ON SOME GENERALIZED EINSTEIN METRIC CONDITIONS ON HYPERSURFACES IN SEMI-RIEMANNIAN SPACE FORMS

BY

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Dedicated to the memory of Professor Stanisław Golab

Abstract. Solutions of the P. J. Ryan problem as well as investigations of curvature properties of Cartan hypersurfaces and Ricci-pseudosymmetric hypersurfaces lead to curvature identities holding on every hypersurface \( M \) isometrically immersed in a semi-Riemannian space form. These identities, under some assumptions, give rises to new generalized Einstein metric conditions on \( M \). We investigate hypersurfaces satisfying such curvature conditions.

1. Some generalized Einstein metric conditions. In [14, Theorem 3.1] a curvature property of pseudosymmetry type of Einstein manifolds was found. It was shown that on any semi-Riemannian Einstein manifold \((M,g), n \geq 4\), the following identity holds:

\[
R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n} Q(g,R) = \frac{\kappa}{(n-1)n} Q(g,C).
\]

For precise definitions of the symbols used we refer to Sections 2 and 3 of the present paper. The above theorem gives rise to a family of curvature conditions of pseudosymmetry type ([14]). In particular, curvature properties of non-Einstein and non-conformally flat semi-Riemannian manifolds of dimension \( \geq 4 \) satisfying at every point the condition: the tensors \( R \cdot C - C \cdot R \) and \( Q(g,C) \) are linearly dependent, were investigated in [14]. This condition is equivalent on \( U_C = \{ x \in M \mid C \neq 0 \text{ at } x \} \) to

\[
R \cdot C - C \cdot R = L_1 Q(g,C),
\]


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where $L_1$ is some function on $\mathcal{U}_C$. In [14, Theorem 4.1] it was shown that if $(M, g), n \geq 4,$ is a semi-Riemannian manifold satisfying (1) then on $\mathcal{U}_S \cap \mathcal{U}_C$ we have $R \cdot R = L_1 Q(g, R)$ and $C \cdot R = 0$, where $\mathcal{U}_S = \{ x \in M \mid S - \frac{k}{n} g \neq 0 \text{ at } x \}$.

Curvature properties of semi-Riemannian manifolds satisfying at every point the condition: the tensors $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent, were investigated in [12]. This condition is equivalent on $\mathcal{U}_R = \{ x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x \}$ to

\[(2) \quad R \cdot C - C \cdot R = L_2 Q(g, R),\]

where $L_2$ is some function on $\mathcal{U}_R$. In [12, Theorem 4.2] it was shown that if $(M, g), n \geq 4,$ is a semi-Riemannian manifold satisfying (2) then $R \cdot R = 0$ on $\mathcal{U}_S \cap \mathcal{U}_C$.

The study of semi-Riemannian manifolds satisfying at every point the condition: the tensors $R \cdot C - C \cdot R$ and $Q(S, R)$ are linearly dependent, was initiated in [22]. This condition is equivalent on $\mathcal{U}_3 = \{ x \in M \mid Q(S, R) \neq 0 \text{ at } x \}$ to

\[(3) \quad R \cdot C - C \cdot R = L_3 Q(S, R),\]

where $L_3$ is some function on $\mathcal{U}_3$. In [22] it was shown that if $(M, g), n \geq 4,$ is a Ricci-semisymmetric $(R \cdot S = 0)$ semi-Riemannian manifold satisfying (3) then at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ at which $L_3$ does not vanish we have

\[(4) \quad R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R).\]

In Section 5 we consider hypersurfaces of semi-Euclidean spaces $\mathbb{E}^{n+1}_s$ with signature $(s, n + 1 - s), n \geq 4,$ satisfying (4).

We can also investigate semi-Riemannian manifolds satisfying at every point the condition: the tensors $R \cdot C - C \cdot R$ and $Q(S, C)$ are linearly dependent. This condition is equivalent on $\mathcal{U}_4 = \{ x \in M \mid Q(S, C) \neq 0 \text{ at } x \}$ to

\[(5) \quad R \cdot C - C \cdot R = L_4 Q(S, C),\]

where $L_4$ is some function on $\mathcal{U}_4$. In this paper we present results on hypersurfaces of $\mathbb{E}^{n+1}_s, n \geq 4,$ satisfying (5). Semi-Riemannian manifolds satisfying (5) will be investigated in subsequent papers.

(1)–(5) as well as other conditions of this kind are called generalized Einstein metric conditions ([12], [14]) and also curvature conditions of pseudosymmetry type. Recently, a review of results on semi-Riemannian manifolds satisfying such conditions was given in [3] (see also [6] and [24]).

Let $M$ be a hypersurface in a semi-Riemannian space of constant curvature $N^{n+1}_s(c)$ with signature $(s, n + 1 - s), n \geq 4.$ We denote by $\mathcal{U}_H$ the set of all points of $M$ at which the tensor $H^2$ is not a linear combination of
the metric tensor $g$ and the second fundamental tensor $H$ of $M$. It is known that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C$.

Let now $M$ be a hypersurface in a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 4$. The following results pertain to (4).

**Theorem 1.1.** Let $M$ be a Ricci-semisymmetric hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$.

(i) ([13, Lemma 3.1]) On $\mathcal{U}_H \subset M$ we have $H^3 = \text{tr}(H)H^2 + \lambda H$ and

$$R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) - \frac{1}{n-2} \left( \varepsilon \lambda + \frac{\kappa}{n-1} \right) Q(g, R),$$

where $\lambda$ is some function on $\mathcal{U}_H$.

(ii) ([15, Theorem 5.1]) In addition, if $M$ is a quasi-Einstein hypersurface then on $\mathcal{U}_H$, (6) reduces to (4).

Curvature properties of Ricci-pseudosymmetric hypersurfaces in semi-Riemannian spaces of constant curvature $N^{n+1}_s(c)$, $n \geq 4$, were investigated in [4], [8], [9], [18] and [19], among others. From Proposition 3.2 and Theorem 3.1 of [4] it follows that for every Ricci-pseudosymmetric hypersurface $M$ in $N^{n+1}_s(c)$, $n \geq 4$, on the set $\mathcal{U}_H \subset M$ we have

$$R \cdot S = \frac{\tau}{n(n+1)} Q(g, S),$$

where $\tau$ is the scalar curvature of the ambient space. In [21] a curvature characterization of pseudosymmetry type of Ricci-pseudosymmetric hypersurfaces $M$ in $N^{n+1}_s(c)$, $n \geq 4$, was found. Namely, we have

**Theorem 1.2 ([21, Proposition 5.1(iii) and Theorem 6.1]).** Let $M$ be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$. On $\mathcal{U}_H \subset M$, (7) is equivalent to

$$R \cdot C = Q(S, R) - \frac{(n-2)\tau}{n(n+1)} Q(g, R) - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G).$$

Cartan hypersurfaces are Ricci-pseudosymmetric ([18], [19]). In [8] further curvature properties of pseudosymmetry type for Cartan hypersurfaces of dimension $\geq 6$ were found.

**Theorem 1.3 ([8, Theorem 4.3]).** On every Cartan hypersurface $M$ in $S^{n+1}(c)$, $n = 6, 12$ or $24$, we have: (7), (8),

$$C \cdot R = \frac{n-3}{n-2} Q(S, R) - \frac{(n-3)\tau}{(n-1)(n+1)} Q(g, R)$$

$$- \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G),$$

$$R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) - \frac{2\tau}{(n-1)n(n+1)} Q(g, R).$$
In Section 3 we consider an extension of the standard Kulkarni–Nomizu product $E \wedge F$ of two $(0,2)$-tensors $E$ and $F$. Namely, we define the Kulkarni–Nomizu product $Q(E, T)$ of a $(0,2)$-tensor $E$ and a $(0,k)$-tensor $T$, $k \geq 2$ (see [8]). We present some properties of this product. We use these properties to prove (see Theorem 3.1) that on any hypersurface $M$ in $N_{s}^{n+1}(c), n \geq 4$, the following identities hold:

\[
R \cdot C = Q(S, R) - \frac{(n-2)\tau}{n(n+1)} Q(g, R) - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G) + \frac{1}{n-2} g \wedge Q(H, A),
\]

\[
C \cdot R = \frac{n-3}{n-2} Q(S, R) - \frac{(n^2 - 3n + 3)\tau}{(n-2)n(n+1)} Q(g, R) - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G) + \frac{1}{n-2} H \wedge Q(g, A),
\]

\[
R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R) + \frac{1}{n-2} (g \wedge Q(H, A) - H \wedge Q(g, A)),
\]

where $\tau, g$ and $H$ are the scalar curvature of $N_{s}^{n+1}(c)$, the metric tensor of $M$ and the second fundamental tensor of $M$, respectively. The $(0,2)$-tensor $A$ is defined by

\[
A = H^3 - \text{tr}(H) H^2 + \frac{\varepsilon \kappa}{n-1} H.
\]

We mention that from Theorem 5.1 of [15] it follows that $A$ vanishes on the subset $\mathcal{U}_H$ of any quasi-Einstein Ricci-semisymmetric hypersurface $M$ in $E_{s}^{n+1}, n \geq 4$. In Section 5 we prove that (4) holds on the subset $\mathcal{U}_H$ of a hypersurface $M$ in $E_{s}^{n+1}, n \geq 4$, if and only if $A = 0$ on $\mathcal{U}_H$. We also present examples of hypersurfaces with nonzero $A$.

From Proposition 5.2 of [21] it follows that if on the subset $\mathcal{U}_H$ of a hypersurface $M$ in $N_{s}^{n+1}(c), n \geq 4$, we have

\[
\sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = 0,
\]

then

\[
A = \left( \lambda + \frac{\varepsilon \kappa}{n-1} \right) H + \varrho g,
\]

\[
\varrho = \frac{1}{n} \left( \text{tr}(A) - \left( \lambda + \frac{\varepsilon \kappa}{n-1} \right) \text{tr}(H) \right),
\]

on $\mathcal{U}_H$, where $\lambda$ is some function on $\mathcal{U}_H$. In Section 4 we prove (see Propo-
tion 4.1) that the following conditions: (14),
\begin{align}
(16) \quad & \sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (R \cdot C - C \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0, \\
(17) \quad & \sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (C \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0
\end{align}
are equivalent on any semi-Riemannian manifold of dimension \( \geq 4 \). Thus on the subset \( \mathcal{U}_H \) of a hypersurface \( M \) in \( N_s^{n+1}(c) \), \( n \geq 4 \), each of the condition (14), (16), (17) implies (15) on \( \mathcal{U}_H \) (see Theorem 4.1).

2. Preliminaries. Throughout this paper all manifolds are assumed to be connected paracompact of class \( C^\infty \). Let \( (M, g) \) be an \( n \)-dimensional, \( n \geq 3 \), semi-Riemannian manifold. We denote by \( \nabla \), \( R \), \( C \), \( S \) and \( \kappa \) the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of \( (M, g) \), respectively. The Ricci operator \( S \) is defined by \( g(SX, Y) = S(X, Y) \), where \( X, Y \in \mathcal{E}(M) \), \( \mathcal{E}(M) \) being the Lie algebra of vector fields on \( M \).

We define the endomorphisms \( X \wedge_A Y, \mathcal{R}(X, Y) \) and \( \mathcal{C}(X, Y) \) of \( \mathcal{E}(M) \) by \( (X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y \), \( \mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \) and
\[
\mathcal{C}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2} \left( X \wedge_g SY + SX \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y \right) Z,
\]
where \( X, Y, Z \in \mathcal{E}(M) \) and \( A \) is a symmetric (0,2)-tensor. Now the Riemann–Christoffel curvature tensor \( R \), the Weyl conformal curvature tensor \( C \) and the (0,4)-tensor \( G \) of \( (M, g) \) are defined by
\[
R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4), \\
C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4), \\
G(X_1, X_2, X_3, X_4) = g((X_1 \wedge g) X_2)X_3, X_4),
\]
where \( X, Y, Z, X_1, X_2, \ldots \in \mathcal{E}(M) \). Let \( \mathcal{B}(X, Y) \) be a skew-symmetric endomorphism of \( \mathcal{E}(M) \) and let \( B \) be the (0,4)-tensor associated with \( \mathcal{B}(X, Y) \) by
\[
B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4).
\]
\( B \) is said to be a generalized curvature tensor if
\[
B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0, \\
B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2).
\]
Clearly, \( R, C \) and \( G \) are generalized curvature tensors.

Let \( \mathcal{B}(X, Y) \) be a skew-symmetric endomorphism of \( \mathcal{E}(M) \) and let \( B \) be the tensor defined by (18). We extend the endomorphism \( \mathcal{B}(X, Y) \) to a derivation \( \mathcal{B}(X, Y) \cdot f = 0 \) for any smooth function.
on $M$. Now for a $(0, k)$-tensor field $T$, $k \geq 1$, we can define the $(0, k + 2)$-tensor $B \cdot T$ by
\[
(B \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{B}(X, Y) \cdot T)(X_1, \ldots, X_k; X, Y)
\]
\[
= -T(\mathcal{B}(X, Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{B}(X, Y)X_k).
\]
In addition, if $A$ is a symmetric $(0, 2)$-tensor then we define the $(0, k + 2)$-tensor $Q(A, T)$ by
\[
Q(A, T)(X_1, \ldots, X_k; X, Y) = (X \wedge A \cdot Y \cdot T)(X_1, \ldots, X_k; X, Y)
\]
\[
= -T((X \wedge A)Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge A)Y)X_k).
\]
In particular, in this manner, we obtain the $(0, 6)$-tensors $B \cdot B$ and $Q(A, B)$. Setting in the above formulas $\mathcal{B} = \mathcal{R}$ or $C$, $T = R$, $C$ or $S$, $A = g$ or $S$, we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(g, S)$.

Let $M$, $n = \dim M \geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, g^N)$. We denote by $g$ the metric tensor of $M$ induced from $g^N$. Further, we denote by $\nabla$ and $\nabla^N$ the Levi-Civita connections corresponding to $g$ and $g^N$, respectively. Let $\xi$ be a local unit normal vector field on $M$ in $N$ and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can write the Gauss formula and the Weingarten formula of $(M, g)$ in $(N, g^N)$ in the forms $\nabla^N_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\nabla_X \xi = -AX$, respectively, where $X, Y$ are vector fields tangent to $M$, $H$ is the second fundamental tensor of $(M, g)$ in $(N, g^N)$, $A$ is the shape operator and $H^k(X, Y) = g(A^kX, Y)$, $k \geq 1$, $H^1 = H$ and $A^1 = A$. We denote by $R$ and $R^N$ the Riemann–Christoffel curvature tensors of $(M, g)$ and $(N, g^N)$, respectively. The Gauss equation of $(M, g)$ in $(N, g^N)$ has the form $R(X_1, \ldots, X_4) = R^N(X_1, \ldots, X_4) + \frac{\varepsilon}{2} (H \wedge H)(X_1, \ldots, X_4)$, where $X_1, \ldots, X_4$ are vector fields tangent to $M$.

Let the equations $x^r = x^r(y^k)$ be the local parametric expression of $(M, g)$ in $(N, g^N)$, where $y^k$ and $x^r$ are the local coordinates of $M$ and $N$, respectively, and $h, i, j, k \in \{1, \ldots, n\}$ and $p, r, t, u \in \{1, \ldots, n+1\}$. Now the Gauss equation yields
\[
R_{hijk} = R^N_{prtu}B^p_hB^t_rB^u_jB^i_k + \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}),
\]
where $B^r_k = \partial x^r/\partial y^k$, $R^N_{rstu}$, $R_{hijk}$ and $H_{hk}$ are the local components of the tensors $R^N$, $R$ and $H$, respectively. If $M$ is a hypersurface in $N^{n+1}(c)$, $n \geq 4$, then (19) becomes
\[
R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\tau}{n(n+1)} G_{hijk},
\]
where $\tau$ is the scalar curvature of the ambient space and $G_{hijk}$ are the local components of the tensor $G$. Contracting (20) with $g^{ij}$ and $g^{kh}$, respectively,
we obtain
\[ S_{hk} = \varepsilon (\text{tr}(H) H_{hk} - H^2_{hk}) + \frac{(n-1)\tau}{n(n+1)} g_{hk} \]  
and
\[ \kappa = \varepsilon (\text{tr}(H)^2 - \text{tr}(H^2)) + \frac{(n-1)\tau}{n+1}, \]
respectively, where \( \text{tr}(H) = g^{hk} H_{hk}, \text{tr}(H^2) = g^{hk} H^2_{hk} \) and \( S_{hk} \) are the local components of the Ricci tensor \( S \) of \( M \). Using (21) and Theorem 4.1 of [17] we can deduce that
\[ U \cdot H \cdot U = - \frac{(n-2)\tau}{n(n+1)} Q(g, C). \]
Evidently, if the ambient space is \( \mathbb{E}^{n+1}_s \) then (22) reduces to \( R \cdot R = Q(S, R) \).

3. The basic identities. For symmetric \((0,2)\)-tensors \( E \) and \( F \) we define their Kulkarni–Nomizu product \( E \wedge F \) by
\[ (E \wedge F)(X_1, X_2, X_3, X_4) = E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \]
\[ - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3). \]
The tensor \( E \wedge F \) is also a generalized curvature tensor. For a symmetric \((0,2)\)-tensor \( E \) we define the \((0,4)\)-tensor \( \overline{E} \) by \( \overline{E} = \frac{1}{2} E \wedge E \). In particular, \( \overline{g} = G = \frac{1}{2} g \wedge g \). We note that the Weyl tensor \( C \) can be represented in the form
\[ C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G. \]
We also have (see e.g. [9, Section 3])
\[ Q(E, E \wedge F) = -Q(F, \overline{E}). \]

**Lemma 3.1.** Let \( E \) be a symmetric \((0,2)\)-tensor at a point \( x \) of a semi-Riemannian manifold \((M, g), n \geq 3\).

(i) ([2, Lemma 2.2]) If
\[ E = \alpha g + \beta u \otimes u, \quad \alpha, \beta \in \mathbb{R} \quad u \in T_x^* M, \]
then at \( x \) we have
\[ E^2 = \tilde{\alpha} E + \tilde{\beta} g, \quad \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}. \]
(ii) ([20, Lemma 3.1]) Let \( \mathcal{U}_E \) be the set of all points of \( M \) at which \( E \) is not proportional to \( g \). If, at some \( x \in \mathcal{U}_E \),
\[ E \wedge E = 2\alpha g \wedge E + 2\beta G, \quad \alpha, \beta \in \mathbb{R}, \]
then at \( x \) we have (25) with \( \alpha^2 = -\beta \).
According to [8], for a symmetric (0, 2)-tensor $E$ and a (0, $k$)-tensor $T$, $k \geq 2$, we define their Kulkarni–Nomizu product $E \wedge T$ by
\[
(E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \ldots, Y_k) = E(X_1, X_4)T(X_2, X_3, Y_3, \ldots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \ldots, Y_k) - E(X_1, X_3)T(X_2, X_4, Y_3, \ldots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \ldots, Y_k).
\]
Using the above definitions we can prove the following

**Lemma 3.2 ([21]).** Let $E_1$, $E_2$ and $F$ be symmetric (0, 2)-tensors at a point $x$ of a semi-Riemannian manifold $(M, g)$, $n \geq 3$. Then at $x$ we have
\[
E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) = -Q(F, E_1 \wedge E_2).
\]
If $E = E_1 = E_2$ then
\[
E \wedge Q(E, F) = -Q(F, E).
\]
(28)

As an immediate consequence of (24) and (28) we have
\[
E \wedge Q(E, F) = Q(E, E \wedge F).
\]
(29)

By making use of (15), Propositions 5.1 and 5.2 of [21] imply

**Proposition 3.1.** Let $M$ be a hypersurface in $N_{s}^{n+1}(c)$, $n \geq 4$.

(i) $R \cdot S = Q(A, H) + \frac{\tau}{n(n+1)} Q(g, S)$ on $M$.

(ii) If, in addition, $M$ is a Ricci-pseudosymmetric manifold then (7) holds on $U_{H}$.

(iii) On $M$,
\[
R \cdot C = Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H)
+ \frac{\tau}{n(n+1)} \left( \frac{1}{n-2} Q(S, G) - (n-2) Q(g, C) \right).
\]
(30)

(iv) In particular, if $M$ is a Ricci-pseudosymmetric hypersurface in $E_{s}^{n+1}$, $n \geq 4$, then $R \cdot C = Q(S, R)$ on $U_{H}$.

(v) Let $M$ be a hypersurface in $E_{s}^{n+1}$, $n \geq 4$, satisfying (14). Then on $U_{H}$ we have (15) and
\[
R \cdot S = -\frac{\mu}{n} Q(g, H), \quad R \cdot C = Q(S, R) - \frac{\mu}{(n-2)n} Q(H, G),
\]
where $\lambda$ is some function on $U_{H}$ and $\mu = \text{tr}(H) \text{tr}(H^2) - \text{tr}(H^3) + \lambda \text{tr}(H)$.

**Theorem 3.1.** The identities (10)–(12) hold on every hypersurface $M$ in $N_{s}^{n+1}(c)$, $n \geq 4$. In particular, on every hypersurface $M$ in $E_{s}^{n+1}$, $n \geq 4$, we have
\[
R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R)
+ \frac{1}{n-2} (g \wedge Q(H, A) - H \wedge Q(g, A)).
\]
(31)
Proof. Applying the relations (23) and (24) in (30) we get (10) easily. From (20), by transvection with $H^h_l = g^h_l H_{tr}$, we obtain

$$H^l_i R_{rijkl} = \varepsilon (H_{ij} H^l_k - H_{ik} H^l_j) + \frac{\tau}{n(n+1)} (g_{ij} H^l_k - g_{ik} H^l_j),$$

which implies

$$R \cdot H = \varepsilon Q(H, H^2) + \frac{\tau}{n(n+1)} Q(g, H). \tag{32}$$

Further, from (20) we also get

$$R - \frac{1}{n-2} \left( g \wedge S + \frac{\kappa}{n-1} G \right)$$

$$= \varepsilon \overline{H} - \frac{1}{n-2} g \wedge S + \left( \frac{\kappa}{(n-2)(n-1)} + \frac{\tau}{n(n+1)} \right) G,$$

which, by making use of (21) and (23), turns into

$$C = \varepsilon \overline{H} + \frac{\varepsilon}{n-2} g \wedge (H^2 - \text{tr}(H) H)$$

$$+ \frac{1}{n-2} \left( \frac{\kappa}{n-1} - \frac{\tau}{n+1} \right) G. \tag{33}$$

(33), by suitable transvection and application of (13) and the definitions of $R \cdot T$ and $Q(E, T)$, leads to

$$C \cdot H = \frac{n-3}{n-2} \varepsilon Q(H, H^2) + \frac{\varepsilon}{n-2} Q(g, A)$$

$$- \frac{\tau}{(n-2)(n+1)} Q(g, H). \tag{34}$$

But (34), in view of (32), yields (see Theorem 3.4 of [2])

$$C \cdot H = \frac{n-3}{n-2} R \cdot H + \frac{\varepsilon}{n-2} Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)} Q(g, H).$$

Using this, (20), (22) and (28) we find

$$C \cdot R = \varepsilon H \wedge (C \cdot H) = \frac{(n-3)\varepsilon}{n-2} H \wedge (R \cdot H)$$

$$+ \frac{1}{n-2} H \wedge Q(g, A) - \frac{(2n-3)\varepsilon \tau}{(n-2)n(n+1)} H \wedge Q(g, H)$$

$$= \frac{(n-3)\varepsilon}{n-2} (R \cdot \overline{H}) + \frac{1}{n-2} H \wedge Q(g, A)$$

$$- \frac{(2n-3)\varepsilon \tau}{(n-2)n(n+1)} Q(g, \overline{H})$$

$$= \frac{n-3}{n-2} (R \cdot R) + \frac{1}{n-2} H \wedge Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)} Q(g, R).$$
From this, by making use of (23) and (24), we get (11). Further, (35) together with (30) yields

\[
R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) - \frac{\tau}{n(n+1)} Q(g, C)
\]

\[
+ \frac{1}{n-2} \left( g \wedge Q(H, A) - H \wedge Q(g, A) \right)
\]

\[
+ \frac{\tau}{(n-2)n(n+1)} Q(S, G) + \frac{(2n-3)\tau}{(n-2)n(n+1)} Q(g, R).
\]

Applying now (23) and (24) we get (12). Finally, we note that (31) is an immediate consequence of (12). Our theorem is thus proved.

**Theorem 3.2.** Let \( M \) be a hypersurface in \( N_s^{n+1}(c), n \geq 4 \). If (15) is satisfied on \( U_H \subset M \) then on this set we have

\[
R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R)
\]

\[
+ \frac{1}{n-2} \left( gQ(H, G) - \varepsilon \left( \lambda + \frac{\varepsilon\kappa}{n-1} \right) Q(g, R) \right).
\]

**Proof.** This is a consequence of Theorem 3.1 and (20) and (28).

4. **Some curvature conditions.** Let \((M, g)\) be covered by a system of charts \(\{W; x^k\}\). We denote by \(g_{ij}, R_{hijk}, S_{ij}, G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}\) and

\[
C_{hijk} = R_{hijk} + \frac{\kappa}{(n-2)(n-1)} G_{hijk}
\]

\[
- \frac{1}{n-2} (g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj})
\]

the local components of the tensors \(g, R, S, G\) and \(C\), respectively. Further, we denote by \(S^2_{ij} = S_{ir}S_{j}^\tau\) and \(S^2_{ij} = g^{r\tau}S_{ir}\) the local components of the tensor \(S^2\) defined by \(S^2(X, Y) = S(SX, Y)\), and of the Ricci operator \(S\), respectively. Let \((R \cdot C)_{hijklm}\) and \((C \cdot R)_{hijklm}\) denote the local components of \(R \cdot C\) and \(C \cdot R\), respectively. We have

\[
(R \cdot C)_{hijklm} = g^{rs}(C_{rj;k}R_{sh;lm} + C_{hr;jk}R_{s;ilm} + C_{hirk}R_{sj;lm} + C_{hijr}R_{sk;lm})
\]

\[
(C \cdot R)_{hijklm} = g^{rs}(R_{rj;k}C_{sh;lm} + R_{hr;jk}C_{s;ilm} + R_{hirk}C_{sj;lm} + R_{hijr}C_{sk;lm})
\]

respectively. Applying (37) in (38) and (39) we get
(40) \[(R \cdot C)_{hijkl} = (R \cdot R)_{hijkl} \]
\[ - \frac{1}{n-2} (g_{ij}(V_{hkln} + V_{khlm}) + g_{hk}(V_{ijlm} + V_{jilm}) \]
\[ - g_{kk}(V_{hjlm} + V_{jhlm}) - g_{hj}(V_{iklm} + V_{kilm}), \]

(41) \[(C \cdot R)_{hijkl} = (R \cdot R)_{hijkl} - \frac{1}{n-2} Q(S, R)_{hijkl} \]
\[ + \frac{\kappa}{(n-1)(n-2)} Q(g, R)_{hijkl} \]
\[ - \frac{1}{n-2} (ghV_{mijk} - ghV_{ijlk} - giV_{mjhk} + giV_{ljhk} \]
\[ + gjV_{mkhi} - gjV_{lkhi} - gkV_{mjhi} + gkV_{ljhi}), \]

(42) \[V_{mijk} = S_n^a R_{sjik},\]

where \((R \cdot R)_{hijkl}, Q(S, R)_{hijkl}, Q(g, R)_{hijkl}\) and \(Q(g, C)_{hijkl}\) are the local components of the respective tensors. Using (40) and (41) we obtain ([13, Section 2])

(43) \[(n-2)(R \cdot C - C \cdot R)_{hijkl} = Q(S, R)_{hijkl} \]
\[ - \frac{\kappa}{n-1} Q(g, R)_{hijkl} + ghV_{mijk} - ghV_{ijlk} - giV_{mjhk} + giV_{ljhk} \]
\[ + gjV_{mkhi} - gjV_{lkhi} - gkV_{mjhi} + gkV_{ljhi} \]
\[ - g_{ij}(V_{hkln} + V_{khlm}) - g_{hk}(V_{ijlm} + V_{jilm}) \]
\[ + g_{ik}(V_{hjlm} + V_{jhlm}) + g_{hj}(V_{iklm} + V_{kilm}). \]

**Lemma 4.1** ([5, Lemma 1.1(iii)]). Let \(B\) be a generalized curvature tensor on a semi-Riemannian manifold \((M, g)\), \(n \geq 3\). The tensor \(Q(g, B)\) vanishes at a point \(x \in M\) if and only if \(B = \frac{\kappa(B)}{(n-1)n} G\) at \(x\).

**Lemma 4.2** (cf. [10, Lemma 3.4]). Let \((M, g), n \geq 3\), be a semi-Riemannian manifold. Let \(E\) be a nonzero symmetric \((0, 2)\)-tensor at a point \(x \in M\) and let \(B\) be a generalized curvature tensor such that \(Q(E, B) = 0\) at \(x\). Moreover, let \(Y\) be a vector at \(x\) such that the scalar \(q = a(Y)\) is nonzero, where \(a\) is the covector defined by \(a(X) = E(X, Y), X \in T_xM\). Then at \(x\) we have two possibilities:

(i) the tensor \(E\) is of rank one (precisely, \(E = \frac{1}{\delta} a \otimes a\)), or

(ii) the tensor \(E - \frac{1}{\delta} a \otimes a\) is nonzero and \(B = \frac{\gamma}{2} E \wedge E, \gamma \in \mathbb{R}\).

Using the above lemma and Lemmas 3.1 and 3.2 we can prove

**Lemma 4.3**. Let \((M, g), n \geq 3\), be a semi-Riemannian manifold. Let \(E\) be a nonzero symmetric \((0, 2)\)-tensor at a point \(x \in M\). If at \(x\) we have \(Q(E - \alpha g, g \wedge E) = 0, \alpha \in \mathbb{R}\), then (26) holds at \(x\).
Let $Q(E, B)_{hijklm}$ be the local components of $Q(E, B)$. We have ([5, Lemma 1.1(i)])

\begin{equation}
Q(E, B)_{hijklm} + Q(E, B)_{jklmhi} + Q(E, B)_{lmhijk} = 0.
\end{equation}

On $M$ we also have the well known Walker identity

\begin{equation}
(R \cdot R)_{hijklm} + (R \cdot R)_{jklmhi} + (R \cdot R)_{lmhijk} = 0.
\end{equation}

**Proposition 4.1.** Let $(M, g)$, $n \geq 4$, be a semi-Riemannian manifold. The equalities (14), (16) and (17) are equivalent on $M$.

**Proof.** We set

\[
P_{hijklm} = \frac{1}{n-2} \left( ((g_{ij}(V_{hklm} + V_{khlm})) + g_{hk}(V_{ijlm} + V_{jilm})
\right.
\]

\[
- g_{ik}(V_{hjlm} + V_{jhlm}) - g_{jh}(V_{iklm} + V_{klm})
\]

\[
+ g_{kl}(V_{mjhi} + V_{jmhi}) + g_{jm}(V_{klhi} + V_{khhi})
\]

\[
- g_{km}(V_{jlhi} + V_{ljhi}) - g_{jl}(V_{kmhi} + V_{mkhi})
\]

\[
+ g_{mh}(V_{lijk} + V_{lhij}) + g_{li}(V_{mhjk} + V_{hmjk})
\]

\[
- g_{ni}(V_{lhjk} + V_{lhjk}) - g_{lh}(V_{mijn} + V_{imjk}),
\]

where $V_{hijk}$ are defined by (42). Symmetrizing (41) with respect to the pairs $(h, i)$, $(j, k)$ and $(l, m)$ and applying (44) and (45) we obtain

\[
(C \cdot R)_{hijklm} + (C \cdot R)_{jklmhi} + (C \cdot R)_{lmhijk} = P_{hijklm}.
\]

In the same way, using (40), we have

\[
(R \cdot C)_{hijklm} + (R \cdot C)_{jklmhi} + (R \cdot C)_{lmhijk} = -P_{hijklm}.
\]

From the last two relations we get

\[
(R \cdot C - C \cdot R)_{hijklm} + (R \cdot C - C \cdot R)_{jklmhi} + (R \cdot C - C \cdot R)_{lmhijk} = -2P_{hijklm}.
\]

Now our assertion is obvious.

Proposition 5.2 of [21] and Proposition 4.1 yield

**Theorem 4.1.** If on the subset $\mathcal{U}_H$ in a hypersurface $M$ of $N^{n+1}_{s}(c)$, $n \geq 4$, one of the conditions (14), (16) or (17) is satisfied then (15) holds on $\mathcal{U}_H$.

Using (44), (45), Proposition 4.1 and Theorem 4.1 we immediately get

**Corollary 4.1.** If on the subset $\mathcal{U}_H$ in a hypersurface $M$ of $N^{n+1}_{s}(c)$, $n \geq 4$, one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is a linear combination of $R \cdot R$ and of a finite sum of tensors of the form $Q(E, B)$, where $E$ is a symmetric $(0, 2)$-tensor and $B$ a generalized curvature tensor, then (15) holds on $\mathcal{U}_H$. 
THEOREM 4.2. Let $M$ be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$. If at every point of $M$ the following two tensors are linearly dependent:

(i) $R \cdot C - C \cdot R$ and $Q(g, C)$, or
(ii) $R \cdot C - C \cdot R$ and $Q(g, R)$, or
(iii) $R \cdot C - C \cdot R$ and $Q(S, R)$, or
(iv) $R \cdot C - C \cdot R$ and $Q(S, C)$,

then (15) and (36) hold on $U_H \subset M$.

Proof. In case (i), resp. (ii), on $U_H \subset M$ we have (1), resp. (2). Now, in view of Corollary 4.1, (15) holds on $U_H$.

Consider case (iii) and let $x \in U_H$. Assume that $Q(S, R)$ vanishes at $x$. Then (22) becomes

$$R \cdot C = R \cdot R - \frac{1}{n-2} g \wedge (R \cdot S) = -\frac{(n-2)\tau}{n(n+1)} Q(g, C).$$

Applying Corollary 4.1 we get (15). Clearly, if $Q(S, R)$ is nonzero at a point $x$ then $x \in U_H$. Thus (3) holds at $x$. Now Corollary 4.1 again implies (15).

Finally, consider case (iv) and let $x \in U_H$. If $Q(S, C)$ is nonzero at $x$ then (5) holds at $x$ and Corollary 4.1 implies (15). Assume now that $Q(S, C) = 0$ at $x$. In view of Theorem 3.1 of [11], we get $R \cdot R = \frac{\kappa}{n-1} Q(g, R)$. This yields $R \cdot S = \frac{\kappa}{n-1} Q(g, S)$. Using (23) and (29) we find

$$R \cdot C = R \cdot R - \frac{1}{n-2} g \wedge (R \cdot S)$$

$$= \frac{\kappa}{n-1} \left( Q(g, R) - \frac{1}{n-2} g \wedge Q(g, S) \right)$$

$$= \frac{\kappa}{n-1} \left( Q(g, R) - \frac{1}{n-2} Q(g, g \wedge S) \right)$$

$$= \frac{\kappa}{n-1} \left( Q(g, R) - \frac{1}{n-2} g \wedge S \right)$$

$$= \frac{\kappa}{n-1} \left( Q(g, R) - \frac{1}{n-2} g \wedge S + \frac{\kappa}{n-1} G \right) = \frac{\kappa}{n-1} Q(g, C).$$

Now, in view of Corollary 4.1, we obtain (15) on $U_H$. Finally, from Theorem 3.2 it follows that (36) holds on $U_H$. This completes the proof.

PROPOSITION 4.2. Let $(M, g)$, $n \geq 4$, be a semi-Riemannian manifold. If at a point $x \in U_S \cap U_C$ its curvature tensor $R$ is of the form

$$R = \phi \mathcal{S} + \mu g \wedge S + \eta G, \quad \phi, \mu, \eta \in \mathbb{R},$$

then at $x$ we have

$$R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) + \left( \frac{(n-1)\mu-1}{(n-2)\phi} + \frac{\kappa}{n-1} \right) Q(g, R)$$

$$+ \frac{\mu((n-1)\mu-1) - (n-1)\phi\eta}{(n-2)\phi} Q(S, G),$$
\[ R \cdot C - C \cdot R = \left( \frac{1}{\phi} \left( \mu - \frac{1}{n-2} \right) + \frac{\kappa}{n-1} \right) Q(g, R) + \left( \frac{\mu}{\phi} \left( \mu - \frac{1}{n-2} \right) - \eta \right) Q(S, G). \]

**Proof.** As shown in [12], (46) implies

\[ V_{mijk} = (\alpha + \mu)(S_{mk}S_{ij} - S_{mij}S_{ik}) + \left( \frac{\alpha \mu}{\phi} + \eta \right) (S_{mk}g_{ij} - S_{mij}g_{ik}) + \beta(g_{mk}S_{ij} - g_{mij}S_{ik}) + \frac{\beta \mu}{\phi} G_{mijk}, \]

where \( \alpha = \phi \kappa - 1 + (n-2)\mu, \beta = \mu \kappa + (n-1)\eta \) and

\[ R \cdot S = (n-2) \left( \frac{\mu}{\phi} \left( \mu - \frac{1}{n-2} \right) - \eta \right) Q(g, S). \]

Substituting (49) and (50) into (43) we get

\[ (n-2)(R \cdot C - C \cdot R) = Q(S, R) - \frac{\kappa}{n-1} Q(g, R) + (\alpha + \mu) Q(g, S) - \left( \frac{\alpha \mu}{\phi} + \eta \right) Q(S, G) - (n-2) \left( \frac{\mu}{\phi} \left( \mu - \frac{1}{n-2} \right) - \eta \right) g \wedge Q(g, S). \]

But (46) implies

\[ Q(g, S) = \frac{1}{\phi} Q(g, R) - \frac{\mu}{\phi} Q(g, g \wedge S) = \frac{1}{\phi} Q(g, R) + \frac{\mu}{\phi} Q(S, G). \]

Substituting (52) and the identity \( g \wedge Q(g, S) = -Q(S, G) \) (see (28)) into (51), we get (47). Using now (24), (46) and (52) we obtain

\[ Q(S, R) = Q(S, \phi \overline{S} + \mu g \wedge S + \eta G) = \mu Q(S, g \wedge S) + \eta Q(S, G) = -\mu Q(g, \overline{S}) + \eta Q(S, G) = -\frac{\mu}{\phi} Q(g, R) - \frac{\mu^2}{\phi} Q(S, G) + \eta Q(S, G) = -\frac{\mu}{\phi} Q(g, R) + \left( \eta - \frac{\mu^2}{\phi} \right) Q(S, G). \]

Thus, in view of the above equality, (47) takes the form (48). This completes the proof.
Remark 4.1. (i) (cf. [12, Proposition 4.2]) Under the assumptions of the above proposition, if additionally 
\( \mu \left( \mu - \frac{1}{n-2} \right) = \eta \phi \) at \( x \), then at this point we have

\[
R \cdot C - C \cdot R = \left( \frac{1}{\phi} \left( \mu - \frac{1}{n-2} \right) + \frac{\kappa}{n-1} \right) Q(g, R).
\]

(ii) An example of a warped product manifold satisfying (46) is given in [23].

5. Hypersurfaces with \( H^3 = \text{tr}(H)H^2 - \frac{2\kappa}{n-1} H \). Let \( M \) be a hypersurface in a semi-Riemannian space of constant curvature \( N_{s}^{n+1}(c), n \geq 4 \). We now present examples of hypersurfaces satisfying (15).

Example 5.1. (i) From Theorem 5.1 of [15] it follows that on the subset \( U_H \) of a quasi-Einstein hypersurface \( M \) in \( \mathbb{E}_{s}^{n+1}, n \geq 4 \), \( R \cdot S = 0 \) if and only if \( A = 0 \). Evidently, the last relation can be written on \( U_H \) in the form (15), where \( \lambda = -\frac{2\kappa}{n-1} \). Examples of such hypersurfaces are given in [1] and [7].

(ii) ([7, Example 4.3]) Let \( M \) be a hypersurface in a Euclidean space \( \mathbb{E}^{n+1}, n \geq 4 \), having three principal curvatures: 0, \( \sqrt{\gamma} \) and \( -\sqrt{\gamma} \) with multiplicities \( \frac{n+2p}{3}, \frac{n-p}{3} \) and \( \frac{n-p}{3} \), respectively, where \( n - p = 3, 6, 12 \) or 24, \( p \geq 1 \), and \( \gamma \) is a positive function on \( M \). The hypersurface \( M \) is a non-quasi-Einstein Ricci-semisymmetric manifold. Moreover, if \( n - p = 6, 12 \) or 24 then \( M \) is a non-semisymmetric manifold. It is easy to check that on \( M \) we have:

\[
\text{tr}(H) = 0, \quad S = -H^2, \quad \kappa = -\frac{2(n-p)\gamma}{3},
\]

\[
H^3 = \text{tr}(H)H^2 + \gamma H = -\frac{3\kappa}{2(n-p)} H.
\]

Now the relation \( H^3 = \lambda H \), where \( \lambda = -\frac{3\kappa}{2(n-p)} \), yields (15).

(iii) Let \( M \) be the Cartan hypersurface of dimension \( n = 6, 12 \) or 24. It is known that on \( M \) the following relations hold (see e.g. [8, Section 4]):

\[
(53) \quad H^3 = \frac{3\tau}{n(n+1)} H, \quad \text{tr}(H) = 0, \quad \kappa = \frac{(n-3)\tau}{n+1}, \quad \varepsilon = 1.
\]

Applying (53) to (13) we obtain \( A = \left( \lambda + \frac{\kappa}{n(n+1)} \right) H \) and \( \lambda = \frac{3\tau}{n(n+1)} \). In addition, using the above relations we find \( A = \frac{(n^2-3)\tau}{(n-1)n(n+1)} H \). Substituting this into (12) and using (28) we find (9) easily.

(iv) Examples of hypersurfaces in \( N_{s}^{n+1}(c), n \geq 4 \), satisfying an equation of the form \( A = \alpha H + \beta g \), where \( \alpha \) and \( \beta \) are some functions on \( M \), will be given in [16].
Theorem 5.1. Let $M$ be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$. On $U_H \subset M$ the condition $A = 0$ is equivalent to

\begin{equation}
R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R).
\end{equation}

Proof. Clearly, $A = 0$, by (12), implies (54). Now assume that (54) holds on $U_H$. Then (12) reduces to $g \wedge Q(H, A) - H \wedge Q(g, A) = 0$, which in virtue of (15) and (29) can be written in the form

\[
\begin{align*}
\varrho g \wedge Q(H, g) &= \left( \lambda + \frac{\epsilon \kappa}{n-1} \right) H \wedge Q(g, H) \\
&= -\varrho g \wedge Q(g, H) + \left( \lambda + \frac{\epsilon \kappa}{n-1} \right) H \wedge Q(H, g) \\
&= -\varrho Q(g, g \wedge H) + \left( \lambda + \frac{\epsilon \kappa}{n-1} \right) Q(H, g \wedge H) = 0.
\end{align*}
\]

Thus we have

\begin{equation}
Q\left( \left( \lambda + \frac{\epsilon \kappa}{n-1} \right) H - \varrho g, g \wedge H \right) = 0.
\end{equation}

Let $x \in U_H$. We prove that $A = (\lambda + \frac{\epsilon \kappa}{n-1}) H - \varrho g$ vanishes at $x$. First we assert that

\begin{equation}
\lambda + \frac{\epsilon \kappa}{n-1} = 0.
\end{equation}

Suppose not; then we can write (55) in the form $Q(H - \alpha g, g \wedge H) = 0$, $\alpha \in \mathbb{R}$. Applying Lemmas 3.2 and 4.2 we deduce that $x \in M - U_H$, a contradiction. Thus we have (56), and (55) now reduces to $\varrho Q(g, g \wedge H) = 0$. Supposing that $\varrho \neq 0$ we get $Q(g, g \wedge H) = 0$ and, by (28), $Q(H, G) = 0$. Applying Lemmas 3.1 and 4.2 we deduce that $x \in M - U_H$, a contradiction. So we have $\varrho = 0$ and $A = 0$. Our theorem is thus proved.

Corollary 5.1. Let $M$ be a hypersurface in $\mathbb{R}^{n+1}_s$, $n \geq 4$. The conditions $A = 0$ and (4) are equivalent on $U_H \subset M$.

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