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ON THE NONEXISTENCE OF STABLE MINIMAL SUBMANIFOLDS AND THE LAWSON–SIMONS CONJECTURE

BY

ZE-JUN HU and GUO-XIN WEI (Zhengzhou)

Abstract. Let $\overline{M}$ be a compact Riemannian manifold with sectional curvature $K_{\overline{M}}$ satisfying $1/5 < K_{\overline{M}} \le 1$ (resp. $2 \le K_{\overline{M}} < 10$), which can be isometrically immersed as a hypersurface in the Euclidean space (resp. the unit Euclidean sphere). Then there exist no stable compact minimal submanifolds in $\overline{M}$. This extends Shen and Xu's result for $\frac{1}{4}$-pinched Riemannian manifolds and also suggests a modified version of the well-known Lawson–Simons conjecture.

1. Introduction. A *minimal submanifold* M in a Riemannian manifold $\overline{M}$ is a critical point of the volume functional of $\overline{M}$. M is said to be *stable* if the second variation of the volume is always nonnegative for any normal deformation of M in $\overline{M}$ with compact support; otherwise M is called *unstable*. It was proved by Simons [11] that there are no compact stable minimal submanifolds in the Euclidean spheres. In [5], Lawson and Simons made the following:

CONJECTURE. *Let $\overline{M}$ be a compact simply-connected Riemannian manifold with sectional curvature $K_{\overline{M}}$ satisfying $1/4 < K_{\overline{M}} \le 1$. Then every compact minimal submanifold M in $\overline{M}$ is unstable.*

Apparently this conjecture arose in connection with the sphere theorem. There are several results supporting this conjecture (cf. [1, 3, 5, 8, 10]). In particular, Aminov [1] proved that if M is homeomorphic to a two-sphere, then the conjecture is true; Shen and Xu [10] proved that if $0.77 \le K_{\overline{M}} \le 1$, then the conclusion of the conjecture holds. Note that a Riemannian manifold M is called *δ-pinched* ($\delta > 0$) if the sectional curvature K_M satisfies $\delta \le K_M \le 1$.

We know from [5, 7] that the rank one symmetric spaces $\mathbb{CP}^m$, $\mathbb{HP}^m$ and $\mathbb{C}\mathrm{a}\mathbb{P}^2$, whose sectional curvatures are $\frac{1}{4}$-pinched, admit stable compact

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minimal submanifolds. Thus the pinched condition for sectional curvature in the Lawson–Simons conjecture cannot be weakened in general.

For the special case that $\overline{M}$ is an m-dimensional compact Riemannian manifold isometrically immersed in an $(m+1)$-dimensional Euclidean space E^{m+1}, some better results are known (cf. [4, 6, 7, 9–11]). In particular, Shen and Xu [10] proved that the conjecture is true for $\frac{1}{4}$-pinched $\overline{M}$. Note that from the classical Hadamard theorem, every compact connected hypersuface M in E^{m+1} $(m \geq 2)$ is diffeomorphic to the m-sphere provided that $K_M > 0$ (cf. [2]). The Lawson–Simons conjecture together with this version of the sphere theorem suggests that the following modified conjecture might be correct.

Conjecture*. *Let $\overline{M}$ be a compact connected hypersurface in E^{m+1} $(m \geq 3)$ satisfying $K_{\overline{M}} > 0$. Then every compact minimal submanifold in $\overline{M}$ is unstable.*

Since $\overline{M}$ is compact, we can assume without loss of generality that $\overline{M}$ is δ-pinched for some constant $\delta > 0$. Thus, Shen and Xu's result [10] says that Conjecture* is correct for $\delta = 1/4$. In this paper, we shall give a further positive answer to Conjecture* by establishing Theorem 1* (see Section 3), and as a special case we obtain

Theorem 1. *Let $\overline{M}$ be a compact connected hypersurface in E^{m+1} $(m \geq 3)$ with sectional curvature satisfying $1/5 < K_{\overline{M}} \leq 1$. Then there exist no compact stable minimal submanifolds in $\overline{M}$.*

We note that Theorem 1 combined with the argument in the proof of Proposition 1 of [6] implies the following

Proposition 1. *Let $\overline{M}$ be the m-dimensional $(m \geq 3)$ ellipsoid in E^{m+1}:*

$$\frac{x_1^2}{a_1^2} + \ldots + \frac{x_{m+1}^2}{a_{m+1}^2} = 1, \qquad 1 \leq a_1 \leq \ldots \leq a_{m+1}.$$

If $1 \leq a_{m+1} < \sqrt[3]{5}$ and $a_1 \geq \sqrt{a_{m+1}}$, then there exist no compact stable minimal submanifolds in $\overline{M}$.

Remark 1. It can be proved in a similar way that the above result remains valid for *stable n-currents* on $\overline{M}$. In fact, we can state the counterpart of Theorem 1 as follows (for the concept of stable current and related results, see Lawson–Simons [5]), which is better than the result in [9].

Theorem 1′. *Let $\overline{M}$ be a compact connected hypersurface in E^{m+1} $(m \geq 3)$ with sectional curvature satisfying $1/5 < K_{\overline{M}} \leq 1$. Then there exist no compact stable n-currents in $\overline{M}$ for each n with $1 \leq n \leq m-1$.*

Remark 2. From Simons [11] we know that if $\overline{M}$ is a hypersurface in the $(m+1)$-sphere S^{m+1} with constant sectional curvature, then it does not admit compact stable minimal submanifolds. Using the same method as in the proof of Theorem 1, we can strengthen this result by establishing:

THEOREM 2*. *For $m > n \geq 2$, there is a positive constant $\sigma_{m,n} \geq 3$ such that if $\overline{M}$ is a compact connected hypersurface in the unit sphere S^{m+1} with sectional curvature satisfying $2 \leq K_{\overline{M}} < 1 + \sigma_{m,n}^2$, then every compact n-dimensional minimal submanifold in $\overline{M}$ is unstable.*

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2. Basic formulas and notations. In this paper, we use the following convention on the ranges of indices:

$$1 \leq \alpha, \beta, \gamma, \ldots \leq m; \quad 1 \leq i, j, \ldots \leq n; \quad n+1 \leq r, s, \ldots \leq m.$$

Let $M^n \hookrightarrow \overline{M}^m$ be an n-dimensional compact minimal submanifold isometrically immersed in $\overline{M}$. Let $N(M)$ be the normal bundle of M in $\overline{M}$ and V be a cross-section in $N(M)$ with compact support. Then the second variational formula for V is (cf. [10])

$$I(V,V) = \int_M \Big\{ \sum_{i=1}^n \|\nabla_{e_i}^\perp V\|^2 - \sum_{i,j=1}^n \langle B(e_i,e_j), V\rangle^2 - \sum_{i=1}^n \langle \overline{R}(e_i,V)V, e_i\rangle \Big\}\, dv, \tag{2.1}$$

where dv is the volume element of M, $\nabla^\perp$ and B are the normal connection and the second fundamental form of M in $\overline{M}$ respectively, $\overline{R}$ is the curvature tensor of $\overline{M}$, and $\{e_i\}$ is a local orthonormal frame field on M.

Now suppose $\overline{M}$ is isometrically immersed in the space forms $R^{m+1}(c)$ $(c = 0, 1)$ as a hypersurface, with $R^{m+1}(0) = E^{m+1}$ being the Euclidean space and $R^{m+1}(1) = S^{m+1}(1)$ the unit sphere. Denote by D the canonical connection of $R^{m+1}(c)$ and by $\overline{\nabla}$ the induced connection on $\overline{M}$. Then the second fundamental form h of $\overline{M}$ in $R^{m+1}(c)$ is defined as

$$h(X,Y)e_{m+1} = D_X Y - \overline{\nabla}_X Y \quad \text{for } X, Y \in T\overline{M},$$

where e_{m+1} is a unit normal vector field to $\overline{M}$ in $R^{m+1}(c)$. The following proposition is well known (see, e.g., [5, 6]).

PROPOSITION 2. *Let $\overline{M}$ be an m-dimensional compact hypersurface in $R^{m+1}(c)$ with second fundamental form h. If M is an n-dimensional compact* *minimal submanifold in $\overline{M}$ and satisfies*

$$\int_M \Big\{-n(m-n)c+\sum_{r,i}[2h(e_i,e_r)^2-h(e_i,e_i)h(e_r,e_r)]\Big\}\,dv<0 \tag{2.2}$$

for any local orthonormal frame field $\{e_i, e_r\}$ on $\overline{M}$ such that $\{e_r\}$ is normal to M, then M is unstable.

At a given point $p \in M$ in $\overline{M}$, let λ_α be the principal curvatures of $\overline{M}$ corresponding to the principal directions $\{e^*_\alpha\}$ which form an orthonormal basis of $T_p\overline{M}$ and satisfy $h(e^*_\alpha, e^*_\beta) = \lambda_\alpha \delta_{\alpha\beta}$. Thus, from the Gauss equation of $\overline{M}$ in $R^{m+1}(c)$ we have

$$\overline{R}_{\alpha\beta\alpha\beta} = c + \lambda_\alpha\lambda_\beta \quad (\alpha \neq \beta) \tag{2.3}$$

at $p \in \overline{M}$, where $\overline{R}_{\alpha\beta\alpha\beta} = \langle \overline{R}(e^*_\alpha, e^*_\beta)e^*_\beta, e^*_\alpha\rangle$. Let $\{e_i, e_r\}$ be an arbitrary local orthonormal frame around $p \in M$. We can write

$$e_\alpha = \sum_\beta A^\beta_\alpha e^*_\beta \tag{2.4}$$

for some special orthogonal matrix (A^β_α). Then we have

$$\begin{aligned} \Delta &:= -n(m-n)c + \sum_{r,i}\{2h(e_i,e_r)^2 - h(e_i,e_i)h(e_r,e_r)\} \\ &= -n(m-n)c + \sum_{\alpha,i,r}(\lambda_\alpha)^2(A^\alpha_i)^2(A^\alpha_r)^2 + F(A,\Lambda), \end{aligned} \tag{2.5}$$

where

$$F(A,\Lambda) = \sum_{\alpha\neq\beta}\lambda_\alpha\lambda_\beta\Big\{2\sum_{i,r}A^\alpha_i A^\beta_i A^\alpha_r A^\beta_r - \sum_{i,r}(A^\alpha_i)^2(A^\beta_r)^2\Big\}. \tag{2.6}$$

3. Proof of the results. We first prove Theorem 1. In fact, we will prove the following stronger version.

THEOREM 1*. *For $m > n \geq 2$, there is a positive constant $\sigma_{m,n} \leq 1/\sqrt{5}$ such that if $\overline{M}$ is a compact connected hypersurface in E^{m+1} with sectional curvature satisfying $\sigma^2_{m,n} < K_{\overline{M}} \leq 1$, then every compact n-dimensional minimal submanifold in $\overline{M}$ is unstable.*

To prove Theorem 1*, we adopt an argument similar to that of [10]. Let $\overline{M}$ be σ^2-pinched for some $\sigma > 0$ and M be an n-dimensional compact minimal submanifold in $\overline{M}$. Our aim below is to find the least possible constant σ so that $\Delta < 0$ on M.

We fix an arbitrary point $p \in M \hookrightarrow \overline{M}$; all calculations are carried out at p. From (2.3) we have

$$\sigma^2 \leq \lambda_\alpha\lambda_\beta \leq 1 \quad \text{for } \alpha \neq \beta, \tag{3.1}$$

which implies that all λ_α are nonzero and have the same sign. Without loss of generality, we may assume that

$$0 < \lambda_1 \le \ldots \le \lambda_m. \tag{3.2}$$

From (3.1) and (3.2), one can see that

$$\lambda_\alpha \ge \sigma \quad \text{for } \alpha \ne 1; \qquad \lambda_\alpha \le 1 \quad \text{for } \alpha \ne m; \tag{3.3}$$

$$\lambda_m \le 1/\sigma \quad \text{and} \quad \lambda_1 \ge \sigma^2, \quad \text{since } m \ge 3. \tag{3.4}$$

Since $(A_\alpha^\beta) \in \mathrm{SO}(m)$, we have

$$\sum_\gamma A_\gamma^\alpha A_\gamma^\beta = \sum_i A_i^\alpha A_i^\beta + \sum_r A_r^\alpha A_r^\beta = \delta_{\alpha\beta}, \tag{3.5}$$

$$\Big(\sum_i A_i^\alpha A_i^\beta\Big)^2 + \Big(\sum_r A_r^\alpha A_r^\beta\Big)^2 + 2\sum_{i,r} A_i^\alpha A_i^\beta A_r^\alpha A_r^\beta = \delta_{\alpha\beta}. \tag{3.6}$$

Define

$$\begin{aligned} G_{\alpha,m} = {} & 2\Big(\sum_i A_i^\alpha A_i^m\Big)^2 + 2\Big(\sum_r A_r^\alpha A_r^m\Big)^2 \\ & + \sum_{r,i}\{(A_i^\alpha)^2(A_r^m)^2 + (A_i^m)^2(A_r^\alpha)^2\}, \end{aligned} \tag{3.7}$$

which is nonnegative. Then, by (3.5) and (3.6), we have

$$\begin{aligned} \sum_{\alpha\ne m} G_{\alpha,m} &= \sum_\alpha G_{\alpha,m} - 2 + 2\sum_{r,i}(A_i^m)^2(A_r^m)^2 \\ &= n\sum_r (A_r^m)^2 + (m-n)\sum_i (A_i^m)^2 + 2\sum_{r,i}(A_i^m)^2(A_r^m)^2 \\ &\ge \min\{n, m-n\} =: T_{m,n} \ge 1. \end{aligned} \tag{3.8}$$

By (2.6), (3.1) and (3.5)–(3.7), it follows that

$$\begin{aligned} & F(A,\Lambda) \\ & = \sum_{\alpha\ne m}\lambda_\alpha\lambda_m\sum_{r,i}\{4A_i^\alpha A_i^m A_r^\alpha A_r^m - (A_i^\alpha)^2(A_r^m)^2 - (A_i^m)^2(A_r^\alpha)^2\} \\ & \quad + \sum_{\alpha,\beta\ne m,\, \alpha\ne\beta}\lambda_\alpha\lambda_\beta\sum_{r,i}\{2A_i^\alpha A_i^\beta A_r^\alpha A_r^\beta - (A_i^\alpha)^2(A_r^\beta)^2\} \\ & = -\sum_{\alpha\ne m}\lambda_\alpha\lambda_m G_{\alpha,m} \\ & \quad - \sum_{\alpha,\beta\ne m,\, \alpha\ne\beta}\lambda_\alpha\lambda_\beta\Big\{\Big(\sum_i A_i^\alpha A_i^\beta\Big)^2 + \Big(\sum_r A_r^\alpha A_r^\beta\Big)^2 + \sum_{r,i}(A_i^\alpha)^2(A_r^\beta)^2\Big\} \end{aligned}$$

$$\leq -\sum_{\alpha\neq m}\lambda_\alpha\lambda_m G_{\alpha,m}$$

$$-\sigma^2\sum_{\alpha,\beta\neq m,\,\alpha\neq\beta}\Big\{\Big(\sum_i A_i^\alpha A_i^\beta\Big)^2+\Big(\sum_r A_r^\alpha A_r^\beta\Big)^2+\sum_{r,i}(A_i^\alpha)^2(A_r^\beta)^2\Big\},$$

and then

$$\begin{aligned}(3.9)\quad & F(A,\Lambda)\\ &\leq \sum_{\alpha\neq m}(\sigma^2-\lambda_\alpha\lambda_m)G_{\alpha,m}\\ &\quad -\sigma^2\sum_{\alpha\neq\beta}\Big\{\Big(\sum_i A_i^\alpha A_i^\beta\Big)^2+\Big(\sum_r A_r^\alpha A_r^\beta\Big)^2+\sum_{r,i}(A_i^\alpha)^2(A_r^\beta)^2\Big\}\\ &\leq T_{m,n}(\sigma^2-\lambda_1\lambda_m)+\sigma^2\sum_{\alpha\neq\beta}\sum_{r,i}\{2A_i^\alpha A_i^\beta A_r^\alpha A_r^\beta-(A_i^\alpha)^2(A_r^\beta)^2\}\\ &= T_{m,n}(\sigma^2-\lambda_1\lambda_m)-\sigma^2\sum_{\alpha,r,i}(A_i^\alpha)^2(A_r^\alpha)^2\\ &\quad +\sigma^2\sum_{\alpha,\beta,r,i}\{2A_i^\alpha A_i^\beta A_r^\alpha A_r^\beta-(A_i^\alpha)^2(A_r^\beta)^2\}\\ &= T_{m,n}(\sigma^2-\lambda_1\lambda_m)-\sigma^2 n(m-n)-\sigma^2\sum_{\alpha,r,i}(A_i^\alpha)^2(A_r^\alpha)^2.\end{aligned}$$

Thus, we obtain an estimate for Δ:

$$\begin{aligned}(3.10)\quad \Delta &= \sum_{\alpha,r,i}(\lambda_\alpha)^2(A_i^\alpha)^2(A_r^\alpha)^2+F(A,\Lambda)\\ &\leq T_{m,n}(\sigma^2-\lambda_1\lambda_m)-\sigma^2 n(m-n)+\sum_{\alpha,r,i}(\lambda_\alpha^2-\sigma^2)(A_i^\alpha)^2(A_r^\alpha)^2.\end{aligned}$$

Since $(A_\alpha^\beta)\in \mathrm{SO}(m)$, by (3.6), we see that, for any α,

$$\sum_{r,i}(A_i^\alpha)^2(A_r^\alpha)^2\leq\frac{1}{2}\Big(\sum_i(A_i^\alpha)^2\Big)^2+\frac{1}{2}\Big(\sum_r(A_r^\alpha)^2\Big)^2=\frac{1}{2}-\sum_{r,i}(A_i^\alpha)^2(A_r^\alpha)^2,$$

i.e.,

$$(3.11)\qquad \sum_{r,i}(A_i^\alpha)^2(A_r^\alpha)^2\leq\frac{1}{4}.$$

According to (3.1)–(3.4), we have four cases: (1) $\lambda_1\geq\sigma$ and $\sigma\leq\lambda_m\leq 1$; (2) $\lambda_1\geq\sigma$ and $1<\lambda_m\leq\sigma^{-1}$; (3) $\lambda_1<\sigma$ and $\sigma<\lambda_m\leq 1$; (4) $\lambda_1<\sigma$ and $1<\lambda_m\leq\sigma^{-1}$.

CASE (1): $\lambda_1 \geq \sigma$ and $\sigma \leq \lambda_m \leq 1$. From (3.10) and (3.11), we have

$$\begin{aligned}\Delta &\leq T_{m,n}(\sigma^2 - \sigma\lambda_m) - \sigma^2 n(m-n) + \frac{m}{4}(\lambda_m^2 - \sigma^2) \\ &= \frac{m}{4}\lambda_m^2 - T_{m,n}\sigma\lambda_m - \frac{m}{4}\sigma^2 - \sigma^2 n(m-n) + T_{m,n}\sigma^2 \\ &=: g_1(\lambda_m),\end{aligned}$$

where $g_1(x) = \frac{m}{4}x^2 - T_{m,n}\sigma x - \frac{m}{4}\sigma^2 - \sigma^2 n(m-n) + T_{m,n}\sigma^2$ satisfies $g_1''(x) > 0$, $g_1(\sigma) = -n(m-n)\sigma^2 < 0$ and

$$g_1(1) = -[m/4 + n(m-n) - T_{m,n}]\sigma^2 - T_{m,n}\sigma + m/4 :\equiv h(\sigma)$$

with $h(0) = m/4 > 0$, $h(1) = -n(m-n) < 0$ and $h''(x) < 0$. Set

$$\sigma_{m,n}^{(1)} = \frac{m}{2T_{m,n} + \sqrt{(m-2T_{m,n})^2 + 4mn(m-n)}} > 0. \tag{3.12}$$

An easy verification shows that

$$\sigma_{m,n}^{(1)} < \frac{1}{\sqrt{5}} \qquad \text{for all } m > n \geq 2, \tag{3.13}$$

$$\lim_{m\to\infty} \sigma_{m,n}^{(1)} = \frac{1}{\sqrt{1+4n}} \qquad \text{for each fixed } n \geq 2, \tag{3.14}$$

and $h(\sigma_{m,n}^{(1)}) = 0$. Hence $g_1(1) < 0$ for all $\sigma > \sigma_{m,n}^{(1)}$. This proves the following

CLAIM 1. *For any $\sigma > \sigma_{m,n}^{(1)}$, if $\lambda_1 \geq \sigma$ and $\sigma \leq \lambda_m \leq 1$, then $\Delta < 0$.*

CASE (2): $\lambda_1 \geq \sigma$ and $1 < \lambda_m \leq \sigma^{-1}$. From (3.10) and (3.11), using $\lambda_\alpha \leq \lambda_m^{-1}$ for $\alpha \neq m$, we have

$$\begin{aligned}\Delta &\leq T_{m,n}(\sigma^2 - \sigma\lambda_m) - \sigma^2 n(m-n) + \frac{1}{4}(\lambda_m^2 - \sigma^2) + \frac{1}{4}\sum_{\alpha\neq m}(\lambda_\alpha^2 - \sigma^2) \\ &\leq T_{m,n}(\sigma^2 - \sigma\lambda_m) - \sigma^2 n(m-n) + \frac{1}{4}(\lambda_m^2 - \sigma^2) + \frac{m-1}{4}(\lambda_m^{-2} - \sigma^2) \\ &=: \frac{1}{4\lambda_m^2}\, g_2(\lambda_m),\end{aligned}$$

where $g_2(x) = x^4 - 4T_{m,n}\sigma x^3 + [4T_{m,n} - 4n(m-n) - m]\sigma^2 x^2 + m - 1$ satisfies $g_2(0) > 0$, $g_2(+\infty) = +\infty$ and has exactly one critical point (minimum) for $x > 0$. Note also that

$$g_2(1) = [4T_{m,n} - 4n(m-n) - m]\sigma^2 - 4T_{m,n}\sigma + m = 4h(\sigma) < 0$$

for every $\sigma > \sigma_{m,n}^{(1)}$, and

$$g_2(\sigma^{-1}) = \sigma^{-4} - 4T_{m,n}\sigma^{-2} + 4T_{m,n} - 4n(m-n) - 1 < 0$$

for every $\sigma > \sigma^{(2)}_{m,n}$, where

$$\sigma^{(2)}_{m,n} = \frac{1}{\sqrt{2T_{m,n} + \sqrt{4T^2_{m,n} + 4n(m-n) + 1 - 4T_{m,n}}}},$$

which satisfies

$$\sigma^{(2)}_{m,n} \le \frac{1}{\sqrt{5}} \quad \text{for all } m > n \ge 2, \tag{3.15}$$

$$\lim_{m\to\infty} \sigma^{(2)}_{m,n} = 0 \quad \text{for each fixed } n \ge 2. \tag{3.16}$$

Thus, we have

Claim 2. *For any $\sigma > \max\{\sigma^{(1)}_{m,n}, \sigma^{(2)}_{m,n}\}$, if $\lambda_1 \ge \sigma$ and $1 < \lambda_m \le \sigma^{-1}$, then $\Delta < 0$.*

Case (3): $\lambda_1 < \sigma$ and $\sigma < \lambda_m \le 1$. From (3.10) and (3.11), we have

$$\Delta \le -\sigma^2 n(m-n) + \frac{1}{4}\sum_{\alpha\neq 1}(\alpha^2_\alpha - \sigma^2) \le -\sigma^2 n(m-n) + \frac{m-1}{4}(1-\sigma^2) < 0$$

for $\sigma > \sqrt{\frac{m-1}{m-1+4n(m-n)}} =: \sigma^{(3)}_{m,n}$, and

$$\sigma^{(3)}_{m,n} \le \frac{1}{\sqrt{5}} \quad \text{for all } m > n \ge 2, \tag{3.17}$$

$$\lim_{m\to\infty} \sigma^{(3)}_{m,n} = \frac{1}{\sqrt{1+4n}} \quad \text{for each fixed } n \ge 2. \tag{3.18}$$

Thus, we have

Claim 3. *For any $\sigma > \sigma^{(3)}_{m,n}$, if $\lambda_1 < \sigma$ and $\sigma < \lambda_m \le 1$, then $\Delta < 0$.*

Case (4): $\lambda_1 < \sigma$ and $1 < \lambda_m \le \sigma^{-1}$. From (3.10) and (3.11), using $\lambda_\alpha \le 1/\lambda_m$ $(\alpha \neq m)$ and $\lambda_1 \ge \sigma^2/\lambda_2 \ge \lambda_m\sigma^2$, we have

$$\begin{aligned}\Delta &\le T_{m,n}(\sigma^2 - \lambda_1\lambda_m) - \sigma^2 n(m-n) + \sum_{\alpha\neq 1; r,i}(\lambda^2_\alpha - \sigma^2)(A^\alpha_i)^2(A^\alpha_r)^2\\
&\le T_{m,n}(\sigma^2 - \sigma^2\lambda^2_m) - \sigma^2 n(m-n) + \frac{1}{4}(\lambda^2_m - \sigma^2) + \frac{m-2}{4}\left(\frac{1}{\lambda^2_m} - \sigma^2\right)\\
&=: \frac{1}{4\lambda^2_m} g_3(\lambda^2_m),\end{aligned}$$

where $g_3(x) = (1 - 4T_{m,n}\sigma^2)x^2 + [4T_{m,n} - 4n(m-n) - m + 1]\sigma^2 x + m - 2$ satisfies

$$g_3(1) = -[4n(m-n) + m - 1]\sigma^2 + m - 1 < 0 \qquad \text{for } \sigma > \sigma^{(3)}_{m,n},$$
$$g_3(\sigma^{-2}) = \sigma^{-4} - 4T_{m,n}\sigma^{-2} + 4T_{m,n} - 4n(m-n) - 1 < 0 \qquad \text{for } \sigma > \sigma^{(2)}_{m,n}.$$

Note that if $1 - 4T_{m,n}\sigma^2 < 0$, then $g_3''(x) < 0$ and $g_3(x)$ has no critical points for $x > 0$; if $1 - 4T_{m,n}\sigma^2 > 0$, then $g_3''(x) > 0$ and $g_3(x)$ has exactly one critical point (minimum) for $x > 0$. These facts imply that $g_3(x) < 0$ provided that $1 \le x \le \sigma^{-2}$ and $\sigma > \max\{\sigma_{m,n}^{(2)}, \sigma_{m,n}^{(3)}\}$. Thus, we have

CLAIM 4. *For any* $\sigma > \max\{\sigma_{m,n}^{(2)}, \sigma_{m,n}^{(3)}\}$, *if* $\lambda_1 < \sigma$ *and* $1 < \lambda_m \le \sigma^{-1}$, *then* $\Delta < 0$.

Summing up Claims 1–4 and choosing $\sigma_{m,n} = \max_{1\le i\le 3} \sigma_{m,n}^{(i)}$, we complete the proof of Theorem 1*.

*Proof of Theorem 2**. Suppose $2 \le K_{\overline{M}} \le 1 + \sigma^2$. Then, from (2.3), for any $\alpha \ne \beta$ we have

$$1 \le \lambda_\alpha \lambda_\beta \le \sigma^2. \tag{3.19}$$

By assuming $0 < \lambda_1 \le \ldots \le \lambda_m$, we can see that

$$\lambda_\alpha \ge 1 \quad \text{for } \alpha \ne 1; \quad \lambda_\alpha \le \sigma \quad \text{for } \alpha \ne m;$$
$$\lambda_m \le \sigma^2 \quad \text{and} \quad \lambda_1 \ge 1/\sigma, \quad \text{since } m \ge 3.$$

From these and by the same procedure as in deriving (3.10), we can obtain an estimate for Δ:

$$\Delta \le T_{m,n}(1 - \lambda_1\lambda_m) - 2n(m-n) + \sum_{\alpha,r,i} (\lambda_\alpha^2 - 1)(A_i^\alpha)^2(A_r^\alpha)^2. \tag{3.20}$$

Our aim below is to find the greatest possible σ so that $\Delta < 0$ at $p \in M$. Now we also have four cases: (i) $\lambda_1 \ge 1$ and $1 \le \lambda_m \le \sigma$; (ii) $\lambda_1 \ge 1$ and $\sigma < \lambda_m \le \sigma^2$; (iii) $\lambda_1 < 1$ and $1 < \lambda_m \le \sigma$; (iv) $\lambda_1 < 1$ and $\sigma < \lambda_m \le \sigma^2$.

CASE (i): $\lambda_1 \ge 1$ and $1 \le \lambda_m \le \sigma$. From (3.11) and (3.20), we have

$$\Delta \le T_{m,n}(1 - \lambda_m) - 2n(m-n) + \frac{m}{4}(\lambda_m^2 - 1).$$

Let $\sigma_{m,n}^{(1)} = m^{-1}\{2T_{m,n} + \sqrt{8mn(m-n) + (2T_{m,n} - m)^2}\}$. Then $\Delta < 0$ if $\sigma < \sigma_{m,n}^{(1)}$ in this case.

CASE (ii): $\lambda_1 \ge 1$ and $\sigma < \lambda_m \le \sigma^2$. From (3.11) and (3.20), using $\lambda_\alpha \le \sigma^2/\lambda_m$ for $\alpha \ne m$, we have

$$\Delta \le T_{m,n}(1 - \lambda_m) - 2n(m-n) + \frac{1}{4}(\lambda_m^2 - 1) + \frac{m-1}{4}\left(\frac{\sigma^4}{\lambda_m^2} - 1\right).$$

Let $\sigma_{m,n}^{(2)} = 2T_{m,n} + \sqrt{(2T_{m,n} - 1)^2 + 8n(m-n)}$. Then $\Delta < 0$ if $\sigma < \min\{\sigma_{m,n}^{(1)}, \sigma_{m,n}^{(2)}\}$ in this case.

CASE (iii): $\lambda_1 < 1$ and $1 < \lambda_m \le \sigma$. From (3.11) and (3.20), we have

$$\Delta \le -2n(m-n) + \frac{m-1}{4}(\lambda_m^2 - 1).$$

Let $\sigma_{m,n}^{(3)} = \sqrt{\frac{8n(m-n)+m-1}{m-1}}$. Then $\Delta < 0$ if $\sigma < \sigma_{m,n}^{(3)}$ in this case.

CASE (iv): $\lambda_1 < 1$ and $\sigma < \lambda_m \le \sigma^2$. From (3.11) and (3.20), using $\lambda_1 \ge \lambda_2^{-1} \ge \lambda_m \sigma^{-2}$ and $\lambda_\alpha \le \sigma^2 \lambda_m^{-1}$ for $\alpha \ne m$, we have

$$\begin{aligned}\Delta &\le T_{m,n}(1 - \lambda_1\lambda_m) - 2n(m-n) + \sum_{\alpha\ne 1;r,i} (\lambda_\alpha^2 - 1)(A_i^\alpha)^2(A_r^\alpha)^2 \\ &\le T_{m,n}\left(1 - \frac{\lambda_m^2}{\sigma^2}\right) - 2n(m-n) + \frac{1}{4}(\lambda_m^2 - 1) + \frac{m-2}{4}\left(\frac{\sigma^4}{\lambda_m^2} - 1\right).\end{aligned}$$

Then, by the same argument as in case (4), we conclude that if $\sigma < \min\{\sigma_{m,n}^{(2)}, \sigma_{m,n}^{(3)}\}$, then $\Delta < 0$ in this case.

It is easily seen that $\sigma_{m,n}^{(i)} \ge 3$ for $i = 1, 2, 3$. Thus we complete the proof of Theorem 2* by choosing $\sigma_{m,n} = \min_{1\le i\le 3} \sigma_{m,n}^{(i)}$.

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Department of Mathematics
Zhengzhou University
Zhengzhou 450052, P.R. China
E-mail: huzj@zzu.edu.cn

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