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NOTE ON A HYPOTHESIS IMPLYING THE NON-VANISHING OF DIRICHLET L-SERIES $L(s,\chi)$ FOR $s>0$ AND REAL CHARACTERS χ

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Abstract. We prove that if χ is a real non-principal Dirichlet character for which $L(1,\chi)\le 1-\log 2$, then Chowla's hypothesis is not satisfied and we cannot use Chowla's method for proving that $L(s,\chi)>0$ for $s>0$.

1. Introduction. Let χ be a real non-principal Dirichlet character (we do not assume that χ is primitive). Set

$$S_\chi^{(0)}(n)=\chi(n)\quad\text{and}\quad S_\chi^{(m+1)}(n)=\sum_{a=0}^{n}S_\chi^{(m)}(a)\qquad(n\ge 0\text{ and }m\ge 0),$$

and

$$m(\chi):=\min\{m\ge 1;\ S_\chi^{(m)}(n)\ge 0\text{ for all }n\ge 0\}$$

if this set is non-empty, and $m(\chi)=\infty$ otherwise. Since

$$\Gamma(s)L(s,\chi)=\int_0^\infty\Big(\sum_{n\ge 1}S_\chi^{(m)}(n)e^{-nt}\Big)(1-e^{-t})^m t^{s-1}\,dt\qquad(s>0\text{ and }m\ge 1),$$

we see that if $m(\chi)<\infty$ then $L(s,\chi)>0$ for all $s>0$ (see [Cho]). S. Chowla believed that $m(\chi)<\infty$ for all real non-principal Dirichlet characters. However, noticing that

$$f_\chi(t):=\sum_{n\ge 1}\chi(n)t^n=(1-t)^m\sum_{n\ge 1}S_\chi^{(m)}(n)t^n\qquad(0\le t<1\text{ and }m\ge 0)$$

we deduce that if $f_\chi(t_0)<0$ for some $t_0\in[0,1)$, then $m(\chi)=\infty$. In that way, H. Heilbronn proved that there are infinitely many (primitive) quadratic characters χ for which $m(\chi)=\infty$ (see [Hei]). Following Heilbronn's result, it has then been conjectured that for any non-principal real character ψ there exists some induced character χ for which $m(\chi)=1$ if ψ is odd and $m(\chi)=2$ if ψ is even, which implies $L(s,\chi)>0$ for $s>0$ and $L(s,\psi)>0$ for $s>0$ (see [CD], [CDH], [CH] and [Ros]). No counterexample

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to this conjecture is known. Since

$$L(1,\chi)=\sum_{n\geq 1}\frac{m!}{n(n+1)\dots(n+m)}\,S_\chi^{(m)}(n)\qquad (m\geq 0) \tag{1}$$

and $S_\chi^{(m)}(1)=1$, we obtain $L(1,\chi)\geq 1/(1+m(\chi))$ (see [Cho]). Hence, if $L(1,\chi)$ is small then $m(\chi)$ must be large. We improve upon this result:

Theorem 1. *If $L(1,\chi)\leq 1-\log 2=0.306852\dots$ then $m(\chi)=\infty$ (i.e., there does not exist any $m\geq 0$ such that $S_\chi^{(m)}(n)\geq 0$ for all $n\geq 1$).*

It follows that there are infinitely many (primitive) quadratic characters χ for which $m(\chi)=\infty$ (by [CE]), a result proved by H. Heilbronn (see [Hei]). Even though we do not know the value of the largest constant $c\geq 1-\log 2$ for which Theorem 1 holds true for all (or for all but finitely many) real non-principal Dirichlet characters χ, there is not much room for improving Theorem 1:

Theorem 2. *Let χ_3 be the Dirichlet character mod 3 defined by $\chi_3(n)=0$, 1 or -1 according as $n\equiv 0$, 1 or 2 $\pmod 3$. Let $p\neq 3$ be a prime, and let χ_{3p} denote the real non-principal Dirichlet character mod $3p$ induced by χ_3. Then*

$$L(1,\chi_{3p})=\left(1-\frac{\chi_3(p)}{p}\right)L(1,\chi_3)=\left(1-\frac{\chi_3(p)}{p}\right)\frac{\pi}{3\sqrt{3}}\leq\frac{\pi}{2\sqrt{3}}=0.906899\dots$$

is asymptotic to $\pi/(3\sqrt{3})=0.604599\dots$ as $p\to\infty$ but $m(\chi_{3p})<\infty$, for

$$m(\chi_{3p})=\begin{cases}1 & \text{if } p\equiv 2 \pmod 3.\\ 3 & \text{if } p\equiv 1 \pmod 3.\end{cases}$$

In particular, for $p=7$ we have $m(\chi)<\infty$ and $L(1,\chi)=2\pi/(7\sqrt{3})=0.518228\dots$

2. Proof of Theorem 1. Theorem 1 follows from the following more precise result:

Proposition 3. *Let χ be a real non-principal Dirichlet character. If $S_\chi^{(m)}(n)\geq 0$ for all $n\geq 1$, then*

$$L(1,\chi)\geq 1-\sum_{n=1}^{m}\frac{1}{n+m}=\sum_{n=2}^{2m}\frac{(-1)^n}{n}>1-\log 2=0.306852\dots$$

Proof. Set $f(n)=1$ for $n=1$ and $f(n)=-1$ for $n>1$. Then $\chi\geq f$ yields $S_\chi^{(m)}(n)\geq S_f^{(m)}(n)$ for all $n\geq 1$. By induction on $m\geq 0$, we easily obtain

$$\begin{aligned}\sum_{n\geq 1} S_f^{(m)}(n)t^n &= (1-t)^{-m}\sum_{n\geq 1} f(n)t^n\\ &= -\frac{1}{(1-t)^{m+1}} + 3\,\frac{1}{(1-t)^m} - 2\,\frac{1}{(1-t)^{m-1}}.\end{aligned}$$

By looking at the values at $t=0$ of the nth derivative of this equality, we obtain

$$S_f^{(m)}(n) = \frac{n(n+1)\dots(n+m)}{m!}\,\frac{m+1-n}{(n+m-1)(n+m)} \qquad (m\geq 1 \text{ and } n\geq 1).$$

Now assume that $S_\chi^{(m)}(n)\geq 0$ for all $n\geq 1$. Using (1), we obtain

$$\begin{aligned}L(1,\chi) &\geq \sum_{n=1}^{m}\frac{m!}{n(n+1)\dots(n+m)}\,S_\chi^{(m)}(n)\\ &\geq \sum_{n=1}^{m}\frac{m!}{n(n+1)\dots(n+m)}\,S_f^{(m)}(n)\\ &= \sum_{n=1}^{m}(m+1-n)\left(\frac{1}{n+m-1}-\frac{1}{n+m}\right) = 1-\sum_{n=1}^{m}\frac{1}{n+m},\end{aligned}$$

where we have used $S_\chi^{(m)}(n)\geq S_f^{(m)}(n)$ for $1\leq n\leq m$. ∎

3. Proof of Theorem 2

LEMMA 4. *Let χ be an odd real non-principal Dirichlet character mod f. Assume that $L(0,\chi)=0$. Then $S_\chi^{(m)}(f)=0$ and $n\mapsto S_\chi^{(m)}(n)$ is f-periodic for $0\leq m\leq 3$. Hence, $S_\chi^{(3)}(n)\geq 0$ for all $n\geq 1$ if and only if $S_\chi^{(3)}(n)\geq 0$ for $1\leq n\leq f$.*

Proof. Since χ is non-principal, we have $\sum_{n=1}^{f}\chi(n)=0$ and $fL(0,\chi)=-\sum_{n=1}^{f}n\chi(n)$ (see [Wa, Theorem 4.2]). Hence, $\sum_{n=1}^{f}n\chi(n)=0$ and

$$\sum_{n=1}^{f-1}n^2\chi(n)=\sum_{n=1}^{f-1}(f-n)^2\chi(f-n)=-\sum_{n=1}^{f-1}(f^2-2fn+n^2)\chi(n)=-\sum_{n=1}^{f-1}n^2\chi(n)$$

yields $\sum_{n=1}^{f}n^2\chi(n)=0$. It follows that $S_\chi^{(1)}(f)=0$,

$$S_\chi^{(2)}(f)=\sum_{a=1}^{f}\sum_{b=1}^{a}\chi(b)=\sum_{b=1}^{f}(f+1-b)\chi(b)=0$$

and

$$S_\chi^{(3)}(f) = \sum_{a=1}^{f}\sum_{b=1}^{a}\sum_{c=1}^{b}\chi(c) = \sum_{c=1}^{f}\frac{(f+1-c)(f+2-c)}{2}\chi(c) = 0.$$

Finally, for the f-periodicity of $S_\chi^{(m)}$ for $0 \leq m \leq 3$, we notice that

$$S_\chi^{(m+1)}(f+n) = S_\chi^{(m+1)}(f) + \sum_{m=1}^{n} S_\chi^{(m)}(f+m) \qquad (n \geq 0 \text{ and } m \geq 0).$$

Hence, if $S_\chi^{(m)}$ is f-periodic and $S_\chi^{(m+1)}(f) = 0$ then $S_\chi^{(m+1)}$ is f-periodic. ∎

We are now in a position to proceed with the proof of Theorem 2. By induction on $n \geq 1$, we have

$$S_{\chi_3}^{(1)}(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{2}$$

$$S_{\chi_3}^{(2)}(n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3, \\ (n+2)/3 & \text{if } n \equiv 1 \pmod 3, \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{3}$$

$$S_{\chi_3}^{(3)}(n) = \begin{cases} (n^2+3n)/6 & \text{if } n \equiv 0 \pmod 3, \\ (n^2+3n+2)/6 & \text{if } n \equiv 1 \pmod 3, \\ (n^2+3n+2)/6 & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{4}$$

Let ψ_{3p} be $3p$-periodic and defined by $\psi_{3p}(n) = 0$ if p does not divide n and $\psi_{3p}(n) = \chi_3(n)$ if p divides n. Then

$$S_{\psi_{3p}}^{(1)}(n) = \chi_3(p)\cdot\begin{cases} 0 & \text{if } 1 \leq n \leq p-1, \\ 1 & \text{if } p \leq n \leq 2p-1, \\ 0 & \text{if } 2p \leq n \leq 3p-1, \end{cases} \tag{5}$$

$$S_{\psi_{3p}}^{(2)}(n) = \chi_3(p)\cdot\begin{cases} 0 & \text{if } 1 \leq n \leq p-1, \\ n+1-p & \text{if } p \leq n \leq 2p-1, \\ p & \text{if } 2p \leq n \leq 3p-1, \end{cases} \tag{6}$$

$$S_{\psi_{3p}}^{(3)}(n) = \chi_3(p)\cdot\begin{cases} 0 & \text{if } 1 \leq n \leq p-1, \\ (n+1-p)(n+2-p)/2 & \text{if } p \leq n \leq 2p-1, \\ p(p+1)/2 + p(n+1-2p) & \text{if } 2p \leq n \leq 3p-1, \end{cases} \tag{7}$$

$\chi_{3p}(n) = \chi_3(n) - \psi_{3p}(n)$ and

$$S_{\chi_{3p}}^{(m)} = S_{\chi_3}^{(m)} - S_{\psi_{3p}}^{(m)}. \tag{8}$$

By (2), (5) and (8), it follows that $S^{(1)}_{\chi_{3p}}(n) \ge 0$ for all $n \ge 1$ if $\chi_3(p) = -1$, i.e. if $p \equiv 2 \pmod 3$. From now on, we assume that $\chi_3(p) = +1$, i.e. that $p \equiv 1 \pmod 3$. Then $L(0, \chi_{3p}) = (1 - \chi_3(p))L(0, \chi_3) = 0$ and it suffices to prove that $S^{(3)}_{\chi_{3p}}(n) \ge 0$ for $1 \le n \le 3p - 1$, by Lemma 4.

1. If $1 \le n \le p - 1$, then $S^{(3)}_{\chi_{3p}}(n) = S^{(3)}_{\chi_3}(n) \ge 0$, by (4), (7) and (8).
2. If $p \le n \le 2p - 1$, then

$$\begin{aligned} S^{(3)}_{\chi_{3p}}(n) &\ge \frac{n^2 + 3n}{6} - \frac{(n + 1 - p)(n + 2 - p)}{2} =: f(n) \\ &\ge \min(f(p), f(2p - 1)) \\ &= \min((p^2 + 3p - 6)/6, (p^2 - p - 2)/6) \ge 0, \end{aligned}$$

by (4), (7) and (8) (notice that $f''(x) = -2/3 \le 0$).

3. If $2p \le n \le 3p - 1$, then

$$S^{(3)}_{\chi_{3p}}(n) = \begin{cases} (n - (3p - 3))(n - 3p)/6 & \text{if } n \equiv 0 \pmod 3, \\ (n - (3p - 2))(n - (3p - 1))/6 & \text{if } n \equiv 1,\ 2 \pmod 3, \end{cases}$$

by (4), (7) and (8), and $S^{(3)}_{\chi_{3p}}(n) \ge 0$.

Finally, since $(3p+1)/2 \equiv 2 \pmod 3$, we obtain $S^{(1)}_{\chi_{3p}}((3p+1)/2) = 0 - 1 = -1$ by (2), (5) and (8), and

$$S^{(2)}_{\chi_{3p}}\left(\frac{3p + 1}{2}\right) = \frac{p + 1}{2} - \left(\frac{3p + 1}{2} + 1 - p\right) = -1$$

by (3), (6) and (8).

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