GLOBAL PINCHING THEOREMS FOR MINIMAL SUBMANIFOLDS IN SPHERES

BY

KAIREN CAI (Hangzhou)

Abstract. Let $M$ be a compact submanifold with parallel mean curvature vector embedded in the unit sphere $S^{n+p}(1)$. By using the Sobolev inequalities of P. Li to get $L_p$ estimates for the norms of certain tensors related to the second fundamental form of $M$, we prove some rigidity theorems. Denote by $H$ and $\|\sigma\|_p$ the mean curvature and the $L_p$ norm of the square length of the second fundamental form of $M$. We show that there is a constant $C$ such that if $\|\sigma\|_{n/2} < C$, then $M$ is a minimal submanifold in the sphere $S^{n+p-1}(1 + H^2)$ with sectional curvature $1 + H^2$.

1. Introduction and results. Inspired by the well-known results about minimal submanifolds in a sphere due to J. Simons [7], the investigation of submanifolds with parallel mean curvature vector in a sphere has made big progress [5, 6, 8, 9]. However, most of these works estimate some kind of curvature of a manifold in order to obtain some pinching condition in a pointwise manner. Recently C. L. Shen [6] has obtained some global pinching theorems for minimal hypersurfaces in a sphere. He has proven that if $M$ is a compact minimal hypersurface with nonnegative Ricci curvature embedded in the unit sphere $S^{n+1}(1)$, then there exists a constant $A$ such that if $\|\sigma\|_{n/2} < A$, then $M$ must be totally geodesic, where $\sigma$ is the square length of the second fundamental form of $M$ and $\|\sigma\|_{n/2}$ is the $L_{n/2}$ norm of $\sigma$.

The purpose of the paper is to extend Shen’s result to submanifolds in a sphere with constant mean curvature vector. Also we notice that H. Alencar and M. do Carmo [1] study hypersurfaces with constant mean curvature $H$ by introducing a tensor $\phi$, related to $H$ and to the second fundamental form. By obtaining an $L_p$ estimate of $\phi$ and $\sigma$, we will prove the following.

Theorem 1. Let $M$ be an $n$-dimensional compact submanifold embedded in the unit sphere $S^{n+p}(1)$ ($n \geq 3$, $p > 1$). Suppose that $M$ has parallel mean curvature vector. Denote by $\sigma$ the square length of the second fundamental form of $M$. Then there is a constant $C$ (see (3.18) below) such that
if \((\int_M \sigma^{n/2})^{2/n} < C\), then \(M\) must be quasiumbilical. Furthermore \(M\) is a minimal submanifold in the sphere \(S^{n+p-1}(1+H^2)\) with sectional curvature \(1+H^2\).

In the case of surfaces in the sphere \(S^{p+2}(1)\) \((p > 1)\) we have the following result where the constant depends only on the mean curvature and the lower bound on the Gauss curvature.

**Theorem 2.** Let \(M\) be a compact surface with parallel mean curvature vector and zero genus embedded in the sphere \(S^{p+2}(1)\) \((p > 1)\). Suppose that the Gauss curvature of \(M\) has a positive lower bound \(k\). If

\[
\int_M \sigma^2 < \frac{4k^7}{\pi^{11}(1+H^2)^6},
\]

where \(H\) is the mean curvature and \(\sigma\) the square length of the second fundamental form of \(M\), then \(M\) is a quasiumbilical surface in the unit sphere \(S^{p+1}(1+H^2)\) with sectional curvature \(1+H^2\).

**2. Preliminaries.** The Sobolev inequality obtained by P. Li \([4, 6]\) states: Suppose that \(M\) is a compact oriented connected Riemannian manifold. For every \(f \in H_{1,2}(M^n)\), \(n = \dim M > 2\), we have

\[
(2.1) \quad \int_M |\nabla f|^2 \geq \left(\frac{n-2}{2(n-1)}\right)^2 \times C_0^{2/n} \left\{ 2^{-\frac{n+2}{n}}\|f\|^2_{2n/(n-2)} - (\operatorname{vol} M)^{-2/n} 2^{E(n)} \|f\|_2^2 \right\},
\]

where

\[
(2.2) \quad \|f\|_p = \left(\int_M |f|^p\right)^{1/p},
\]

\[
(2.3) \quad E(n) = \begin{cases} 
(n-4)(n-2)/2 & \text{if } n > 3, \\
1 & \text{if } n = 3
\end{cases}
\]

and the best Sobolev constant \(C_0\) satisfies

\[
(2.4) \quad C_1 \leq C_0 \leq 2C_1.
\]

In (2.4), the isoperimetric constant of \(M^n\) is defined by

\[
(2.5) \quad C_1 = \inf \frac{(\operatorname{area}(S))^n}{(\min(\operatorname{vol} M_1, \operatorname{vol} M_2))^{n-1}}
\]

where \(S\) ranges over all hypersurfaces of \(M\), \(S\) divides \(M\) into two parts \(M_1, M_2\), and \(\operatorname{area}(S)\) is the \((n-1)\)-dimensional volume of \(S\). Let
\begin{equation}
k_1 = 2^{-3-2/n} \left( \frac{n-2}{n-1} \right)^2 C_1^{2/n},
\end{equation}

\begin{equation}
k_2 = 2^{E(n)+2/n-2} \left( \frac{n-2}{n-1} \right)^2 C_1^{2/n} (\text{vol } M)^{-2/n}.
\end{equation}

Then we have
\begin{equation}
\int_M |\nabla f|^2 \geq k_1 \|f\|^2_{2n/(n-2)} - k_2 \|f\|^2_2.
\end{equation}

Let \( S^{n+p} \) be an \((n + p)\)-dimensional standard sphere in the Euclidean space \( \mathbb{R}^{n+p+1} \) and \( M \) a compact submanifold isometrically embedded in \( S^{n+p}(1) \). We choose a local orthonormal frame field \( \{e_A\}, 1 \leq A \leq n + p \), in \( S^{n+p} \) such that when restricted to \( M \), the vectors \( \{e_i\}, 1 \leq i \leq n \), are tangent to \( M \). We denote the second fundamental form of \( M \) by
\begin{equation}
B = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,
\end{equation}
where \( \{\omega_i\} \) is the dual frame of \( \{e_i\}, 1 \leq i, j \leq n, n + 1 \leq \alpha \leq n + p \). The Weingarten transformation \( H_\alpha \) corresponding to the normal vector \( e_\alpha \) is defined by
\begin{equation}
\langle H_\alpha(X), Y \rangle = \langle B(X, Y), e_\alpha \rangle,
\end{equation}
where \( X, Y \) are tangent vectors to \( M \). Denote the mean curvature vector of \( M \) by
\begin{equation}
\xi = \frac{1}{n} \sum_\alpha (\text{tr } H_\alpha) e_\alpha,
\end{equation}
where \( \text{tr } H_\alpha \) is the trace of the transformation \( H_\alpha \). Then the mean curvature \( H \) and the square length \( \sigma \) of the second fundamental form of \( M \) can be expressed as
\begin{equation}
H = |\xi| = \frac{1}{n} \sqrt{\sum_\alpha (\text{tr } H_\alpha)^2}, \quad \sigma = \sum_\alpha \text{tr}(H_\alpha^2).
\end{equation}
If we choose \( e_{n+1} \) such that \( He_{n+1} = \xi \), then
\begin{equation}
\text{tr } H_{n+1} = nH, \quad \text{tr } H_\beta = 0, \quad n + 2 \leq \beta \leq n + p.
\end{equation}
Furthermore, \( M \) is quasiumbilical if and only if \( H_{n+1} = HI \), where \( I \) is the identical mapping. \( M \) is called a manifold with parallel mean curvature vector if \( \xi \) is parallel in the normal bundle of \( M \), i.e., \( \nabla_X^\perp \xi = 0 \) for any tangent vector \( X \) to \( M \) where \( \nabla^\perp \) is the connection of the normal bundle. From \( \nabla_X(\xi, \xi) = 0 \) it is easy to check that then the mean curvature of \( M \) is a constant. The Gauss equation of \( M \) in the sphere \( S^{n+p}(1) \) is given by (see [9, I, pp. 348–349])
(2.13) \[ R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{\alpha} (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \]

From the Ricci identity we get
(2.14) \[ h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}. \]

It is known from [9, II, p. 78] that if \( e_{n+1} \) is the normalized mean curvature normal vector, then
(2.15) \[ H_\beta H_{n+1} = H_{n+1}H_\beta \]
and
(2.16) \[ R_{n+1\beta kl} = 0. \]

Since \( M \) has constant mean curvature we have
(2.17) \[ \Delta h^{n+1}_{ij} = \sum_k h^{n+1}_{ijkk} = \sum_{m,k} (h^{n+1}_{km}R_{mijk} + h^{n+1}_{im}R_{mkjk}). \]

Here we denote the components of the Riemannian curvature tensor of \( M \) immersed in \( S^{n+p}(1) \) by \( R_{ijkl} \) and \( R_{\alpha\beta kl} \). Thus
(2.18) \[ \frac{1}{2} \Delta \sum_{i,j} (h^{n+1}_{ij})^2 = \sum_{i,j,k} (h^{n+1}_{ijk})^2 + \sum_{i,j,k,m} h^{n+1}_{ij}(h^{n+1}_{km}R_{mijk} + h^{n+1}_{im}R_{mkjk}). \]

It follows from the Gauss equation (2.13) and (2.15) that
\[
\sum_{i,j,k,m} h^{n+1}_{ij}h^{n+1}_{km}R_{mijk} = \sum_{i,j} (h^{n+1}_{ij})^2 - n^2H^2 + \text{tr}(H^4_{n+1})
- (\text{tr}(H^2_{n+1}))^2 + \sum_{\beta \neq n+1} \text{tr}((H_{n+1}H_\beta)^2) - \sum_{\beta \neq n+1} (\text{tr}(H_{n+1}H_\beta))^2,
\]
(2.19) \[ \sum_{i,j,k,m} h^{n+1}_{ij}h^{n+1}_{im}R_{mkjk} = (n - 1) \sum_{i,j} (h^{n+1}_{ij})^2 
+ nH \text{tr}(H^3_{n+1}) - \text{tr}(H^4_{n+1}) - \sum_{\beta \neq n+1} \text{tr}((H_{n+1}H_\beta)^2). \]

From (2.18) and (2.19) the Laplacian of the function \( \text{tr}(H^2_{n+1}) \) can be expressed as
(2.20) \[ \frac{1}{2} \Delta \sum_{i,j} (h^{n+1}_{ij})^2 = \sum_{i,j,k} (h^{n+1}_{ijk})^2 + n \text{tr}(H^2_{n+1}) - n^2H^2 
+ nH \text{tr}(H^3_{n+1}) - (\text{tr}(H^2_{n+1}))^2 - \sum_{\beta \neq n+1} (\text{tr}(H_{n+1}H_\beta))^2. \]
3. Proof of Theorem 1. We choose a local orthonormal frame field \( \{e_A\}, 1 \leq A \leq n+p \) in \( S^{n+p}(1) \) such that when restricted to \( M \), the vectors \( \{e_i\}, 1 \leq i \leq n \), are tangent to \( M \). Furthermore \( H e_{n+1} = \xi \), where \( \xi \) is the mean curvature vector of \( M \). Now we define two tensors \( \phi \) and \( \psi \) of type \((1,2)\) by

\[
\phi = \sum_{i,j} (h^{n+1}_{ij} - H \delta_{i,j}) \omega_i \otimes \omega_j \otimes e_{n+1},
\]

(3.1)

\[
\psi = \sum_{i,j,\beta} h^\beta_{ij} \omega_i \otimes \omega_j \otimes \epsilon_\beta,
\]

(3.2)

where \( \{\omega_i\} \) is the dual frame to \( \{e_i\}, 1 \leq i, j \leq n, n+2 \leq \beta \leq n+p \). Denote by \( \sigma_H \) the square length of the second fundamental form in the direction of the normal vector \( \xi \). It is easily checked that \( \operatorname{tr} \phi = 0 \) and \( |\phi|^2 = \sigma_H - nH^2 \), where \( \operatorname{tr} \phi \) is the trace of \( \phi \). Then \( M \) is a quasiumbilical submanifold if and only if \( |\phi|^2 = 0 \). The square norm of \( \psi \) is given by

\[
|\psi|^2 = \sum_{\beta \neq n+1} \operatorname{tr}(H^2_\beta).
\]

(3.3)

We have

\[
\sigma = |\phi|^2 + |\psi|^2 + nH^2,
\]

(3.4)

\[
\sigma^2_H = (\operatorname{tr} H^2_{n+1})^2 = |\phi|^4 - 2nH^2|\phi|^2 + n^2H^4,
\]

(3.5)

\[
\sum_{\beta \neq n+1} (\operatorname{tr}(H_{n+1}H_\beta))^2 = \sum_{\beta \neq n+1} (\operatorname{tr}((H_{n+1} - HI)H_\beta))^2 \leq |\phi|^2|\psi|^2.
\]

(3.6)

Since \( \operatorname{tr} \phi = 0 \), we can use Lemma (2.6) of [1] to obtain

\[
|\operatorname{tr} \phi^3| \leq \frac{n - 2}{\sqrt{n(n-1)}} |\phi|^3.
\]

A direct calculation shows that

\[
\operatorname{tr}(H^3_{n+1}) = nH^3 + 3H|\phi|^2 - \operatorname{tr} \phi^3 \geq nH^3 + 3H|\phi|^2 - \frac{n - 2}{\sqrt{n(n-1)}} |\phi|^3.
\]

(3.7)

Substituting (3.5)–(3.7) in (2.20), we get

\[
\frac{1}{2} \Delta \sum_{i,j} (h^{n+1}_{ij})^2 \geq \sum_{i,j,k} (h^{n+1}_{ijk})^2 + |\phi|^2 \left( n + nH^2 - \frac{n - 2}{\sqrt{n-1}} \sqrt{nH^2} |\phi| - |\phi|^2 - |\psi|^2 \right).
\]

(3.8)
Since $H$ is constant, we have $\sum_{i,j,k} (h_{ijk}^{n+1})^2 = |\nabla \phi|^2$. This yields
\begin{equation}
\frac{1}{2} \Delta |\phi|^2 \geq |\nabla \phi|^2 + |\phi|^2 \left( n + nH^2 - \frac{n-2}{\sqrt{n-1}} \sqrt{nH^2} |\phi| - |\phi|^2 - |\psi|^2 \right).
\end{equation}

Let us consider a quadratic form $F$ with eigenvalues $\pm \frac{n}{2\sqrt{n-1}}$:
\begin{equation}
F(x, y) = x^2 - \frac{n-2}{\sqrt{n-1}} xy - y^2.
\end{equation}
Then there exists an orthogonal transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\psi(x, y) = (u, v)$, such that
\begin{equation}
F(x, y) = \frac{n}{2\sqrt{n-1}} (u^2 - v^2).
\end{equation}
If $x = \sqrt{nH^2}$, $y = |\phi|$, from
\begin{equation}
u^2 + v^2 = nH^2 + |\phi|^2 = \sigma - |\psi|^2
\end{equation} it follows that
\begin{equation}
F(\sqrt{nH^2}, |\phi|) \geq -\frac{n}{2\sqrt{n-1}} (u^2 + v^2) = -\frac{n}{2\sqrt{n-1}} (\sigma - |\psi|^2).
\end{equation}
Since $|\nabla |\phi||^2 \leq |\nabla \phi|^2$, from (3.9) we have
\begin{equation}
\frac{1}{2} \Delta |\phi|^2 \geq |\nabla |\phi||^2 + |\phi|^2 \left( n - \frac{n}{2\sqrt{n-1}} \sigma + \left( \frac{n}{2\sqrt{n-1}} - 1 \right) |\psi|^2 \right)
\geq |\nabla |\phi||^2 + n|\phi|^2 - \frac{n}{2\sqrt{n-1}} \sigma |\phi|^2.
\end{equation}
It follows from (2.7) that
\begin{equation}
\int_M |\nabla |\phi||^2 \geq k_1 ||\phi||^2_{2n/(n-2)} - k_2 ||\phi||^2_2,
\end{equation}
where $k_1, k_2$ have been defined by (2.6). Integrating both sides of (3.14) and applying (3.15) and the inequality
\begin{equation}
||\sigma|\phi||^2_1 \leq ||\sigma||_{n/2} ||\phi||^2_{2n/(n-2)},
\end{equation}
we get
\begin{equation}
0 \geq (n - k_2) ||\phi||^2_2 + k_1 ||\phi||^2_{2n/(n-2)} - \frac{n}{2\sqrt{n-1}} ||\sigma|\phi||^2_1
\geq (n - k_2) ||\phi||^2_2 + \left( k_1 - \frac{n}{2\sqrt{n-1}} ||\sigma||_{n/2} \right) ||\phi||^2_{2n/(n-2)}.
\end{equation}
Let
\begin{equation}
||\sigma||_{n/2} < \min \left\{ \frac{2\sqrt{n-1}}{n} k_1, 2\sqrt{n-1} \frac{k_1}{k_2} \right\}.
\end{equation}
From (3.14), (3.16) and (3.18) we can easily obtain
\begin{equation}
0 \geq n\|\phi\|_2^2 - \frac{n}{2\sqrt{n-1}} \|\sigma\|_1 \|\phi\|^2_1
\end{equation}
\begin{equation}
\geq n\|\phi\|_2^2 - \frac{n}{2\sqrt{n-1}} \|\phi\|^2_{2n/(n-2)} \|\sigma\|_{n/2}
\end{equation}
\begin{equation}
\geq n\|\phi\|_2^2 - n\frac{k_1}{k_2} \|\phi\|^2_{2n/(n-2)}.
\end{equation}
If \(|\phi|^2 \neq 0\), it follows from (3.17)–(3.19) that
\begin{equation}
0 \geq (n - k_2)\|\phi\|_2^2 + \left( k_1 - \frac{n}{2\sqrt{n-1}} \|\sigma\|_{n/2} \right) \frac{k_2}{k_1} \|\phi\|_2^2
\end{equation}
a contradiction. Hence \(|\phi|^2 = 0\), i.e., \(M\) is quasiumbilical in \(S^{n+p}(1)\). From (3.1) we have \(h_{ij}^{n+1} = H \delta_{ij}\). The mean curvature vector \(\xi\) can be treated as a subbundle of the normal bundle \(T^\perp M\) with base \(M\) embedded in the sphere \(S^{n+p}(1)\) with fiber dimension 1. Now \(M\) is umbilical with respect to \(\xi\), and \(\xi\) is parallel in \(T^\perp M\). According to a theorem of Yau ([9, I, p. 351]), we derive that \(M\) lies in an \(n + p - 1\)-dimensional umbilical hypersurface with \(\xi\) perpendicular to the umbilical hypersurface. Furthermore, the Gauss equation of \(M\) becomes
\begin{equation}
R_{ijkl} = (1 + H^2)(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \sum_{\beta}(h_{ij}^{\beta} h_{jl}^{\beta} - h_{il}^{\beta} h_{jk}^{\beta}),
\end{equation}
so the hypersurface must be \(S^{n+p-1}(1 + H^2)\). Since we know from (2.12) that \(\text{tr} H_\beta = 0\) for every normal vector \(e_\beta\) of \(M\) in \(S^{n+p-1}(1 + H^2), n + 2 \leq \beta \leq n + p\), we derive that \(M\) is a minimal submanifold in \(S^{n+p-1}(1 + H^2)\). Thus we conclude that Theorem 1 holds.

4. Proof of Theorem 2. When \(n = 2\), it follows from (3.4) and (3.9) that
\begin{equation}
\frac{1}{2} \Delta|\phi|^2 \geq |\nabla|\phi||^2 + 2(1 + 2H^2)|\phi|^2 - \sigma|\phi|^2.
\end{equation}
Furthermore, P. Li [4] obtained another Sobolev inequality for \(\dim M = 2\): For every \(f \in H_{1,2}(M^2)\), we have
\begin{equation}
\int_M |\nabla f|^2 \geq \frac{C_0}{4} \left\{ (\text{vol } M)^{-1/2} \left( \int_M f^4 \right)^{1/2} - (\text{vol } M)^{-1} \int_M f^2 \right\}
\end{equation}
\begin{equation}
\geq \tilde{k}_1 \|f\|_4^2 - \tilde{k}_2 \|f\|_2^2,
\end{equation}
where
\[
\tilde{k}_1 = \frac{C_1}{4} (\text{vol } M)^{-1/2}, \quad \tilde{k}_2 = \frac{C_1}{2} (\text{vol } M)^{-1},
\]
and \(C_0\) is the best Sobolev constant, \(C_1\) the isoperimetric constant of \(M\), \(C_1 \leq C_0 \leq 2C_1\). Since \(H\) is constant and \(|\nabla \phi|^2 \leq |\phi|^2\), integrating both sides of (4.1) and applying (4.2) we get
\[
\begin{align*}
0 & \geq \tilde{k}_1 \|\phi\|^2 \frac{1}{4} - \tilde{k}_2 \|\phi\|^2 \frac{1}{2} + 2(1 + 2H^2)\|\phi\|^2 \frac{1}{2} - 2\|\phi\|^2 \frac{1}{2} \geq \{2(1 + 2H^2) - \tilde{k}_2\} \|\phi\|^2 \frac{1}{2} + (\tilde{k}_1 - 2\|\phi\|^2)\|\phi\|^2 \frac{1}{4}.
\end{align*}
\]
Suppose that
\[
\|\sigma\|_2 < \min \left\{ \tilde{k}_1, 2(1 + 2H^2) \frac{\tilde{k}_1}{\tilde{k}_2} \right\}.
\]
It follows from (4.1) and (4.4) that
\[
\begin{align*}
2(1 + 2H^2)\|\phi\|^2 \frac{1}{2} & \leq \|\phi\|^2 \frac{1}{2} - \|\sigma\|^2 \frac{1}{2} \leq \|\sigma\|^2 \frac{1}{2} \leq 2(1 + 2H^2) \frac{\tilde{k}_1}{\tilde{k}_2} \|\phi\|^2 \frac{1}{4}.
\end{align*}
\]
Hence
\[
\frac{\tilde{k}_2}{\tilde{k}_1} \|\phi\|^2 \frac{1}{2} \leq \|\phi\|^2 \frac{1}{4}.
\]
If \(|\phi|^2 \neq 0\), from (4.3) and (4.6) we derive
\[
\begin{align*}
0 & \geq \left\{ 2(1 + 2H^2) - \frac{\tilde{k}_2}{\tilde{k}_1} \|\sigma\|^2 \right\} \|\phi\|^2 \frac{1}{2} > 0.
\end{align*}
\]
This is a contradiction. Hence \(|\phi|^2 = 0\), i.e., \(M\) is quasiumbilical. By the same reason as in Theorem 1, we conclude that \(M\) is a minimal surface in the sphere \(S^{p+1}(1 + H^2)\).

Let us find a lower bound of the isoperimetric constant \(C_1\) and an upper bound of the volume of \(M\) to obtain a lower bound of the quantity \(\min\{\tilde{k}_1, 2(1 + 2H^2)\tilde{k}_1/\tilde{k}_2\}\) which depends only on \(H\) and \(k\). We will make use of Wang’s argument [8].

From a result of B. Y. Chen [2] we know that for any \(p\)-dimensional compact submanifold \(\overline{M}\) in the Euclidean space \(\mathbb{R}^m\) we have
\[
\int_{\overline{M}} |\overline{H}|^p \geq \omega_p,
\]
where \(\overline{H}\) is the mean curvature of \(\overline{M}\) in \(\mathbb{R}^m\) and \(\omega_p\) is the volume of the unit sphere \(S^p(1)\). In our case that \(M\) is an embedded surface in \(S^{p+1}(1),\)
we have $H^2 = 1 + H^2$, where $H$ is the constant mean curvature of $M$ in $S^{p+1}(1)$. Therefore, we obtain

$$\text{(4.9)} \quad \text{vol} M \geq \frac{\omega_2}{1 + H^2} = \frac{4\pi}{1 + H^2}.$$ 

For any $n$-dimensional manifold $M$ with positive Ricci curvature, a result due to C. B. Croke [3] shows that

$$\text{(4.10)} \quad C_1(M) \geq \frac{(\text{vol} M)^{n+1}}{4\omega_{n-1}\omega_n^{n-1}} \frac{1}{\int_0^d (\sqrt{1/k} \sin \sqrt{k}r)^{n-1} dr} \geq \frac{n^{n+1}(\text{vol} M)^{n+1}}{4d^n(n+1)\omega_{n-1}\omega_n^{n-1}},$$

for $n \geq 2$, where $(n - 1)k$ is the lower bound of the Ricci curvature, $d$ is the diameter of $M$ and $\omega_n$ is the volume of the unit sphere $S^n(1)$. It follows from the Myers theorem that $d \leq \pi/\sqrt{k}$ for a compact manifold whose Ricci curvature has positive lower bound $(n - 1)k$. Then we have

$$\text{(4.11)} \quad C_1(M) > \frac{n^{n+1}k^{n(n+1)/2}}{4\pi^n(n+1)} \frac{(\text{vol} M)^{n+1}}{\omega_{n-1}\omega_n^{n-1}}.$$ 

When $n = 2$, this becomes

$$\text{(4.12)} \quad C_1(M) \geq \frac{16k^3}{\pi^5(1 + H^2)^3}.$$ 

According to the Gauss–Bonnet formula we have

$$\text{(4.13)} \quad k \text{ vol } M \leq \int_M K \text{ d}V = 2\pi \chi(M) = 4\pi,$$

where $k$ is the positive lower bound of the Gauss curvature $K$ of $M$ and $\chi(M)$ is the Euler characteristic of the surface $M$ with genus zero. Thus both $\tilde{k}_1$ and $\tilde{k}_1/\tilde{k}_2$ have positive lower bounds depending only on $H$ and $k$. It follows from (4.9) and (4.13) that $k \leq 1 + H^2$. By a direct calculation from (4.2), (4.9), (4.12) and (4.13) we get

$$\text{(4.14)} \quad \frac{2k^{7/2}}{\pi^{11/2}(1 + H^2)^3} \leq \min \left\{ \frac{\tilde{k}_1}{2(1 + 2H^2)}, \frac{\tilde{k}_1}{\tilde{k}_2} \right\}.$$ 

This completes the proof of Theorem 2.

**Acknowledgements.** The author is grateful to the School of Mathematics, University of Bristol for their hospitality during his visit and to the referee for many beneficial suggestions.
REFERENCES


Department of Mathematics
Hangzhou Teachers’ College
96 Wen Yi Road
Hangzhou 310036, P.R. China
E-mail: kcai@mail.hz.zj.cn

Received 14 March 2002;
revised 22 February 2003