1,X_4) + B(X_3,X_1,X_2,X_4) = 0,\\ &B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2). \end{aligned}$$

Clearly, R, C and G are generalized curvature tensors.

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the tensor defined by (18). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$ for any smooth function on M. Now for a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0,k+2)$-tensor $B \cdot T$ by

$$(B \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{B}(X,Y) \cdot T)(X_1, \ldots, X_k; X, Y)$$
$$= -T(\mathcal{B}(X,Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{B}(X,Y)X_k).$$

In addition, if A is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$Q(A,T)(X_1, \ldots, X_k; X, Y) = (X \wedge_A Y \cdot T)(X_1, \ldots, X_k; X, Y)$$
$$= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).$$

In particular, in this manner, we obtain the $(0,6)$-tensors $B \cdot B$ and $Q(A,B)$. Setting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{C}$, $T = R$, C or S, $A = g$ or S, we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot S$, $C \cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(g,S)$.

Let M, $n = \dim M \geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold (N, g^N). We denote by g the metric tensor of M induced from g^N. Further, we denote by ∇ and ∇^N the Levi-Civita connections corresponding to g and g^N, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can write the Gauss formula and the Weingarten formula of (M,g) in (N, g^N) in the forms $\nabla^N_X Y = \nabla_X Y + \varepsilon H(X,Y)\xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor of (M,g) in (N, g^N), $\mathcal{A}$ is the shape operator and $H^k(X,Y) = g(\mathcal{A}^k X, Y)$, $k \geq 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and R^N the Riemann–Christoffel curvature tensors of (M,g) and (N, g^N), respectively. The *Gauss equation* of (M,g) in (N, g^N) has the form $R(X_1, \ldots, X_4) = R^N(X_1, \ldots, X_4) + \frac{\varepsilon}{2}(H \wedge H)(X_1, \ldots, X_4)$, where $X_1, \ldots, X_4$ are vector fields tangent to M.

Let the equations $x^r = x^r(y^k)$ be the local parametric expression of (M,g) in (N, g^N), where y^k and x^r are the local coordinates of M and N, respectively, and $h, i, j, k \in \{1, \ldots, n\}$ and $p, r, t, u \in \{1, \ldots, n+1\}$. Now the Gauss equation yields

$$R_{hijk} = R^N_{prtu} B^p_h B^r_i B^t_j B^u_k + \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}), \tag{19}$$

where $B_k^{\ r} = \partial x^r / \partial y^k$, R^N_{rstu}, R_{hijk} and H_{hk} are the local components of the tensors R^N, R and H, respectively. If M is a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$, then (19) becomes

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\tau}{n(n+1)} G_{hijk}, \tag{20}$$

where τ is the scalar curvature of the ambient space and G_{hijk} are the local components of the tensor G. Contracting (20) with g^{ij} and g^{kh}, respectively, we obtain

$$S_{hk} = \varepsilon(\mathrm{tr}(H)H_{hk} - H^2_{hk}) + \frac{(n-1)\tau}{n(n+1)}\, g_{hk} \tag{21}$$

and

$$\kappa = \varepsilon((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)) + \frac{(n-1)\tau}{n+1},$$

respectively, where $\mathrm{tr}(H) = g^{hk}H_{hk}$, $\mathrm{tr}(H^2) = g^{hk}H^2_{hk}$ and S_{hk} are the local components of the Ricci tensor S of M. Using (21) and Theorem 4.1 of [17] we can deduce that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$. It is known that at every point of a hypersurface M in $N^{n+1}_s(c)$, $n \geq 4$, the following condition of pseudosymmetry type holds ([6, Section 5.5], [17]): the tensors $R\cdot R - Q(S,R)$ and $Q(g,C)$ are linearly dependent. Precisely, on M we have

$$R\cdot R - Q(S,R) = -\frac{(n-2)\tau}{n(n+1)}\, Q(g,C). \tag{22}$$

Evidently, if the ambient space is $\mathbb{E}^{n+1}_s$ then (22) reduces to $R\cdot R = Q(S,R)$.

3. The basic identities. For symmetric $(0,2)$-tensors E and F we define their Kulkarni–Nomizu product $E \wedge F$ by

$$\begin{aligned}(E\wedge F)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)F(X_2,X_3) + E(X_2,X_3)F(X_1,X_4)\\ &\quad - E(X_1,X_3)F(X_2,X_4) - E(X_2,X_4)F(X_1,X_3).\end{aligned}$$

The tensor $E\wedge F$ is also a generalized curvature tensor. For a symmetric $(0,2)$-tensor E we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E} = \frac{1}{2}\, E\wedge E$. In particular, $\overline{g} = G = \frac{1}{2}\, g\wedge g$. We note that the Weyl tensor C can be represented in the form

$$C = R - \frac{1}{n-2}\, g\wedge S + \frac{\kappa}{(n-2)(n-1)}\, G. \tag{23}$$

We also have (see e.g. [9, Section 3])

$$Q(E, E\wedge F) = -Q(F, \overline{E}). \tag{24}$$

Lemma 3.1. *Let E be a symmetric $(0,2)$-tensor at a point x of a semi-Riemannian manifold (M,g), $n \geq 3$.*

(i) ([2, Lemma 2.2]) *If*

$$E = \alpha g + \beta u\otimes u, \qquad \alpha,\beta \in \mathbb{R}\ u \in T^*_x M, \tag{25}$$

then at x we have

$$E^2 = \widetilde{\alpha}E + \widetilde{\beta}g, \qquad \widetilde{\alpha},\widetilde{\beta} \in \mathbb{R}. \tag{26}$$

(ii) ([20, Lemma 3.1]) *Let $\mathcal{U}_E$ be the set of all points of M at which E is not proportional to g. If, at some $x \in \mathcal{U}_E$,*

$$E\wedge E = 2\alpha g\wedge E + 2\beta G, \qquad \alpha,\beta\in\mathbb{R}, \tag{27}$$

then at x we have (25) *with* $\alpha^2 = -\beta$.

According to [8], for a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k \geq 2$, we define their Kulkarni–Nomizu product $E \wedge T$ by

$$\begin{aligned}
&(E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \dots, Y_k) \\
&\quad = E(X_1, X_4) T(X_2, X_3, Y_3, \dots, Y_k) + E(X_2, X_3) T(X_1, X_4, Y_3, \dots, Y_k) \\
&\qquad - E(X_1, X_3) T(X_2, X_4, Y_3, \dots, Y_k) - E(X_2, X_4) T(X_1, X_3, Y_3, \dots, Y_k).
\end{aligned}$$

Using the above definitions we can prove the following

LEMMA 3.2 ([21]). *Let E_1, E_2 and F be symmetric $(0,2)$-tensors at a point x of a semi-Riemannian manifold (M,g), $n \geq 3$. Then at x we have*

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) = -Q(F, E_1 \wedge E_2).$$

If $E = E_1 = E_2$ then ([8])

$$E \wedge Q(E, F) = -Q(F, \overline{E}). \tag{28}$$

As an immediate consequence of (24) and (28) we have

$$E \wedge Q(E, F) = Q(E, E \wedge F). \tag{29}$$

By making use of (15), Propositions 5.1 and 5.2 of [21] imply

PROPOSITION 3.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*

(i) *$R \cdot S = Q(A, H) + \frac{\tau}{n(n+1)} Q(g, S)$ on M.*

(ii) *If, in addition, M is a Ricci-pseudosymmetric manifold then* (7) *holds on $\mathcal{U}_H$.*

(iii) *On M,*

$$\begin{aligned}
R \cdot C = Q(S, R) - \frac{1}{n-2}\, g \wedge Q(A, H) \\
+ \frac{\tau}{n(n+1)} \left(\frac{1}{n-2}\, Q(S, G) - (n-2)\, Q(g, C) \right).
\end{aligned} \tag{30}$$

(iv) *In particular, if M is a Ricci-pseudosymmetric hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$, then $R \cdot C = Q(S, R)$ on $\mathcal{U}_H$.*

(v) *Let M be a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying* (14). *Then on $\mathcal{U}_H$ we have* (15) *and*

$$R \cdot S = -\frac{\mu}{n}\, Q(g, H), \qquad R \cdot C = Q(S, R) - \frac{\mu}{(n-2)n}\, Q(H, G),$$

where λ is some function on $\mathcal{U}_H$ and $\mu = \operatorname{tr}(H)\operatorname{tr}(H^2) - \operatorname{tr}(H^3) + \lambda \operatorname{tr}(H)$.

THEOREM 3.1. *The identities* (10)–(12) *hold on every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$. In particular, on every hypersurface M in $\mathbb{E}_s^{n+1}$, $n \geq 4$, we have*

$$\begin{aligned}
R \cdot C - C \cdot R = \frac{1}{n-2}\, Q(S, R) \\
+ \frac{1}{n-2}\, (g \wedge Q(H, A) - H \wedge Q(g, A)).
\end{aligned} \tag{31}$$

Proof. Applying the relations (23) and (24) in (30) we get (10) easily. From (20), by transvection with $H_l^h = g^{rh}H_{lr}$, we obtain

$$H_l^r R_{rijk} = \varepsilon(H_{ij}H^2_{lk} - H_{ik}H^2_{lj}) + \frac{\tau}{n(n+1)}\,(g_{ij}H_{lk} - g_{ik}H_{lj}),$$

which implies

$$R \cdot H = \varepsilon Q(H, H^2) + \frac{\tau}{n(n+1)}\,Q(g, H). \tag{32}$$

Further, from (20) we also get

$$\begin{aligned} R - \frac{1}{n-2}\Big(g \wedge S + \frac{\kappa}{n-1}\,G\Big) \\ = \varepsilon\overline{H} - \frac{1}{n-2}\,g \wedge S + \Big(\frac{\kappa}{(n-2)(n-1)} + \frac{\tau}{n(n+1)}\Big)G, \end{aligned}$$

which, by making use of (21) and (23), turns into

$$\begin{aligned} C = \varepsilon\overline{H} + \frac{\varepsilon}{n-2}\,g \wedge (H^2 - \operatorname{tr}(H)H) \\ + \frac{1}{n-2}\Big(\frac{\kappa}{n-1} - \frac{\tau}{n+1}\Big)G. \end{aligned} \tag{33}$$

(33), by suitable transvection and application of (13) and the definitions of $R \cdot T$ and $Q(E, T)$, leads to

$$\begin{aligned} C \cdot H = \frac{n-3}{n-2}\,\varepsilon Q(H, H^2) + \frac{\varepsilon}{n-2}\,Q(g, A) \\ - \frac{\tau}{(n-2)(n+1)}\,Q(g, H). \end{aligned} \tag{34}$$

But (34), in view of (32), yields (see Theorem 3.4 of [2])

$$C \cdot H = \frac{n-3}{n-2}\,R \cdot H + \frac{\varepsilon}{n-2}\,Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)}\,Q(g, H).$$

Using this, (20), (22) and (28) we find

$$\begin{aligned} C \cdot R = \varepsilon H \wedge (C \cdot H) &= \frac{(n-3)\varepsilon}{n-2}\,H \wedge (R \cdot H) \\ &\quad + \frac{1}{n-2}\,H \wedge Q(g, A) - \frac{(2n-3)\varepsilon\tau}{(n-2)n(n+1)}\,H \wedge Q(g, H) \\ &= \frac{(n-3)\varepsilon}{n-2}\,(R \cdot \overline{H}) + \frac{1}{n-2}\,H \wedge Q(g, A) \\ &\quad - \frac{(2n-3)\varepsilon\tau}{(n-2)n(n+1)}\,Q(g, \overline{H}) \\ &= \frac{n-3}{n-2}\,(R \cdot R) + \frac{1}{n-2}\,H \wedge Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)}\,Q(g, R) \end{aligned} \tag{35}$$

$$= \frac{n-3}{n-2}\, Q(S,R) - \frac{(n-3)\tau}{n(n+1)}\, Q(g,C) + \frac{1}{n-2}\, H \wedge Q(g,A) - \frac{(2n-3)\tau}{(n-2)n(n+1)}\, Q(g,R).$$

From this, by making use of (23) and (24), we get (11). Further, (35) together with (30) yields

$$\begin{aligned} R\cdot C - C\cdot R = {} & \frac{1}{n-2}\, Q(S,R) - \frac{\tau}{n(n+1)}\, Q(g,C) \\ & + \frac{1}{n-2}\,(g\wedge Q(H,A) - H\wedge Q(g,A)) \\ & + \frac{\tau}{(n-2)n(n+1)}\, Q(S,G) + \frac{(2n-3)\tau}{(n-2)n(n+1)}\, Q(g,R). \end{aligned}$$

Applying now (23) and (24) we get (12). Finally, we note that (31) is an immediate consequence of (12). Our theorem is thus proved.

THEOREM 3.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If* (15) *is satisfied on $\mathcal{U}_H \subset M$ then on this set we have*

$$\begin{aligned} R\cdot C - C\cdot R = {} & \frac{1}{n-2}\, Q(S,R) + \frac{(n-1)\tau}{(n-2)n(n+1)}\, Q(g,R) \\ & + \frac{1}{n-2}\left(\varrho Q(H,G) - \varepsilon\left(\lambda + \frac{\varepsilon\kappa}{n-1}\right)Q(g,R)\right). \end{aligned} \tag{36}$$

Proof. This is a consequence of Theorem 3.1 and (20) and (28).

4. Some curvature conditions. Let (M,g) be covered by a system of charts $\{W; x^k\}$. We denote by g_{ij}, R_{hijk}, S_{ij}, $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$\begin{aligned} C_{hijk} = {} & R_{hijk} + \frac{\kappa}{(n-2)(n-1)}\, G_{hijk} \\ & - \frac{1}{n-2}\,(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \end{aligned} \tag{37}$$

the local components of the tensors g, R, S, G and C, respectively. Further, we denote by $S^2_{ij} = S_{ir}S_j{}^r$ and $S_i{}^j = g^{jr}S_{ir}$ the local components of the tensor S^2 defined by $S^2(X,Y) = S(\mathcal{S}X,Y)$, and of the Ricci operator $\mathcal{S}$, respectively. Let $(R\cdot C)_{hijklm}$ and $(C\cdot R)_{hijklm}$ denote the local components of $R\cdot C$ and $C\cdot R$, respectively. We have

$$(R\cdot C)_{hijklm} = g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}), \tag{38}$$

$$(C\cdot R)_{hijklm} = g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}), \tag{39}$$

respectively. Applying (37) in (38) and (39) we get

$$
\begin{aligned}
(40)\qquad (R\cdot C)_{hijklm} &= (R\cdot R)_{hijklm} \\
&\quad - \frac{1}{n-2}\,(g_{ij}(V_{hklm}+V_{khlm}) + g_{hk}(V_{ijlm}+V_{jilm}) \\
&\quad - g_{ik}(V_{hjlm}+V_{jhlm}) - g_{hj}(V_{iklm}+V_{kilm}),
\end{aligned}
$$

$$
\begin{aligned}
(41)\qquad (C\cdot R)_{hijklm} &= (R\cdot R)_{hijklm} - \frac{1}{n-2}\,Q(S,R)_{hijklm} \\
&\quad + \frac{\kappa}{(n-1)(n-2)}\,Q(g,R)_{hijklm} \\
&\quad - \frac{1}{n-2}\,(g_{hl}V_{mijk} - g_{hm}V_{lijk} - g_{il}V_{mhjk} + g_{im}V_{lhjk} \\
&\quad + g_{jl}V_{mkhi} - g_{jm}V_{lkhi} - g_{kl}V_{mjhi} + g_{km}V_{ljhi}),
\end{aligned}
$$

$$
(42)\qquad V_{mijk} = S_m^{\ s}R_{sijk},
$$

where $(R\cdot R)_{hijklm}$, $Q(S,R)_{hijklm}$, $Q(g,R)_{hijklm}$ and $Q(g,C)_{hijklm}$ are the local components of the respective tensors. Using (40) and (41) we obtain ([13, Section 2])

$$
\begin{aligned}
(43)\qquad (n-2)(R\cdot C - C\cdot R)_{hijklm} &= Q(S,R)_{hijklm} \\
&\quad - \frac{\kappa}{n-1}\,Q(g,R)_{hijklm} + g_{hl}V_{mijk} - g_{hm}V_{lijk} - g_{il}V_{mhjk} \\
&\quad + g_{im}V_{lhjk} + g_{jl}V_{mkhi} - g_{jm}V_{lkhi} - g_{kl}V_{mjhi} + g_{km}V_{ljhi} \\
&\quad - g_{ij}(V_{hklm}+V_{khlm}) - g_{hk}(V_{ijlm}+V_{jilm}) \\
&\quad + g_{ik}(V_{hjlm}+V_{jhlm}) + g_{hj}(V_{iklm}+V_{kilm}).
\end{aligned}
$$

LEMMA 4.1 ([5, Lemma 1.1(iii)]). *Let B be a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n \geq 3$. The tensor $Q(g,B)$ vanishes at a point $x \in M$ if and only if $B = \frac{\kappa(B)}{(n-1)n}\,G$ at x.*

LEMMA 4.2 (cf. [10, Lemma 3.4]). *Let (M,g), $n \geq 3$, be a semi-Riemannian manifold. Let E be a nonzero symmetric $(0,2)$-tensor at a point $x \in M$ and let B be a generalized curvature tensor such that $Q(E,B) = 0$ at x. Moreover, let Y be a vector at x such that the scalar $\varrho = a(Y)$ is nonzero, where a is the covector defined by $a(X) = E(X,Y)$, $X \in T_xM$. Then at x we have two possibilities:*

(i) *the tensor E is of rank one (precisely, $E = \frac{1}{\varrho}\,a\otimes a$), or*

(ii) *the tensor $E - \frac{1}{\varrho}\,a\otimes a$ is nonzero and $B = \frac{\gamma}{2}\,E\wedge E$, $\gamma \in \mathbb{R}$.*

Using the above lemma and Lemmas 3.1 and 3.2 we can prove

LEMMA 4.3. *Let (M,g), $n \geq 3$, be a semi-Riemannian manifold. Let E be a nonzero symmetric $(0,2)$-tensor at a point $x \in M$. If at x we have $Q(E-\alpha g,\, g\wedge E) = 0$, $\alpha \in \mathbb{R}$, then (26) holds at x.*

Let $Q(E,B)_{hijklm}$ be the local components of $Q(E,B)$. We have ([5, Lemma 1.1(i)])

$$Q(E,B)_{hijklm} + Q(E,B)_{jklmhi} + Q(E,B)_{lmhijk} = 0. \tag{44}$$

On M we also have the well known Walker identity

$$(R\cdot R)_{hijklm} + (R\cdot R)_{jklmhi} + (R\cdot R)_{lmhijk} = 0. \tag{45}$$

PROPOSITION 4.1. *Let* (M,g), $n \geq 4$, *be a semi-Riemannian manifold. The equalities* (14), (16) *and* (17) *are equivalent on* M.

Proof. We set

$$\begin{aligned} P_{hijklm} = \frac{1}{n-2}\,((&g_{ij}(V_{hklm}+V_{khlm}) + g_{hk}(V_{ijlm}+V_{jilm}) \\ &- g_{ik}(V_{hjlm}+V_{jhlm}) - g_{hj}(V_{iklm}+V_{kilm}) \\ &+ g_{kl}(V_{mjhi}+V_{jmhi}) + g_{jm}(V_{klhi}+V_{lkhi}) \\ &- g_{km}(V_{jlhi}+V_{ljhi}) - g_{jl}(V_{kmhi}+V_{mkhi}) \\ &+ g_{mh}(V_{lijk}+V_{iljk}) + g_{li}(V_{mhjk}+V_{hmjk}) \\ &- g_{mi}(V_{lhjk}+V_{hljk}) - g_{lh}(V_{mijk}+V_{imjk})), \end{aligned}$$

where V_{hijk} are defined by (42). Symmetrizing (41) with respect to the pairs (h,i), (j,k) and (l,m) and applying (44) and (45) we obtain

$$(C\cdot R)_{hijklm} + (C\cdot R)_{jklmhi} + (C\cdot R)_{lmhijk} = P_{hijklm}.$$

In the same way, using (40), we have

$$(R\cdot C)_{hijklm} + (R\cdot C)_{jklmhi} + (R\cdot C)_{lmhijk} = -P_{hijklm}.$$

From the last two relations we get

$$\begin{aligned} (R\cdot C - C\cdot R)_{hijklm} &+ (R\cdot C - C\cdot R)_{jklmhi} \\ &+ (R\cdot C - C\cdot R)_{lmhijk} = -2\,P_{hijklm}. \end{aligned}$$

Now our assertion is obvious.

Proposition 5.2 of [21] and Proposition 4.1 yield

THEOREM 4.1. *If on the subset* $\mathcal{U}_H$ *in a hypersurface* M *of* $N_s^{n+1}(c)$, $n \geq 4$, *one of the conditions* (14), (16) *or* (17) *is satisfied then* (15) *holds on* $\mathcal{U}_H$.

Using (44), (45), Proposition 4.1 and Theorem 4.1 we immediately get

COROLLARY 4.1. *If on the subset* $\mathcal{U}_H$ *in a hypersurface* M *of* $N_s^{n+1}(c)$, $n \geq 4$, *one of the tensors* $R\cdot C$, $C\cdot R$ *or* $R\cdot C - R\cdot C$ *is a linear combination of* $R\cdot R$ *and of a finite sum of tensors of the form* $Q(E,B)$, *where* E *is a symmetric* $(0,2)$-*tensor and* B *a generalized curvature tensor*, *then* (15) *holds on* $\mathcal{U}_H$.

THEOREM 4.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If at every point of M the following two tensors are linearly dependent*:

(i) $R \cdot C - C \cdot R$ *and* $Q(g, C)$, *or*
(ii) $R \cdot C - C \cdot R$ *and* $Q(g, R)$, *or*
(iii) $R \cdot C - C \cdot R$ *and* $Q(S, R)$, *or*
(iv) $R \cdot C - C \cdot R$ *and* $Q(S, C)$,

then (15) *and* (36) *hold on* $\mathcal{U}_H \subset M$.

Proof. In case (i), resp. (ii), on $\mathcal{U}_H \subset M$ we have (1), resp. (2). Now, in view of Corollary 4.1, (15) holds on $\mathcal{U}_H$.

Consider case (iii) and let $x \in \mathcal{U}_H$. Assume that $Q(S, R)$ vanishes at x. Then (22) becomes $R \cdot R = -\frac{(n-2)\tau}{n(n+1)} Q(g, C)$, whence $R \cdot S = 0$. Thus

$$R \cdot C = R \cdot R - \frac{1}{n-2}\, g \wedge (R \cdot S) = -\frac{(n-2)\tau}{n(n+1)}\, Q(g, C).$$

Applying Corollary 4.1 we get (15). Clearly, if $Q(S, R)$ is nonzero at a point x then $x \in \mathcal{U}_H$. Thus (3) holds at x. Now Corollary 4.1 again implies (15).

Finally, consider case (iv) and let $x \in \mathcal{U}_H$. If $Q(S, C)$ is nonzero at x then (5) holds at x and Corollary 4.1 implies (15). Assume now that $Q(S, C) = 0$ at x. In view of Theorem 3.1 of [11], we get $R \cdot R = \frac{\kappa}{n-1} Q(g, R)$. This yields $R \cdot S = \frac{\kappa}{n-1} Q(g, S)$. Using (23) and (29) we find

$$\begin{aligned}
R \cdot C &= R \cdot R - \frac{1}{n-2}\, g \wedge (R \cdot S) \\
&= \frac{\kappa}{n-1} \left(Q(g, R) - \frac{1}{n-2}\, g \wedge Q(g, S) \right) \\
&= \frac{\kappa}{n-1} \left(Q(g, R) - \frac{1}{n-2}\, Q(g, g \wedge S) \right) \\
&= \frac{\kappa}{n-1} \left(Q(g, R) - \frac{1}{n-2}\, g \wedge S \right) \\
&= \frac{\kappa}{n-1} \left(Q(g, R) - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{n-1}\, G \right) = \frac{\kappa}{n-1}\, Q(g, C)\,.
\end{aligned}$$

Now, in view of Corollary 4.1, we obtain (15) on $\mathcal{U}_H$. Finally, from Theorem 3.2 it follows that (36) holds on $\mathcal{U}_H$. This completes the proof.

PROPOSITION 4.2. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold. If at a point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ its curvature tensor R is of the form*

$$R = \phi \overline{S} + \mu g \wedge S + \eta G, \qquad \phi, \mu, \eta \in \mathbb{R}, \tag{46}$$

then at x we have

$$\begin{aligned}
R \cdot C - C \cdot R &= \frac{1}{n-2}\, Q(S, R) + \left(\frac{(n-1)\mu - 1}{(n-2)\phi} + \frac{\kappa}{n-1} \right) Q(g, R) \\
&\quad + \frac{\mu((n-1)\mu - 1) - (n-1)\phi\eta}{(n-2)\phi}\, Q(S, G),
\end{aligned} \tag{47}$$

$$R \cdot C - C \cdot R = \left(\frac{1}{\phi}\left(\mu - \frac{1}{n-2}\right) + \frac{\kappa}{n-1}\right) Q(g,R) + \left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right) Q(S,G). \tag{48}$$

Proof. As shown in [12], (46) implies

$$V_{mijk} = (\alpha + \mu)(S_{mk}S_{ij} - S_{mj}S_{ik}) + \left(\frac{\alpha\mu}{\phi} + \eta\right)(S_{mk}g_{ij} - S_{mj}g_{ik}) + \beta(g_{mk}S_{ij} - g_{mj}S_{ik}) + \frac{\beta\mu}{\phi} G_{mijk}, \tag{49}$$

where $\alpha = \phi\kappa - 1 + (n-2)\mu$, $\beta = \mu\kappa + (n-1)\eta$ and

$$R \cdot S = (n-2)\left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right) Q(g,S). \tag{50}$$

Substituting (49) and (50) into (43) we get

$$(n-2)(R \cdot C - C \cdot R) = Q(S,R) - \frac{\kappa}{n-1} Q(g,R) + (\alpha+\mu)\, Q(g,\overline{S}) - \left(\frac{\alpha\mu}{\phi} + \eta\right) Q(S,G) - (n-2)\left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right) g \wedge Q(g,S). \tag{51}$$

But (46) implies

$$Q(g,\overline{S}) = \frac{1}{\phi} Q(g,R) - \frac{\mu}{\phi} Q(g, g \wedge S) = \frac{1}{\phi} Q(g,R) + \frac{\mu}{\phi} Q(S,G). \tag{52}$$

Substituting (52) and the identity $g \wedge Q(g,S) = -Q(S,G)$ (see (28)) into (51), we get (47). Using now (24), (46) and (52) we obtain

$$\begin{aligned} Q(S,R) &= Q(S, \phi\overline{S} + \mu g \wedge S + \eta G) \\ &= \mu Q(S, g \wedge S) + \eta Q(S,G) \\ &= -\mu Q(g,\overline{S}) + \eta Q(S,G) \\ &= -\frac{\mu}{\phi} Q(g,R) - \frac{\mu^2}{\phi} Q(S,G) + \eta Q(S,G) \\ &= -\frac{\mu}{\phi} Q(g,R) + \left(\eta - \frac{\mu^2}{\phi}\right) Q(S,G). \end{aligned}$$

Thus, in view of the above equality, (47) takes the form (48). This completes the proof.

REMARK 4.1. (i) (cf. [12, Proposition 4.2]) Under the assumptions of the above proposition, if additionally $\mu\big(\mu - \frac{1}{n-2}\big) = \eta\phi$ at x, then at this point we have

$$R \cdot C - C \cdot R = \left(\frac{1}{\phi}\left(\mu - \frac{1}{n-2}\right) + \frac{\kappa}{n-1}\right) Q(g, R).$$

(ii) An example of a warped product manifold satisfying (46) is given in [23].

5. Hypersurfaces with $H^3 = \mathrm{tr}(H)H^2 - \frac{\varepsilon\kappa}{n-1}\, H$. Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$. We now present examples of hypersurfaces satisfying (15).

EXAMPLE 5.1. (i) From Theorem 5.1 of [15] it follows that on the subset $\mathcal{U}_H$ of a quasi-Einstein hypersurface M in $\mathbb{E}_s^{n+1}$, $n \geq 4$, $R \cdot S = 0$ if and only if $A = 0$. Evidently, the last relation can be written on $\mathcal{U}_H$ in the form (15), where $\lambda = -\frac{\varepsilon\kappa}{n-1}$. Examples of such hypersurfaces are given in [1] and [7].

(ii) ([7, Example 4.3]) Let M be a hypersurface in a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, having three principal curvatures: 0, $\sqrt{\gamma}$ and $-\sqrt{\gamma}$ with multiplicities $\frac{n+2p}{3}$, $\frac{n-p}{3}$ and $\frac{n-p}{3}$, respectively, where $n - p = 3, 6, 12$ or 24, $p \geq 1$, and γ is a positive function on M. The hypersurface M is a non-quasi-Einstein Ricci-semisymmetric manifold. Moreover, if $n - p = 6, 12$ or 24 then M is a non-semisymmetric manifold. It is easy to check that on M we have:

$$\mathrm{tr}(H) = 0, \qquad S = -H^2, \qquad \kappa = -\frac{2(n-p)\gamma}{3},$$
$$H^3 = \mathrm{tr}(H)H^2 + \gamma H = -\frac{3\kappa}{2(n-p)}\, H.$$

Now the relation $H^3 = \lambda H$, where $\lambda = -\frac{3\kappa}{2(n-p)}$, yields (15).

(iii) Let M be the Cartan hypersurface of dimension $n = 6, 12$ or 24. It is known that on M the following relations hold (see e.g. [8, Section 4]):

$$H^3 = \frac{3\tau}{n(n+1)}\, H, \qquad \mathrm{tr}(H) = 0, \qquad \kappa = \frac{(n-3)\tau}{n+1}, \qquad \varepsilon = 1. \tag{53}$$

Applying (53) to (13) we obtain $A = \big(\lambda + \frac{\kappa}{n(n+1)}\big)H$ and $\lambda = \frac{3\tau}{n(n+1)}$. In addition, using the above relations we find $A = \frac{(n^2-3)\tau}{(n-1)n(n+1)}\, H$. Substituting this into (12) and using (28) we find (9) easily.

(iv) Examples of hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying an equation of the form $A = \alpha H + \beta g$, where α and β are some functions on M, will be given in [16].

THEOREM 5.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. On $\mathcal{U}_H \subset M$ the condition $A = 0$ is equivalent to*

$$R \cdot C - C \cdot R = \frac{1}{n-2}\, Q(S,R) + \frac{(n-1)\tau}{(n-2)n(n+1)}\, Q(g,R). \tag{54}$$

Proof. Clearly, $A = 0$, by (12), implies (54). Now assume that (54) holds on $\mathcal{U}_H$. Then (12) reduces to $g \wedge Q(H,A) - H \wedge Q(g,A) = 0$, which in virtue of (15) and (29) can be written in the form

$$\begin{aligned} \varrho g \wedge Q(H,g) - \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right) H \wedge Q(g,H) \\ &= -\varrho g \wedge Q(g,H) + \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right) H \wedge Q(H,g) \\ &= -\varrho Q(g, g \wedge H) + \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right) Q(H, g \wedge H) = 0. \end{aligned}$$

Thus we have

$$Q\left(\left(\lambda + \frac{\varepsilon\kappa}{n-1}\right) H - \varrho g, g \wedge H\right) = 0. \tag{55}$$

Let $x \in \mathcal{U}_H$. We prove that $A = \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right) H - \varrho g$ vanishes at x. First we assert that

$$\lambda + \frac{\varepsilon\kappa}{n-1} = 0. \tag{56}$$

Suppose not; then we can write (55) in the form $Q(H - \alpha g, g \wedge H) = 0$, $\alpha \in \mathbb{R}$. Applying Lemmas 3.2 and 4.2 we deduce that $x \in M - \mathcal{U}_H$, a contradiction. Thus we have (56), and (55) now reduces to $\varrho Q(g, g \wedge H) = 0$. Supposing that $\varrho \neq 0$ we get $Q(g, g \wedge H) = 0$ and, by (28), $Q(H,G) = 0$. Applying Lemmas 3.1 and 4.2 we deduce that $x \in M - \mathcal{U}_H$, a contradiction. So we have $\varrho = 0$ and $A = 0$. Our theorem is thus proved.

COROLLARY 5.1. *Let M be a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$. The conditions $A = 0$ and (4) are equivalent on $\mathcal{U}_H \subset M$.*

REFERENCES

[1] B. E. Abdalla and F. Dillen, *A Ricci-semi-symmetric hypersurface of the Euclidean space which is not semi-symmetric*, Proc. Amer. Math. Soc. 130 (2002), 1805–1808.

[2] K. Arslan, R. Deszcz, R. Ezentaş, C. Murathan, and C. Ozgür, *On pseudosymmetry type hypersurfaces of semi-Euclidean spaces, I*, Acta Math. Sci. 22 B (2002), 346–358.

[3] M. Belkhelfa, R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk, and L. Verstraelen, *A review on pseudosymmetry type manifolds*, in: Banach Center Publ. 57, Inst. Math., Polish Acad. Sci., 2002, 179–194.

[4] F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen, and Ş. Yaprak, *On Ricci-pseudosymmetric hypersurfaces in spaces of constant curvature*, Results Math. 27 (1995), 227–236.

[5] J. Deprez, R. Deszcz, and L. Verstraelen, *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math. 17 (1989), 51–65.

[6] R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Belg. Math. Sér. A 44 (1992), 1–34.

[7] R. Deszcz and M. Głogowska, *Examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces*, Colloq. Math. 94 (2002), 87–101.

[8] —, —, *Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces*, Publ. Inst. Math. (Beograd) (N.S.) 72 (86) (2002), 81–94.

[9] R. Deszcz, M. Głogowska, M. Hotloś, and Z. Şentürk, *On certain quasi-Einstein semisymmetric hypersurfaces*, Ann. Univ. Sci. Budapest. Rolando Eötvös Sect. Math. 41 (1998), 151–164.

[10] R. Deszcz and M. Hotloś, *On a certain subclass of pseudosymmetric manifolds*, Publ. Math. Debrecen 53 (1998), 29–48.

[11] —, —, *On a certain extension of the class of semisymmetric manifolds*, Publ. Inst. Math. (Beograd) (N.S.) 63 (77) (1998), 115–130.

[12] —, —, *On some pseudosymmetry type curvature condition*, Tsukuba J. Math. 27 (2003), in print.

[13] R. Deszcz, M. Hotloś, and Z. Şentürk, *Quasi-Einstein hypersurfaces in semi-Riemannian space forms*, Colloq. Math. 89 (2001), 81–97.

[14] —, —, —, *On some family of generalized Einstein metric conditions*, Demonstr. Math. 34 (2001), 943–954.

[15] —, —, —, *On curvature properties of quasi-Einstein hypersurfaces in semi-Euclidean spaces*, Soochow Math. 27 (2001), 375–389.

[16] R. Deszcz and K. Sawicz, *On a class of hypersurfaces in semi-Euclidean spaces*, to appear.

[17] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifolds*, in: Geometry and Topology of Submanifolds, III, World Sci., River Edge, NJ, 1991, 131–147.

[18] R. Deszcz, L. Verstraelen, and Ş. Yaprak, *Pseudosymmetric hypersurfaces in* 4-*dimensional spaces of constant curvature*, Bull. Inst. Math. Acad. Sinica 22 (1994), 167–179.

[19] R. Deszcz and Ş. Yaprak, *Curvature properties of Cartan hypersurfaces*, Colloq. Math. 67 (1994), 91–98.

[20] M. Głogowska, *Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni–Nomizu square*, Publ. Inst. Math. (Beograd) (N.S.) 72 (86) (2002), in print.

[21] —, *On a curvature characterization of Ricci-pseudosymmetric hypersurfaces*, Dept. Math. Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. 97, 2001.

[22] M. Hotloś, *On a certain curvature condition of pseudosymmetry type*, to appear.

[23] D. Kowalczyk, *On the Schwarzschild type spacetimes*, Dept. Math. Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. 83, 2000.

[24] L. Verstraelen, *Comments on pseudo-symmetry in the sense of Ryszard Deszcz*, in: Geometry and Topology of Submanifolds, VI, World Sci., River Edge, NJ, 1994, 199–209.

Ryszard Deszcz and Małgorzata Głogowska
Department of Mathematics
Agricultural University of Wrocław
Grunwaldzka 53
50-357 Wrocław, Poland
E-mail: rysz@ozi.ar.wroc.pl
mglog@ozi.ar.wroc.pl

Leopold Verstraelen
Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Leuven, Belgium
E-mail: Leopold.Verstraelen@wis.kuleuven.ac.be

Marian Hotloś
Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: hotlos@im.pwr.wroc.pl

Received 8 July 2002 (4246)