COLLOQUIUM MATHEMATICUM

VOL. 96 2003 NO. 2

LINEAR LIFTINGS OF AFFINORS TO WEIL BUNDLES

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Abstract. We give a classification of all linear natural operators transforming affinors on each n-dimensional manifold M into affinors on $T^A M$, where T^A is the product preserving bundle functor given by a Weil algebra A, under the condition that $n \geq 2$.

We recall that an *affinor* on a manifold M is a tensor field of type $(1,1)$ on M, which can be interpreted as a linear endomorphism of the tangent bundle TM. We will denote by $\mathrm{aff}(M)$ the vector space of all affinors on M. Let A be a Weil algebra and T^A the Weil functor corresponding to A, which is a product preserving bundle functor (see [3]). Fix also a positive integer n.

A *lifting of affinors to* T^A is, by definition, a system of maps $\Lambda_M : \mathrm{aff}(M) \to \mathrm{aff}(T^A M)$ indexed by n-dimensional manifolds and satisfying for all such manifolds M, N, for every embedding $f : M \to N$ and for all $t \in \mathrm{aff}(M)$, $u \in \mathrm{aff}(N)$ the following implication:

$$Tf \circ t = u \circ Tf \;\Rightarrow\; TT^A f \circ \Lambda_M(t) = \Lambda_N(u) \circ TT^A f.$$

A lifting Λ is said to be *linear* if Λ_M is linear for each n-dimensional manifold M. Of course, all linear liftings of affinors to T^A form a vector space.

We begin by constructing three examples.

Example 1. Let $C \in A$. For every n-dimensional manifold we have the map $b_M : \mathbb{R} \times TM \ni (h, v) \mapsto hv \in TM$. Applying the product preserving functor T^A we obtain $T^A b_M : T^A\mathbb{R} \times T^A TM \to T^A TM$. But $T^A\mathbb{R} = A$ and there is a canonical exchange map between $T^A TM$ and $TT^A M$. Hence $T^A b_M$ can be interpreted as a map $A \times TT^A M \to TT^A M$, and so $TT^A M \ni V \mapsto T^A b_M(C, V) \in TT^A M$ as an affinor on $T^A M$ (this is a natural affinor constructed in [4]). Likewise, for every $t \in \mathrm{aff}(M)$ the map $T^A t : T^A TM \to T^A TM$ can be interpreted as an affinor $TT^A M \to TT^A M$

2000 *Mathematics Subject Classification*: 58A32, 58A20, 53A55.

Key words and phrases: natural operator, affinor, product preserving bundle functor, Weil algebra.

on T^AM (T^At is called the *complete lifting* of t, see [1]). Therefore we can define

$$\widetilde{C}_M(t)(V) = T^A b_M(C, T^A t(V))$$

for $V \in TT^AM$. A trivial verification shows that $\widetilde{C}$ is a linear lifting of affinors to T^A. Clearly, this lifting is the composition of the complete lifting of affinors to affinors on the Weil bundle and a natural affinor on the Weil bundle.

EXAMPLE 2. Let $L : A \to A$ be an $\mathbb{R}$-linear map. For every n-dimensional manifold M and every $t \in \operatorname{aff}(M)$ we have the trace function $\operatorname{tr} t : M \to \mathbb{R}$, and so $T^A \operatorname{tr} t : T^AM \to A$. Let $\pi_{T^AM} : TT^AM \to T^AM$ be the tangent bundle projection. Define

$$\widetilde{L}_M(t)(V) = T^A b_M(L(T^A \operatorname{tr} t(\pi_{T^AM}(V))), V)$$

for $V \in TT^AM$, where b_M is as in Example 1. A trivial verification shows that $\widetilde{L}$ is a linear lifting of affinors to T^A. It is worth pointing out that this lifting is nothing but a sum of products of linear liftings of affinors to functions on the Weil bundle (see [5]) and natural affinors on the Weil bundle.

EXAMPLE 3. Let $D : A \times A \to A$ be an $\mathbb{R}$-bilinear map with the property that $D(P \cdot Q, R) = P \cdot D(Q, R) + D(P, R) \cdot Q$ for $P, Q, R \in A$. For every n-dimensional manifold M and every $t \in \operatorname{aff}(M)$ we have the map $d(T^A \operatorname{tr} t) : TT^AM \to A$, which is the exterior derivative of $T^A \operatorname{tr} t$. Clearly, for every $V \in TT^AM$ the map $r_{t,V} : A \ni P \mapsto D(P, d(T^A \operatorname{tr} t)(V)) \in A$ is a differentiation of the algebra A. It is well known that every differentiation of the Weil algebra A determines in a natural way a vector field on T^AN for each manifold N (see [2] for a construction of such natural vector fields). Denote by $\widetilde{r_{t,V}}_M$ the vector field on T^AM determined by $r_{t,V}$. Define

$$\widetilde{D}_M(t)(V) = \widetilde{r_{t,V}}_M(\pi_{T^AM}(V))$$

for $V \in TT^AM$. A trivial verification shows that $\widetilde{D}$ is a linear lifting of affinors to T^A. Observe that this lifting is nothing but a sum of tensor products of natural vector fields on the Weil bundle and linear liftings of affinors to 1-forms on the Weil bundle (see [5]).

We are now in a position to formulate our main result.

THEOREM. *If $n \geq 2$ then for each linear lifting Λ of affinors to T^A there are $C \in A$, an $\mathbb{R}$-linear map $L : A \to A$ and an $\mathbb{R}$-bilinear map $D : A \times A \to A$ with the property that $D(P \cdot Q, R) = P \cdot D(Q, R) + D(P, R) \cdot Q$ for $P, Q, R \in A$, such that*

$$\Lambda = \widetilde{C} + \widetilde{L} + \widetilde{D}.$$

Moreover, C, L and D are uniquely determined.

The proof will be based on a general lemma proved by Mikulski (see [5]). This lemma applied to our problem says that if Δ and Θ are two linear liftings of affinors to T^A and

$$\Delta_{\mathbb{R}^n}\left(x^1 \frac{\partial}{\partial x^1} \otimes dx^1\right) = \Theta_{\mathbb{R}^n}\left(x^1 \frac{\partial}{\partial x^1} \otimes dx^1\right),$$

then $\Delta = \Theta$.

The proof of our theorem will be divided into several steps, but first we have to establish a few basic facts and introduce some notation.

We first observe that for every open subset U of $\mathbb{R}^n$ and every $t \in \mathrm{aff}(\mathbb{R}^n)$ we have $\Lambda_U(t|_{TU}) = \Lambda_{\mathbb{R}^n}(t)|_{TT^AU}$. This can be easily proved by taking the inclusion $U \to \mathbb{R}^n$ for f in the definition of lifting.

Next, $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and if $s \in \mathrm{aff}(\mathbb{R}^n)$, then $s(x,y) = (x, s_i(x)y^i)$ for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, where $s^i_j : \mathbb{R}^n \to \mathbb{R}$ for $i,j \in \{1,\dots,n\}$ are smooth maps. Similarly, $TA^n = A^n \times A^n$. Let $\mathrm{end}(A)$ denote the vector space of all $\mathbb{R}$-linear endomorphisms of A. Thus, if $S \in \mathrm{aff}(A^n)$, then $S(X,Y) = (X, S_i(X)(Y^i))$ for $(X,Y) \in A^n \times A^n$, where $S^i_j : A^n \to \mathrm{end}(A)$ for $i,j \in \{1,\dots,n\}$ are smooth maps. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map. Since the addition and multiplication in the algebra A are maps obtained by applying T^A to the addition and multiplication in the field $\mathbb{R}$, it is evident that $T^A f(X) = f(X)$ for $X \in A^n$ and

$$TT^A f(X,Y) = \left(f(X), \frac{\partial f}{\partial x^i}(X) \cdot Y^i\right)$$

for $(X,Y) \in A^n \times A^n$.

We will identify each $P \in A$ with the map $A \ni Q \mapsto P \cdot Q \in A$, which is an element of $\mathrm{end}(A)$.

Therefore for every open subset U of $\mathbb{R}^n$, for every polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f|_U$ is an embedding and for all $t, u \in \mathrm{aff}(\mathbb{R}^n)$, if

$$\frac{\partial f^i}{\partial x^j}(x) t^j_k(x) = u^i_j(f(x)) \frac{\partial f^j}{\partial x^k}(x) \tag{1}$$

for $i,k \in \{1,\dots,n\}$ and $x \in U$, then

$$\frac{\partial f^i}{\partial x^j}(X) \circ \Lambda_{\mathbb{R}^n}(t)^j_k(X) = \Lambda_{\mathbb{R}^n}(u)^i_j(f(X)) \circ \frac{\partial f^j}{\partial x^k}(X) \tag{2}$$

for $i,k \in \{1,\dots,n\}$ and $X \in T^AU$.

Finally, let $e \in \mathrm{aff}(\mathbb{R}^n)$ be the affinor from Mikulski's lemma. In other words $e^1_1(x) = x^1$, $e^i_j(x) = 0$ for $i \neq 1$ or $j \neq 1$.

STEP 1. *The maps $\Lambda_{\mathbb{R}^n}(e)^p_q : A^n \to \mathrm{end}(A)$ for $p,q \in \{1,\dots,n\}$ are $\mathbb{R}$-linear.*

Fix $h \in \mathbb{R} \setminus \{0\}$. If $U = \mathbb{R}^n$ and $f(x) = hx$, $t = he$, $u = e$, then (1) holds. Hence (2) holds. Taking $i = p$ and $k = q$ we get $h^2 \Lambda_{\mathbb{R}^n}(e)^p_q(X) =$

$h\Lambda_{\mathbb{R}^n}(e)^p_q(hX)$, and so $h\Lambda_{\mathbb{R}^n}(e)^p_q(X) = \Lambda_{\mathbb{R}^n}(e)^p_q(hX)$. By continuity, the same holds for every $h \in \mathbb{R}$. Applying the homogeneous function theorem (see [3]) we deduce that $\Lambda_{\mathbb{R}^n}(e)^p_q$ is $\mathbb{R}$-linear.

STEP 2. *If* $p, q \in \{1, \ldots, n\}$ *are such that* $p \neq q$, $q \neq 1$, *then* $\Lambda_{\mathbb{R}^n}(e)^p_q = 0$.

Fix $h \in \mathbb{R} \setminus \{0\}$. If $U = \mathbb{R}^n$, $f^q(x) = hx^q$, $f^i(x) = x^i$ for $i \neq q$, $t = e$, $u = e$, then (1) holds. Hence (2) holds. Taking $i = p$ and $k = q$ we get $\Lambda_{\mathbb{R}^n}(e)^p_q(X) = h\Lambda_{\mathbb{R}^n}(e)^p_q(f(X))$. This is still true if $h = 0$, by continuity. Hence $\Lambda_{\mathbb{R}^n}(e)^p_q = 0$.

STEP 3. *There is an* $\mathbb{R}$*-linear map* $E : A \to \operatorname{end}(A)$ *such that* $\Lambda_{\mathbb{R}^n}(e)^1_1(X) = E(X^1)$ *for* $X \in A^n$. *For each* $p \in \{2, \ldots, n\}$ *there is an* $\mathbb{R}$*-linear map* $F^p : A \to \operatorname{end}(A)$ *such that* $\Lambda_{\mathbb{R}^n}(e)^p_p(X) = F^p(X^1)$ *for* $X \in A^n$.

Fix $h \in \mathbb{R} \setminus \{0\}$. If $U = \mathbb{R}^n$ and $f^i(x) = hx^i$ for $i \neq 1$, $f^1(x) = x^1$, $t = e$, $u = e$, then (1) holds. Hence (2) holds.

Taking $i = 1$ and $k = 1$ we get $\Lambda_{\mathbb{R}^n}(e)^1_1(X) = \Lambda_{\mathbb{R}^n}(e)^1_1(f(X))$. This is still true if $h = 0$, by continuity.

Taking $i, k = p$ we get $h\Lambda_{\mathbb{R}^n}(e)^p_p(X) = h\Lambda_{\mathbb{R}^n}(e)^p_p(f(X))$, and so $\Lambda_{\mathbb{R}^n}(e)^p_p(X) = \Lambda_{\mathbb{R}^n}(e)^p_p(f(X))$. This is still true if $h = 0$, by continuity.

But if $h = 0$, then $f^i(X) = 0$ for $i \neq 1$, $f^1(X) = X^1$. Hence the existence of E and F^p is obvious. From Step 1 we see that E and F^p are $\mathbb{R}$-linear.

STEP 4. *For each* $p \in \{2, \ldots, n\}$ *there is an* $\mathbb{R}$*-linear map* $G^p : A \to \operatorname{end}(A)$ *such that* $\Lambda_{\mathbb{R}^n}(e)^p_1(X) = G^p(X^p)$ *for* $X \in A^n$.

Fix $h \in \mathbb{R} \setminus \{0\}$. If $U = \mathbb{R}^n$ and $f^i(x) = hx^i$ for $i \neq p$, $f^p(x) = x^p$, $t = he$, $u = e$, then (1) holds. Hence (2) holds. Taking $i = p$ and $k = 1$ we get $h\Lambda_{\mathbb{R}^n}(e)^p_1(X) = h\Lambda_{\mathbb{R}^n}(e)^p_1(f(X))$, and so $\Lambda_{\mathbb{R}^n}(e)^p_1(X) = \Lambda_{\mathbb{R}^n}(e)^p_1(f(X))$. This is still true if $h = 0$, by continuity. But if $h = 0$, then $f^i(X) = 0$ for $i \neq p$ and $f^p(X) = X^p$. Hence the existence of G^p is obvious. From Step 1 we see that G^p is $\mathbb{R}$-linear.

STEP 5. *There are* $\mathbb{R}$*-linear maps* $F, G : A \to \operatorname{end}(A)$ *such that* $F^p = F$ *and* $G^p = G$ *for all* $p \in \{2, \ldots, n\}$.

Fix $p, q \in \{2, \ldots, n\}$. If $U = \mathbb{R}^n$ and $f^p(x) = x^q$, $f^q(x) = x^p$, $f^i(x) = x^i$ for $i \neq p$ and $i \neq q$, $t = u = e$, then (1) holds. Hence (2) holds.

Taking $i = p$ and $k = q$ we get $\Lambda_{\mathbb{R}^n}(e)^q_q(X) = \Lambda_{\mathbb{R}^n}(e)^p_p(f(X))$, and so $F^q(X^1) = F^p(X^1)$, by Step 3.

Taking $i = p$ and $k = 1$ we get $\Lambda_{\mathbb{R}^n}(e)^q_1(X) = \Lambda_{\mathbb{R}^n}(e)^p_1(f(X))$, and so $G^q(X^q) = G^p(X^q)$, by Step 4.

Hence the existence of F and G is obvious.

STEP 6. *There are an* $\mathbb{R}$*-linear map* $L : A \to A$ *and an* $\mathbb{R}$*-bilinear map* $D : A \times A \to A$ *with the property that* $D(P \cdot Q, R) = P \cdot D(Q, R) + D(P, R) \cdot Q$

for $P, Q, R \in A$, such that $F(P)(Q) = L(P)\cdot Q$ for $P, Q \in A$ and $G(Q)(R) = D(Q, R)$ for $Q, R \in A$.

If $U = \{x \in \mathbb{R}^n : x^2 > 0\}$ and $f = g \times \mathrm{id}_{\mathbb{R}^{n-2}}$, where $g(x^1, x^2) = (x^1, (x^2)^2)$, $t = u = e$, then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 2x^2 \end{bmatrix} \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2x^2 \end{bmatrix}.$$

This clearly holds, hence so does (2). On account of the previous steps, we have

$$\begin{bmatrix} \Lambda_{\mathbb{R}^n}(e)^1_1(X) & \Lambda_{\mathbb{R}^n}(e)^1_2(X) \\ \Lambda_{\mathbb{R}^n}(e)^2_1(X) & \Lambda_{\mathbb{R}^n}(e)^2_2(X) \end{bmatrix} = \begin{bmatrix} E(X^1) & 0 \\ G(X^2) & F(X^1) \end{bmatrix}.$$

Hence from (2) it follows that

$$\begin{bmatrix} 1 & 0 \\ 0 & 2X^2 \end{bmatrix} \circ \begin{bmatrix} E(X^1) & 0 \\ G(X^2) & F(X^1) \end{bmatrix} = \begin{bmatrix} E(X^1) & 0 \\ G((X^2)^2) & F(X^1) \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & 2X^2 \end{bmatrix},$$

and so

$$\begin{bmatrix} E(X^1) & 0 \\ 2X^2 \circ G(X^2) & 2X^2 \circ F(X^1) \end{bmatrix} = \begin{bmatrix} E(X^1) & 0 \\ G((X^2)^2) & F(X^1) \circ 2X^2 \end{bmatrix}$$

for $X \in T^A U$. Therefore

$$Q \circ F(P) = F(P) \circ Q, \tag{3}$$

$$2Q \circ G(Q) = G(Q^2) \tag{4}$$

for $P, Q \in A$ such that $\pi^A_{\mathbb{R}}(Q) > 0$, where $\pi^A_{\mathbb{R}} : T^A\mathbb{R} \to \mathbb{R}$ is the bundle projection.

Replacing $U = \{x \in \mathbb{R}^n : x^2 > 0\}$ by $U = \{x \in \mathbb{R}^n : x^2 < 0\}$ and $g(x^1, x^2) = (x^1, (x^2)^2)$ by $g(x^1, x^2) = (x^1, -(x^2)^2)$ we can obtain (3) and (4) for $P, Q \in A$ such that $\pi^A_{\mathbb{R}}(Q) < 0$ in the same manner. Thus (3) and (4) hold for all $P, Q \in A$, by continuity.

Since (3) means that $F(P)$ is A-linear, it suffices to put $L(P) = F(P)(1)$ for $P \in A$.

Polarization of (4) yields $P \cdot G(Q)(R) + G(P)(R) \cdot Q = G(P \cdot Q)(R)$ for $P, Q, R \in A$. Hence in order to complete Step 6 it suffices to put $D(Q, R) = G(Q)(R)$ for $Q, R \in A$.

Before the final step it is useful to summarize what we have proved up till now. Namely, with the notation $H = E - F - G$ we have $\Lambda_{\mathbb{R}^n}(e)^i_j(X)(Y^j) = \delta^{i1} H(X^1)(Y^1) + L(X^1) \cdot Y^i + D(X^i, Y^1)$. A trivial computation shows that

$$\widetilde{L}_{\mathbb{R}^n}(e)^i_j(X)(Y^j) = L(X^1) \cdot Y^i, \qquad \widetilde{D}_{\mathbb{R}^n}(e)^i_j(X)(Y^j) = D(X^i, Y^1).$$

Write $\Xi = \Lambda - \widetilde{L} - \widetilde{D}$. Then $\Xi_{\mathbb{R}^n}(e)^i_j(X)(Y^j) = \delta^{i1} H(X^1)(Y^1)$. A trivial computation shows that for each $C \in A$ we have

$$\widetilde{C}_{\mathbb{R}^n}(e)^i_j(X)(Y^j) = \delta^{i1} C \cdot X^1 \cdot Y^1.$$

Therefore, by Mikulski's lemma, the proof will be completed as soon as we make the following Step 7.

STEP 7. *There is* $C \in A$ *such that* $H(P)(Q) = C \cdot P \cdot Q$ *for* $P, Q \in A$.

In this step we will apply (1) and (2) for $\Lambda = \Xi$ and $f = g \times \mathrm{id}_{\mathbb{R}^{n-2}}$, where $g : \mathbb{R}^2 \to \mathbb{R}^2$. We will use only affinors $s \in \mathrm{aff}(\mathbb{R}^n)$ with the property that if $i \geq 3$ or $j \geq 3$, then $s^i_j = 0$. Such an affinor s will be written as

$$\begin{bmatrix} s^1_1(x) & s^1_2(x) \\ s^2_1(x) & s^2_2(x) \end{bmatrix}.$$

Similarly, we will only use affinors $S \in \mathrm{aff}(A^n)$ with $S^i_j = 0$ if $i \geq 3$ or $j \geq 3$. Such an affinor S will be written as

$$\begin{bmatrix} S^1_1(X) & S^1_2(X) \\ S^2_1(X) & S^2_2(X) \end{bmatrix}.$$

We have proved

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} H(X^1) & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where $H : A \to \mathrm{end}(A)$ is $\mathbb{R}$-linear.

The proof of Step 7 falls naturally into two parts.

PART 1. *For each* $Q \in A$ *the map* $A \ni P \mapsto H(P)(Q) \in A$ *is* A*-linear.*

If $U = \mathbb{R}^n$ and $g(x^1, x^2) = (1 + x^1, x^2)$, $t = v + u$, where

$$v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 + x^1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 + x^1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since this is indeed true, (2) follows. Hence, by (5),

$$\begin{aligned} \Xi_{\mathbb{R}^n}(v)(X) + \begin{bmatrix} H(X^1) & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} H(1 + X^1) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} H(1 + X^1) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} H(1 + X^1) & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

for $X \in A^n$. Therefore

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} H(1) & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

If $U = \mathbb{R}^n$ and $g(x^1, x^2) = (x^1, x^1 + x^2)$, $t = u - v$, where

$$u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This being true, we have (2). Hence, by (6),

$$\begin{bmatrix} H(1) & 0 \\ 0 & 0 \end{bmatrix} - \Xi_{\mathbb{R}^n}(v)(X) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \circ \begin{bmatrix} H(1) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} H(1) & 0 \\ -H(1) & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} H(1) & 0 \\ -H(1) & 0 \end{bmatrix}$$

for $X \in A^n$. Therefore

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ H(1) & 0 \end{bmatrix}. \tag{7}$$

If $U = \mathbb{R}^n$ and $g(x^1, x^2) = (x^1, x^1 + x^2)$, $t = u - v$, where

$$u = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 & 0 \\ x^1 & 0 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x^1 & 0 \\ -x^1 & 0 \end{bmatrix} = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

As this is true, (2) holds. Hence, by (5),

$$\begin{bmatrix} H(X^1) & 0 \\ 0 & 0 \end{bmatrix} - \Lambda_{\mathbb{R}^n}(v)(X) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \circ \begin{bmatrix} H(X^1) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} H(X^1) & 0 \\ -H(X^1) & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} H(X^1) & 0 \\ -H(X^1) & 0 \end{bmatrix}$$

for $X \in A^n$. Therefore

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} 0 & 0 \\ x^1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ H(X^1) & 0 \end{bmatrix}. \tag{8}$$

If $U = \{x \in \mathbb{R}^n : x^1 \neq 0\}$ and $g(x^1, x^2) = (x^1, x^1 x^2)$,

$$t = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} 0 & 0 \\ x^1 & 0 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ x^2 & x^1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x^2 & x^1 \end{bmatrix},$$

which is true, implying (2). Hence, by (7) and (8),

$$\begin{bmatrix} 1 & 0 \\ X^2 & X^1 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ H(1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ H(X^1) & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ X^2 & X^1 \end{bmatrix},$$

and so

$$\begin{bmatrix} 0 & 0 \\ X^1 \circ H(1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ H(X^1) & 0 \end{bmatrix}$$

for $X \in T^A U$. Therefore $P \circ H(1) = H(P)$ for $P \in A$ such that $\pi^A_{\mathbb{R}}(P) \neq 0$. By continuity, the same holds for every $P \in A$. This proves Part 1.

PART 2. *For each $P \in A$ the map $A \ni Q \mapsto H(P)(Q) \in A$ is A-linear.*

If $U = \{x \in \mathbb{R}^n : x^1 \neq 0\}$ and $g(x^1, x^2) = (x^1, x^1 x^2)$, $t = u - v$, where

$$u = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 & 0 \\ x^2 & 0 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ x^2 & x^1 \end{bmatrix} \begin{bmatrix} x^1 & 0 \\ -x^2 & 0 \end{bmatrix} = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x^2 & x^1 \end{bmatrix}.$$

Since this is the case, (2) follows. Hence, by (5),

$$\begin{bmatrix} H(X^1) & 0 \\ 0 & 0 \end{bmatrix} - \Lambda_{\mathbb{R}^n}(v)(X) = \begin{bmatrix} 1 & 0 \\ \dfrac{-X^2}{X^1} & \dfrac{1}{X^1} \end{bmatrix} \circ \begin{bmatrix} H(X^1) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ X^2 & X^1 \end{bmatrix}$$
$$= \begin{bmatrix} H(X^1) & 0 \\ -\dfrac{X^2}{X^1} \circ H(X^1) & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ X^2 & X^1 \end{bmatrix} = \begin{bmatrix} H(X^1) & 0 \\ -\dfrac{X^2}{X^1} \circ H(X^1) & 0 \end{bmatrix}$$

for $X \in T^A U$. Therefore, by Part 1 and by contunuity,

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} 0 & 0 \\ x^2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ H(X^2) & 0 \end{bmatrix}. \tag{9}$$

If $U = \mathbb{R}^n$ and $g(x^1, x^2) = (x^2, x^1)$,

$$t = \begin{bmatrix} 0 & 0 \\ 0 & x^2 \end{bmatrix}, \qquad u = \begin{bmatrix} x^1 & 0 \\ 0 & 0 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x^2 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This being true, we have (2). Hence, by (5),

$$\Lambda_{\mathbb{R}^n}(t)(X) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} H(X^2) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ H(X^2) & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & H(X^2) \end{bmatrix}$$

for $X \in A^n$. Therefore

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} 0 & 0 \\ 0 & x^2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & H(X^2) \end{bmatrix}. \tag{10}$$

If $U = \mathbb{R}^n$ and $g(x^1, x^2) = (x^1, x^1 + x^2)$,

$$t = \begin{bmatrix} 0 & 0 \\ x^1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x^2 & 0 \end{bmatrix} + v + u,$$

where

$$v = \begin{bmatrix} 0 & 0 \\ 0 & x^1 \end{bmatrix}, \qquad u = \begin{bmatrix} 0 & 0 \\ 0 & x^2 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x^1 + x^2 & x^1 + x^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & x^1 + x^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

which implies (2). Hence, by (8)–(10),

$$\begin{aligned} &\begin{bmatrix} 0 & 0 \\ H(X^1) & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ H(X^2) & 0 \end{bmatrix} + \Xi_{\mathbb{R}^n}(v)(X) + \begin{bmatrix} 0 & 0 \\ 0 & H(X^2) \end{bmatrix} \\ &\qquad = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ 0 & H(X^1 + X^2) \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &\qquad = \begin{bmatrix} 0 & 0 \\ 0 & H(X^1 + X^2) \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ H(X^1 + X^2) & H(X^1 + X^2) \end{bmatrix} \end{aligned}$$

for $X \in A^n$. Therefore

$$\Xi_{\mathbb{R}^n}\left(\begin{bmatrix} 0 & 0 \\ 0 & x^1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & H(X^1) \end{bmatrix}. \tag{11}$$

If $U = \{x \in \mathbb{R}^n : x^1 \neq 0\}$ and $g(x^1, x^2) = (x^1, x^1 x^2)$,

$$t = \begin{bmatrix} 0 & 0 \\ x^2 & x^1 \end{bmatrix}, \qquad u = \begin{bmatrix} 0 & 0 \\ 0 & x^1 \end{bmatrix},$$

then (1) becomes

$$\begin{bmatrix} 1 & 0 \\ x^2 & x^1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x^2 & x^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & x^1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ x^2 & x^1 \end{bmatrix},$$

and thus again (2) holds. Hence, by (9) and (11),

$$\begin{bmatrix} 1 & 0 \\ X^2 & X^1 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ H(X^2) & H(X^1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & H(X^1) \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ X^2 & X^1 \end{bmatrix},$$

and so

$$\begin{bmatrix} 0 & 0 \\ X^1 \circ H(X^2) & X^1 \circ H(X^1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ H(X^1) \circ X^2 & H(X^1) \circ X^1 \end{bmatrix}$$

for $X \in T^A U$. Therefore $P \circ H(Q) = H(P) \circ Q$ and, by Part 1, $Q \circ H(P) = H(P) \circ Q$ for $P, Q \in A$ such that $\pi^A_{\mathbb{R}}(P) \neq 0$. By continuity, the same holds for all $P, Q \in A$. This proves Part 2.

In order to complete Step 7 and the whole proof it suffices to put $C = H(1)(1)$.

Since the formulation of our theorem seems to be somewhat abstract, we end off the paper with an example.

EXAMPLE. The simplest Weil functor is the well known tangent functor T, which corresponds to the Weil algebra of dual numbers (see [6]). The algebra of dual numbers can be represented as the vector space $\mathbb{R}^2$ endowed with the multiplication $(a, b) \cdot (c, d) = (ac, ad + bc)$. We will describe the coordinate form of liftings from Examples 1–3 in the case of the tangent bundle. Fix an n-dimensional manifold M and $t \in \operatorname{aff}(M)$. Then

$$t = t^i_j(q) \, \frac{\partial}{\partial q^i} \otimes dq^j$$

in local coordinates q on M. Furthermore, we have the local coordinates $(q, \dot{q})$ on TM induced by q.

Set $C_1 = (1, 0)$, $C_2 = (0, 1)$. Of course, C_1, C_2 form a basis of $\mathbb{R}^2$. An easy computation shows that

$$\widetilde{C_1}_M(t) = t^i_j(q) \, \frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial t^i_j}{\partial q^k}(q)\dot{q}^k \, \frac{\partial}{\partial \dot{q}^i} \otimes dq^j + t^i_j(q) \, \frac{\partial}{\partial \dot{q}^i} \otimes d\dot{q}^j,$$
$$\widetilde{C_2}_M(t) = t^i_j(q) \, \frac{\partial}{\partial \dot{q}^i} \otimes dq^j.$$

Set $L^1_1(a, b) = (a, 0)$, $L^2_1(a, b) = (b, 0)$, $L^1_2(a, b) = (0, a)$, $L^2_2(a, b) = (0, b)$ for $(a, b) \in \mathbb{R}^2$. Of course, L^1_1, L^2_1, L^1_2, L^2_2 form a basis of the vector space of all $\mathbb{R}$-linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. An easy computation shows that

$$\widetilde{L^1_1}_M(t) = t^i_i(q)\left(\frac{\partial}{\partial q^j} \otimes dq^j + \frac{\partial}{\partial \dot{q}^j} \otimes d\dot{q}^j\right),$$
$$\widetilde{L^2_1}_M(t) = \frac{\partial t^i_i}{\partial q^j}(q)\dot{q}^j\left(\frac{\partial}{\partial q^k} \otimes dq^k + \frac{\partial}{\partial \dot{q}^k} \otimes d\dot{q}^k\right),$$
$$\widetilde{L^1_2}_M(t) = t^i_i(q) \, \frac{\partial}{\partial \dot{q}^j} \otimes dq^j,$$
$$\widetilde{L^2_2}_M(t) = \frac{\partial t^i_i}{\partial q^j}(q)\dot{q}^j \, \frac{\partial}{\partial \dot{q}^k} \otimes dq^k.$$

Set $D^1((a, b), (c, d)) = (0, bc)$, $D^2((a, b), (c, d)) = (0, bd)$ for $(a, b), (c, d) \in \mathbb{R}^2$. It is a simple matter to check that D^1, D^2 form a basis of the vector space of all $\mathbb{R}$-bilinear maps $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with the property that

$D(P \cdot Q, R) = P \cdot D(Q, R) + D(P, R) \cdot Q$ for $P, Q, R \in \mathbb{R}^2$. An easy computation shows that

$$\widetilde{D^1}_M(t) = \frac{\partial t_i^i}{\partial q^j}(q)\dot{q}^k \frac{\partial}{\partial \dot{q}^k} \otimes dq^j,$$

$$\widetilde{D^2}_M(t) = \frac{\partial^2 t_i^i}{\partial q^j \partial q^k}(q)\dot{q}^k \dot{q}^l \frac{\partial}{\partial \dot{q}^l} \otimes dq^j + \frac{\partial t_i^i}{\partial q^j}(q)\dot{q}^k \frac{\partial}{\partial \dot{q}^k} \otimes d\dot{q}^j.$$

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Received 27 February 2002;
revised 20 January 2003 (4173)