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GLOBAL PINCHING THEOREMS FOR MINIMAL SUBMANIFOLDS IN SPHERES

BY

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Abstract. Let M be a compact submanifold with parallel mean curvature vector embedded in the unit sphere $S^{n+p}(1)$. By using the Sobolev inequalities of P. Li to get L_p estimates for the norms of certain tensors related to the second fundamental form of M, we prove some rigidity theorems. Denote by H and $\|\sigma\|_p$ the mean curvature and the L_p norm of the square length of the second fundamental form of M. We show that there is a constant C such that if $\|\sigma\|_{n/2} < C$, then M is a minimal submanifold in the sphere $S^{n+p-1}(1+H^2)$ with sectional curvature $1+H^2$.

1. Introduction and results. Inspired by the well-known results about minimal submanifolds in a sphere due to J. Simons [7], the investigation of submanifolds with parallel mean curvature vector in a sphere has made big progress [5, 6, 8, 9]. However, most of these works estimate some kind of curvature of a manifold in order to obtain some pinching condition in a pointwise manner. Recently C. L. Shen [6] has obtained some global pinching theorems for minimal hypersurfaces in a sphere. He has proven that if M is a compact minimal hypersurface with nonnegative Ricci curvature embedded in the unit sphere $S^{n+1}(1)$, then there exists a constant A such that if $\|\sigma\|_{n/2} < A$, then M must be totally geodesic, where σ is the square length of the second fundamental form of M and $\|\sigma\|_{n/2}$ is the $L_{n/2}$ norm of σ.

The purpose of the paper is to extend Shen's result to submanifolds in a sphere with constant mean curvature vector. Also we notice that H. Alencar and M. do Carmo [1] study hypersurfaces with constant mean curvature H by introducing a tensor ϕ, related to H and to the second fundamental form. By obtaining an L_p estimate of ϕ and σ, we will prove the following.

THEOREM 1. *Let M be an n-dimensional compact submanifold embedded in the unit sphere $S^{n+p}(1)$ $(n \geq 3,\ p > 1)$. Suppose that M has parallel mean curvature vector. Denote by σ the square length of the second fundamental form of M. Then there is a constant C (see* (3.18) *below) such that*

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if $(\int_M \sigma^{n/2})^{2/n} < C$, *then* M *must be quasiumbilical. Furthermore* M *is a minimal submanifold in the sphere* $S^{n+p-1}(1+H^2)$ *with sectional curvature* $1+H^2$.

In the case of surfaces in the sphere $S^{p+2}(1)$ $(p>1)$ we have the following result where the constant depends only on the mean curvature and the lower bound on the Gauss curvature.

THEOREM 2. *Let* M *be a compact surface with parallel mean curvature vector and zero genus embedded in the sphere* $S^{p+2}(1)$ $(p>1)$. *Suppose that the Gauss curvature of* M *has a positive lower bound* k. *If*

$$\int_M \sigma^2 < \frac{4k^7}{\pi^{11}(1+H^2)^6},$$

where H *is the mean curvature and* σ *the square length of the second fundamental form of* M, *then* M *is a quasiumbilical surface in the unit sphere* $S^{p+2}(1)$. *Furthermore* M *is a minimal surface in the sphere* $S^{p+1}(1+H^2)$ *with sectional curvature* $1+H^2$.

2. Preliminaries. The Sobolev inequality obtained by P. Li [4, 6] states: Suppose that M is a compact oriented connected Riemannian manifold. For every $f\in H_{1,2}(M^n)$, $n=\dim M>2$, we have

$$\int_M |\nabla f|^2 \ge \left(\frac{n-2}{2(n-1)}\right)^2 \times C_0^{2/n}\{2^{-(n+2)/n}\|f\|^2_{2n/(n-2)} - (\operatorname{vol} M)^{-2/n}2^{E(n)}\|f\|_2^2\}, \tag{2.1}$$

where

$$\|f\|_p = \Big(\int_M |f|^p\Big)^{1/p}, \tag{2.2}$$

$$E(n) = \begin{cases} (n-4)(n-2)/2 & \text{if } n>3,\\ 1 & \text{if } n=3\end{cases} \tag{2.3}$$

and the best Sobolev constant C_0 satisfies

$$C_1 \le C_0 \le 2C_1. \tag{2.4}$$

In (2.4), the isoperimetric constant of M^n is defined by

$$C_1 = \inf \frac{(\operatorname{area}(S))^n}{(\min(\operatorname{vol} M_1, \operatorname{vol} M_2))^{n-1}} \tag{2.5}$$

where S ranges over all hypersurfaces of M, S divides M into two parts M_1, M_2, and area(S) is the $(n-1)$-dimensional volume of S. Let

$$k_1 = 2^{-3-2/n}\left(\frac{n-2}{n-1}\right)^2 C_1^{2/n},$$
(2.6)
$$k_2 = 2^{E(n)+2/n-2}\left(\frac{n-2}{n-1}\right)^2 C_1^{2/n}(\operatorname{vol} M)^{-2/n}.$$

Then we have

$$\int_M |\nabla f|^2 \geq k_1\|f\|^2_{2n/(n-2)} - k_2\|f\|_2^2. \tag{2.7}$$

Let S^{n+p} be an $(n+p)$-dimensional standard sphere in the Euclidean space $\mathbb{R}^{n+p+1}$ and M a compact submanifold isometrically embedded in $S^{n+p}(1)$. We choose a local orthonormal frame field $\{e_A\}, 1 \leq A \leq n+p$, in S^{n+p} such that when restricted to M, the vectors $\{e_i\}$, $1 \leq i \leq n$, are tangent to M. We denote the second fundamental form of M by

$$B = \sum_{i,j,\alpha} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \tag{2.8}$$

where $\{\omega_i\}$ is the dual frame of $\{e_i\}, 1 \leq i,j \leq n$, $n+1 \leq \alpha \leq n+p$. The Weingarten transformation H_α corresponding to the normal vector e_α is defined by

$$\langle H_\alpha(X), Y\rangle = \langle B(X,Y), e_\alpha\rangle, \tag{2.9}$$

where X, Y are tangent vectors to M. Denote the mean curvature vector of M by

$$\xi = \frac{1}{n}\sum_\alpha (\operatorname{tr} H_\alpha)e_\alpha, \tag{2.10}$$

where $\operatorname{tr} H_\alpha$ is the trace of the transformation H_α. Then the mean curvature H and the square length σ of the second fundamental form of M can be expressed as

$$H = |\xi| = \frac{1}{n}\sqrt{\sum_\alpha (\operatorname{tr} H_\alpha)^2}, \qquad \sigma = \sum_\alpha \operatorname{tr}(H_\alpha^2). \tag{2.11}$$

If we choose e_{n+1} such that $He_{n+1} = \xi$, then

$$\operatorname{tr} H_{n+1} = nH, \qquad \operatorname{tr} H_\beta = 0, \qquad n+2 \leq \beta \leq n+p. \tag{2.12}$$

Furthermore, M is quasiumbilical if and only if $H_{n+1} = HI$, where I is the identical mapping. M is called a *manifold with parallel mean curvature vector* if ξ is parallel in the normal bundle of M, i.e., $\nabla^\perp_X \xi = 0$ for any tangent vector X to M where $\nabla^\perp$ is the connection of the normal bundle. From $\nabla_X\langle\xi,\xi\rangle = 0$ it is easy to check that then the mean curvature of M is a constant. The Gauss equation of M in the sphere $S^{n+p}(1)$ is given by (see [9, I, pp. 348–349])

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}). \tag{2.13}$$

From the Ricci identity we get

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m} h^{\alpha}_{im}R_{mjkl} + \sum_{m} h^{\alpha}_{mj}R_{mikl} - \sum_{\beta} h^{\beta}_{ij}R_{\alpha\beta kl}. \tag{2.14}$$

It is known from [9, II, p. 78] that if e_{n+1} is the normalized mean curvature normal vector, then

$$H_{\beta}H_{n+1} = H_{n+1}H_{\beta} \tag{2.15}$$

and

$$R_{n+1\beta kl} = 0. \tag{2.16}$$

Since M has constant mean curvature we have

$$\Delta h^{n+1}_{ij} = \sum_{k} h^{n+1}_{ijkk} = \sum_{m,k}(h^{n+1}_{km}R_{mijk} + h^{n+1}_{im}R_{mkjk}). \tag{2.17}$$

Here we denote the components of the Riemannian curvature tensor of M immersed in $S^{n+p}(1)$ by R_{ijkl} and $R_{\alpha\beta kl}$. Thus

$$\begin{aligned} &\frac{1}{2}\Delta\sum_{i,j}(h^{n+1}_{ij})^2 \\ &\qquad = \sum_{i,j,k}(h^{n+1}_{ijk})^2 + \sum_{i,j,k,m} h^{n+1}_{ij}(h^{n+1}_{km}R_{mijk} + h^{n+1}_{im}R_{mkjk}). \end{aligned} \tag{2.18}$$

It follows from the Gauss equation (2.13) and (2.15) that

$$\begin{aligned} \sum_{i,j,k,m} h^{n+1}_{ij}h^{n+1}_{km}R_{mijk} &= \sum_{i,j}(h^{n+1}_{ij})^2 - n^2H^2 + \operatorname{tr}(H^4_{n+1}) \\ &\quad - (\operatorname{tr}(H^2_{n+1}))^2 + \sum_{\beta\neq n+1}\operatorname{tr}((H_{n+1}H_{\beta})^2) - \sum_{\beta\neq n+1}(\operatorname{tr}(H_{n+1}H_{\beta}))^2, \\ \sum_{i,j,k,m} h^{n+1}_{ij}h^{n+1}_{im}R_{mkjk} &= (n-1)\sum_{i,j}(h^{n+1}_{ij})^2 \\ &\quad + nH\operatorname{tr}(H^3_{n+1}) - \operatorname{tr}(H^4_{n+1}) - \sum_{\beta\neq n+1}\operatorname{tr}((H_{n+1}H_{\beta})^2). \end{aligned} \tag{2.19}$$

From (2.18) and (2.19) the Laplacian of the function $\operatorname{tr}(H^2_{n+1})$ can be expressed as

$$\begin{aligned} \frac{1}{2}\Delta\sum_{i,j}(h^{n+1}_{ij})^2 &= \sum_{i,j,k}(h^{n+1}_{ijk})^2 + n\operatorname{tr}(H^2_{n+1}) - n^2H^2 \\ &\quad + nH\operatorname{tr}(H^3_{n+1}) - (\operatorname{tr}(H^2_{n+1}))^2 - \sum_{\beta\neq n+1}(\operatorname{tr}(H_{n+1}H_{\beta}))^2. \end{aligned} \tag{2.20}$$

3. Proof of Theorem 1. We choose a local orthonormal frame field $\{e_A\}, 1 \le A \le n+p$, in $S^{n+p}(1)$ such that when restricted to M, the vectors $\{e_i\}, 1 \le i \le n$, are tangent to M. Furthermore $He_{n+1} = \xi$, where ξ is the mean curvature vector of M. Now we define two tensors ϕ and ψ of type $(1,2)$ by

$$\phi = \sum_{i,j} (h_{ij}^{n+1} - H\delta_{i,j})\omega_i \otimes \omega_j \otimes e_{n+1}, \tag{3.1}$$

$$\psi = \sum_{i,j,\beta} h_{ij}^{\beta}\omega_i \otimes \omega_j \otimes e_\beta, \tag{3.2}$$

where $\{\omega_i\}$ is the dual frame to $\{e_i\}, 1 \le i,j \le n, n+2 \le \beta \le n+p$. Denote by σ_H the square length of the second fundamental form in the direction of the normal vector ξ. It is easily checked that $\operatorname{tr}\phi = 0$ and $|\phi|^2 = \sigma_H - nH^2$, where $\operatorname{tr}\phi$ is the trace of ϕ. Then M is a quasiumbilical submanifold if and only if $|\phi|^2 = 0$. The square norm of ψ is given by

$$|\psi|^2 = \sum_{\beta \ne n+1} \operatorname{tr}(H_\beta^2). \tag{3.3}$$

We have

$$\sigma = |\phi|^2 + |\psi|^2 + nH^2, \tag{3.4}$$

$$\sigma_H^2 = (\operatorname{tr} H_{n+1}^2)^2 = |\phi|^4 - 2nH^2|\phi|^2 + n^2H^4, \tag{3.5}$$

$$\sum_{\beta \ne n+1} (\operatorname{tr}(H_{n+1}H_\beta))^2 = \sum_{\beta \ne n+1} (\operatorname{tr}((H_{n+1} - HI)H_\beta))^2 \le |\phi|^2|\psi|^2. \tag{3.6}$$

Since $\operatorname{tr}\phi = 0$, we can use Lemma (2.6) of [1] to obtain

$$|\operatorname{tr}\phi^3| \le \frac{n-2}{\sqrt{n(n-1)}}\,|\phi|^3.$$

A direct calculation shows that

$$\begin{aligned} \operatorname{tr}(H_{n+1}^3) &= nH^3 + 3H|\phi|^2 - \operatorname{tr}\phi^3 \\ &\ge nH^3 + 3H|\phi|^2 - \frac{n-2}{\sqrt{n(n-1)}}\,|\phi|^3. \end{aligned} \tag{3.7}$$

Substituting (3.5)–(3.7) in (2.20), we get

$$\begin{aligned} &\frac{1}{2}\Delta \sum_{i,j} (h_{ij}^{n+1})^2 \\ &\ge \sum_{i,j,k} (h_{ijk}^{n+1})^2 + |\phi|^2\left(n + nH^2 - \frac{n-2}{\sqrt{n-1}}\sqrt{nH^2}\,|\phi| - |\phi|^2 - |\psi|^2\right). \end{aligned} \tag{3.8}$$

Since H is constant, we have $\sum_{i,j,k}(h_{ijk}^{n+1})^2 = |\nabla\phi|^2$. This yields

$$\frac{1}{2}\Delta|\phi|^2 \geq |\nabla\phi|^2 + |\phi|^2\left(n + nH^2 - \frac{n-2}{\sqrt{n-1}}\sqrt{nH^2}\,|\phi| - |\phi|^2 - |\psi|^2\right). \tag{3.9}$$

Let us consider a quadratic form F with eigenvalues $\pm\frac{n}{2\sqrt{n-1}}$:

$$F(x,y) = x^2 - \frac{n-2}{\sqrt{n-1}}\,xy - y^2. \tag{3.10}$$

Then there exists an orthogonal transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\psi(x,y) = (u,v)$, such that

$$F(x,y) = \frac{n}{2\sqrt{n-1}}\,(u^2 - v^2). \tag{3.11}$$

If $x = \sqrt{nH^2}$, $y = |\phi|$, from

$$u^2 + v^2 = nH^2 + |\phi|^2 = \sigma - |\psi|^2 \tag{3.12}$$

it follows that

$$F(\sqrt{nH^2}, |\phi|) \geq -\frac{n}{2\sqrt{n-1}}\,(u^2+v^2) = -\frac{n}{2\sqrt{n-1}}\,(\sigma - |\psi|^2). \tag{3.13}$$

Since $\big|\nabla|\phi|\big|^2 \leq |\nabla\phi|^2$, from (3.9) we have

$$\begin{aligned}\frac{1}{2}\Delta|\phi|^2 &\geq \big|\nabla|\phi|\big|^2 + |\phi|^2\left(n - \frac{n}{2\sqrt{n-1}}\,\sigma + \left(\frac{n}{2\sqrt{n-1}} - 1\right)|\psi|^2\right)\\ &\geq \big|\nabla|\phi|\big|^2 + n|\phi|^2 - \frac{n}{2\sqrt{n-1}}\,\sigma|\phi|^2.\end{aligned} \tag{3.14}$$

It follows from (2.7) that

$$\int_M \big|\nabla|\phi|\big|^2 \geq k_1\|\phi\|_{2n/(n-2)}^2 - k_2\|\phi\|_2^2, \tag{3.15}$$

where k_1, k_2 have been defined by (2.6). Integrating both sides of (3.14) and applying (3.15) and the inequality

$$\|\sigma|\phi|^2\|_1 \leq \|\sigma\|_{n/2}\|\phi\|_{2n/(n-2)}^2, \tag{3.16}$$

we get

$$\begin{aligned}0 &\geq (n-k_2)\|\phi\|_2^2 + k_1\|\phi\|_{2n/(n-2)}^2 - \frac{n}{2\sqrt{n-1}}\,\|\sigma|\phi|^2\|_1\\ &\geq (n-k_2)\|\phi\|_2^2 + \left(k_1 - \frac{n}{2\sqrt{n-1}}\,\|\sigma\|_{n/2}\right)\|\phi\|_{2n/(n-2)}^2.\end{aligned} \tag{3.17}$$

Let

$$\|\sigma\|_{n/2} < \min\left\{\frac{2\sqrt{n-1}}{n}\,k_1, 2\sqrt{n-1}\,\frac{k_1}{k_2}\right\}. \tag{3.18}$$

From (3.14), (3.16) and (3.18) we can easily obtain

$$
\begin{aligned}
(3.19)\qquad 0 &\geq n\|\phi\|_2^2 - \frac{n}{2\sqrt{n-1}}\,\|\sigma|\phi|^2\|_1 \\
&\geq n\|\phi\|_2^2 - \frac{n}{2\sqrt{n-1}}\,\|\phi\|_{2n/(n-2)}^2\|\sigma\|_{n/2} \\
&\geq n\|\phi\|_2^2 - n\,\frac{k_1}{k_2}\,\|\phi\|_{2n/(n-2)}^2.
\end{aligned}
$$

If $|\phi|^2 \neq 0$, it follows from (3.17)–(3.19) that

$$
\begin{aligned}
(3.20)\qquad 0 &\geq (n-k_2)\|\phi\|_2^2 + \left(k_1 - \frac{n}{2\sqrt{n-1}}\,\|\sigma\|_{n/2}\right)\frac{k_2}{k_1}\,\|\phi\|_2^2 \\
&\geq \left(n - \frac{n}{2\sqrt{n-1}}\,\frac{k_2}{k_1}\,\|\sigma\|_{n/2}\right)\|\phi\|_2^2 > 0,
\end{aligned}
$$

a contradiction. Hence $|\phi|^2 = 0$, i.e., M is quasiumbilical in $S^{n+p}(1)$. From (3.1) we have $h_{ij}^{n+1} = H\delta_{ij}$. The mean curvature vector ξ can be treated as a subbundle of the normal bundle $T^{\perp}M$ with base M embedded in the sphere $S^{n+p}(1)$ with fiber dimension 1. Now M is umbilical with respect to ξ, and ξ is parallel in $T^{\perp}M$. According to a theorem of Yau ([9, I, p. 351]), we derive that M lies in an $n+p-1$-dimensional umbilical hypersurface with ξ perpendicular to the umbilical hypersurface. Furthermore, the Gauss equation of M becomes

$$
(3.21)\qquad R_{ijkl} = (1+H^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\beta}(h_{ik}^{\beta}h_{jl}^{\beta} - h_{il}^{\beta}h_{jk}^{\beta}),
$$

so the hypersurface must be $S^{n+p-1}(1+H^2)$. Since we know from (2.12) that $\operatorname{tr} H_\beta = 0$ for every normal vector e_β of M in $S^{n+p-1}(1+H^2), n+2 \leq \beta \leq n+p$, we derive that M is a minimal submanifold in $S^{n+p-1}(1+H^2)$. Thus we conclude that Theorem 1 holds.

4. Proof of Theorem 2. When $n = 2$, it follows from (3.4) and (3.9) that

$$
(4.1)\qquad \frac{1}{2}\Delta|\phi|^2 \geq \big|\nabla|\phi|\big|^2 + 2(1+2H^2)|\phi|^2 - \sigma|\phi|^2.
$$

Furthermore, P. Li [4] obtained another Sobolev inequality for $\dim M = 2$: For every $f \in H_{1,2}(M^2)$, we have

$$
\begin{aligned}
\int_M |\nabla f|^2 &\geq \frac{C_0}{4}\left\{(\operatorname{vol} M)^{-1/2}\left(\int_M f^4\right)^{1/2} - (\operatorname{vol} M)^{-1}\int_M f^2\right\} \\
&\geq \widetilde{k}_1\|f\|_4^2 - \widetilde{k}_2\|f\|_2^2,
\end{aligned}
$$

where

$$\widetilde{k}_1 = \frac{C_1}{4}\,(\operatorname{vol} M)^{-1/2}, \qquad \widetilde{k}_2 = \frac{C_1}{2}\,(\operatorname{vol} M)^{-1}, \tag{4.2}$$

and C_0 is the best Sobolev constant, C_1 the isoperimetric constant of M, $C_1 \le C_0 \le 2C_1$. Since H is constant and $\big|\nabla|\phi|\big|^2 \le |\nabla\phi|^2$, integrating both sides of (4.1) and applying (4.2) we get

$$\begin{aligned} 0 &\ge \widetilde{k}_1\|\phi\|_4^2 - \widetilde{k}_2\|\phi\|_2^2 + 2(1+2H^2)\|\phi\|_2^2 - \|\sigma|\phi|^2\|_1 \\ &\ge \{2(1+2H^2) - \widetilde{k}_2)\}\|\phi\|_2^2 + (\widetilde{k}_1 - \|\sigma\|_2)\|\phi\|_4^2. \end{aligned} \tag{4.3}$$

Suppose that

$$\|\sigma\|_2 < \min\left\{\widetilde{k}_1, 2(1+2H^2)\,\frac{\widetilde{k}_1}{\widetilde{k}_2}\right\}. \tag{4.4}$$

It follows from (4.1) and (4.4) that

$$2(1+2H^2)\|\phi\|_2^2 \le \|\sigma|\phi|^2\|_1 \le \|\sigma\|_2\|\phi\|_4^2 \le 2(1+2H^2)\,\frac{\widetilde{k}_1}{\widetilde{k}_2}\,\|\phi\|_4^2. \tag{4.5}$$

Hence

$$\frac{\widetilde{k}_2}{\widetilde{k}_1}\,\|\phi\|_2^2 \le \|\phi\|_4^2. \tag{4.6}$$

If $|\phi|^2 \ne 0$, from (4.3) and (4.6) we derive

$$0 \ge \left\{2(1+2H^2) - \frac{\widetilde{k}_2}{\widetilde{k}_1}\,\|\sigma\|_2\right\}\|\phi\|_2^2 > 0. \tag{4.7}$$

This is a contradiction. Hence $|\phi|^2 = 0$, i.e., M is quasiumbilical. By the same reason as in Theorem 1, we conclude that M is a minimal surface in the sphere $S^{p+1}(1+H^2)$.

Let us find a lower bound of the isoperimetric constant C_1 and an upper bound of the volume of M to obtain a lower bound of the quantity $\min\{\widetilde{k}_1, 2(1+2H^2)\widetilde{k}_1/\widetilde{k}_2\}$ which depends only on H and k. We will make use of Wang's argument [8].

From a result of B. Y. Chen [2] we know that for any p-dimensional compact submanifold $\overline{M}$ in the Euclidean space $\mathbb{R}^m$ we have

$$\int_{\overline{M}} |\overline{H}|^p \ge \omega_p, \tag{4.8}$$

where $\overline{H}$ is the mean curvature of $\overline{M}$ in $\mathbb{R}^m$ and ω_p is the volume of the unit sphere $S^p(1)$. In our case that M is an embedded surface in $S^{p+1}(1)$, we have $\overline{H}^2 = 1 + H^2$, where H is the constant mean curvature of M in $S^{p+1}(1)$. Therefore, we obtain

$$\operatorname{vol} M \geq \frac{\omega_2}{1+H^2} = \frac{4\pi}{1+H^2}. \tag{4.9}$$

For any n-dimensional manifold M with positive Ricci curvature, a result due to C. B. Croke [3] shows that

$$\begin{aligned} C_1(M) &\geq \frac{(\operatorname{vol} M)^{n+1}}{4\omega_{n-1}\omega_n^{n-1}} \left(\frac{1}{\int_0^d (\sqrt{1/k}\sin\sqrt{k}r)^{n-1}\,dr} \right)^{n+1} \\ &\geq \frac{n^{n+1}(\operatorname{vol} M)^{n+1}}{4d^{n(n+1)}\omega_{n-1}\omega_n^{n-1}} \end{aligned} \tag{4.10}$$

for $n \geq 2$, where $(n-1)k$ is the lower bound of the Ricci curvature, d is the diameter of M and ω_n is the volume of the unit sphere $S^n(1)$. It follows from the Myers theorem that $d \leq \pi/\sqrt{k}$ for a compact manifold whose Ricci curvature has positive lower bound $(n-1)k$. Then we have

$$C_1(M) > \frac{n^{n+1}k^{n(n+1)/2}}{4\pi^{n(n+1)}} \frac{(\operatorname{vol} M)^{n+1}}{\omega_{n-1}\omega_n^{n-1}}. \tag{4.11}$$

When $n = 2$, this becomes

$$C_1(M) \geq \frac{16k^3}{\pi^5(1+H^2)^3}. \tag{4.12}$$

According to the Gauss–Bonnet formula we have

$$k \operatorname{vol} M \leq \int_M K\,dV = 2\pi\chi(M) = 4\pi, \tag{4.13}$$

where k is the positive lower bound of the Gauss curvature K of M and $\chi(M)$ is the Euler characteristic of the surface M with genus zero. Thus both $\widetilde{k}_1$ and $\widetilde{k}_1/\widetilde{k}_2$ have positive lower bounds depending only on H and k. It follows from (4.9) and (4.13) that $k \leq 1 + H^2$. By a direct calculation from (4.2), (4.9), (4.12) and (4.13) we get

$$\frac{2k^{7/2}}{\pi^{11/2}(1+H^2)^3} \leq \min\left\{ \widetilde{k}_1, 2(1+2H^2)\,\frac{\widetilde{k}_1}{\widetilde{k}_2} \right\}. \tag{4.14}$$

This completes the proof of Theorem 2.

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