COLLOQUIUM MATHEMATICUM

VOL. 96 2003 NO. 2

SELFINJECTIVE ALGEBRAS OF WILD CANONICAL TYPE

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Abstract. We develop the representation theory of selfinjective algebras which admit Galois coverings by the repetitive algebras of algebras whose derived category of bounded complexes of finite-dimensional modules is equivalent to the derived category of coherent sheaves on a weighted projective line with virtual genus greater than one.

Introduction. In this paper, by an *algebra* we mean a finite-dimensional basic connected algebra (associative, with an identity) over an algebraically closed field K. For an algebra Λ we denote by $\operatorname{mod}\Lambda$ the category of finite-dimensional right Λ-modules and by $D : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$. If all projective modules in $\operatorname{mod}\Lambda$ are injective, then Λ is called *selfinjective*. The classical examples of selfinjective algebras are provided by the blocks of group algebras KG of finite groups G, or more generally by the Hopf algebras. An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$, where $\widehat{B}$ is the *repetitive algebra* [19] (locally finite-dimensional, without identity)

$$\widehat{B} = \begin{bmatrix} \ddots & \ddots & & & & \\ & Q_{m-1} & B_{m-1} & & & \\ & & Q_m & B_m & & \\ & & & Q_{m+1} & B_{m+1} & \\ & & & & \ddots & \ddots \end{bmatrix}$$

of an algebra B, where $B_m = B$ and $Q_m = D(B)$ for all $m \in \mathbb{Z}$, all the remaining entries are zero, the matrices in $\widehat{B}$ have only finitely many nonzero elements, addition is the usual addition of matrices, multiplication is induced from the B-bimodule structure of $D(B)$ and the zero map $D(B) \otimes_B D(B) \to 0$, and G is an admissible group of K-linear automorphisms of $\widehat{B}$. The identity maps $B_m \to B_{m+1}$ and $Q_m \to Q_{m+1}$ induce an automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$, called the *Nakayama automorphism* of $\widehat{B}$. Then the

2000 *Mathematics Subject Classification*: 16D50, 16G20, 16G70, 18E30, 14H60.

quotient $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $B \ltimes D(B)$ of B by $D(B)$. We note that if B is of finite global dimension then the stable module category $\underline{\mathrm{mod}}\,\widehat{B}$ of $\mathrm{mod}\,\widehat{B}$ is equivalent, as a triangulated category, to the derived category $\mathrm{D^b}(\mathrm{mod}\,B)$ of bounded complexes over $\mathrm{mod}\,B$ (see [16]).

Selfinjective algebras of the form $\widehat{B}/G$ with B a tilted algebra are called *selfinjective algebras of tilted type*, and their module categories $\mathrm{mod}\,\widehat{B}/G$ have been extensively investigated (see [2], [9], [12], [19], [31], [32], [34], [37]–[39]). It is also known that the class of selfinjective algebras of tilted type coincides with the class of selfinjective algebras $\widehat{B}/G$ given by all finite-dimensional algebras B whose derived category $\mathrm{D^b}(\mathrm{mod}\,B)$ is equivalent to the derived category $\mathrm{D^b}(\mathrm{mod}\,H)$ of a hereditary algebra H (see [31]).

In this paper we are interested in the representation theory of the selfinjective algebras $\widehat{B}/G$ given by finite-dimensional algebras B whose derived category $\mathrm{D^b}(\mathrm{mod}\,B)$ is equivalent, as a triangulated category, to the derived category $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ of a canonical algebra Λ. We call such selfinjective algebras *selfinjective algebras of canonical type*. The canonical algebras $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$, depending on a weight sequence $\boldsymbol{p} = (p_1, \ldots, p_t)$ and a parameter sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise distinct elements of the projective line over K, were introduced and studied in [33]. The finite-dimensional representation theory of Λ is completely controlled by the category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves on a nonsingular weighted projective line $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$, since the derived categories $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ and $\mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$ are equivalent as triangulated categories [14]. The curve $\mathbb{X}$ thus attached to Λ has (virtual) genus

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\biggl((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\biggr),$$

where $p = \mathrm{lcm}(p_1, \ldots, p_t)$. The complexity of the classification problem for $\mathrm{coh}\,\mathbb{X}$, equivalently for $\mathrm{mod}\,\Lambda$, is essentially determined by $g_{\mathbb{X}}$.

For $g_{\mathbb{X}} < 1$, the algebra Λ is concealed of extended Dynkin type; accordingly the classification problems for $\mathrm{coh}\,\mathbb{X}$ and $\mathrm{mod}\,\Lambda$ are equivalent to the classification of indecomposable finite-dimensional modules over a tame hereditary algebra [14], [21] or, according to [15], closely related to the classification problem for indecomposable Cohen–Macaulay modules over a simple surface singularity. For $g_{\mathbb{X}} = 1$, the algebra Λ is of tubular type, its representation theory is known from [33] while the classification problem for $\mathrm{coh}\,\mathbb{X}$ [14], [22] relates to Atiyah's classification [5] of vector bundles over an elliptic curve. For $g_{\mathbb{X}} > 1$, the algebra Λ is wild and its module category $\mathrm{mod}\,\Lambda$ has been investigated in [24]. Moreover, as shown in [21], for the base field of complex numbers, the category of vector bundles on $\mathbb{X}$ is equivalent to the category of $\mathbb{Z}$-graded Cohen–Macaulay modules over the algebra of entire automorphic forms attached to a Fuchsian group of signature $(0; p_1, \ldots, p_t; 0)$.

The representation theory of selfinjective algebras of canonical type $\Lambda = \Lambda(p, \lambda)$ with $\mathbb{X} = \mathbb{X}(p, \lambda)$ of genus $g_{\mathbb{X}} \leq 1$ has been established in [2], [6], [7], [29], [34]. The object of our study in this paper is the class of remaining selfinjective algebras of canonical type, called *selfinjective algebras of wild canonical type.*

1. Preliminaries

1.1. Throughout this paper K will denote a fixed algebraically closed field. By an algebra we mean an associative finite-dimensional K-algebra, which we shall assume (without loss of generality) to be basic and connected. For such an algebra A there exists an isomorphism $A \cong KQ/I$ where KQ is the path algebra of the ordinary quiver $Q = Q_A$ of A and I is an admissible ideal of KQ. If the quiver Q_A has no oriented cycles then A is said to be *triangular.* For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules and by $\operatorname{ind} A$ its full subcategory of indecomposable modules. For each vertex i of $Q = Q_A$, we shall denote by $S_A(i)$ the simple A-module at i, by $P_A(i)$ the projective cover of $S_A(i)$, and by $I_A(i)$ the injective envelope of $S_A(i)$ in $\operatorname{mod} A$.

1.2. We shall denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^- the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We shall identify the vertices of Γ_A with the corresponding indecomposable A-modules. By a *component* of Γ_A we mean a connected component of Γ_A. A vertex X of Γ_A is said to be *left stable* (respectively *right stable*) if $\tau_A^n X$ is defined for all integers $n \geq 0$ (respectively $n \leq 0$). Further, X is said to be *stable* if $\tau_A^n X$ is defined for all integers n. Moreover, X is said to be *periodic* if $X \cong \tau_A^n X$ for some $n \geq 1$. For a component $\mathcal{C}$ of Γ_A we denote by $\mathcal{C}^{\mathrm{s}}$ the *stable part* of $\mathcal{C}$ obtained from $\mathcal{C}$ by removing all the nonstable modules and the arrows attached to them. A component of Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is called *quasi-serial.* Similarly, a component $\mathcal{C}$ of Γ_A with $\mathcal{C}^{\mathrm{s}} \cong \mathbb{Z}\mathbb{A}_\infty$ or $\mathcal{C}^{\mathrm{s}} \cong \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is said to be *stably quasi-serial.* A stable module X in a stably quasi-serial component $\mathcal{C}$ of Γ_A is said to be *stably quasi-simple* provided it has exactly one immediate predecessor (equivalently, exactly one immediate successor) in $\mathcal{C}^{\mathrm{s}}$. For such a module X, there are infinite sectional paths

$$\ldots \to [r]X \to [r-1]X \to \ldots \to [2]X \to [1]X = X$$

and

$$X = X[1] \to X[2] \to \ldots \to X[r-1] \to X[r] \to \ldots$$

in $\mathcal{C}^{\mathrm{s}}$. Then any module M in $\mathcal{C}^{\mathrm{s}}$ is of the form $\tau_A^i\, [r]X$ (equivalently, $\tau_A^i\, X[r]$)

for some $i \in \mathbb{Z}$ and some $r \geq 1$, and r is said to be the *stable quasi-length* of $\tau_A^i[r]X$, denoted by $\mathrm{sql}(M)$. Hence $\mathrm{sql}([r]X) = r = \mathrm{sql}(X[r])$.

If $\mathcal{C}$ is quasi-serial then we write $\mathrm{ql}(M)$ instead of $\mathrm{sql}(M)$ and call it the *quasi-length* of M. Moreover, in such a case, a module X in $\mathcal{C}$ with $\mathrm{ql}(X) = 1$ is said to be *quasi-simple*. For a module X in a component $\mathcal{C}$ of Γ_A we denote by $(\rightarrow X)$ the full translation subquiver of $\mathcal{C}$ formed by all the predecessors of X in $\mathcal{C}$. Dually, $(X \rightarrow)$ denotes the full translation subquiver of $\mathcal{C}$ formed by all successors of X in $\mathcal{C}$.

1.3. Let A be a selfinjective algebra, that is, $A_A \cong D(A)_A$. If A and $D(A)$ are isomorphic as A-bimodules, the algebra A is said to be *symmetric*. We shall denote by $\underline{\mathrm{mod}}\, A$ the stable category of $\mathrm{mod}\, A$. Recall that the objects of $\underline{\mathrm{mod}}\, A$ are the objects of $\mathrm{mod}\, A$ without projective direct summands, and for any two objects M and N of $\mathrm{mod}\, A$ the space of morphisms from M to N in $\underline{\mathrm{mod}}\, A$ is the quotient $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/P(M,N)$, where $P(M,N)$ is the subspace of $\mathrm{Hom}_A(M,N)$ consisting of all morphisms which factor through projective A-modules. We have two mutually inverse functors $\tau_A, \tau_A^- : \underline{\mathrm{mod}}\, A \xrightarrow{\sim} \underline{\mathrm{mod}}\, A$, called the Auslander–Reiten translations. We shall also consider Heller's loop and suspension functors $\Omega_A, \Omega_A^- : \underline{\mathrm{mod}}\, A \xrightarrow{\sim} \underline{\mathrm{mod}}\, A$. If A is symmetric then $\tau_A = \Omega_A^2$ and $\tau_A^- = \Omega_A^{-2}$.

Observe that if $\mathcal{C}$ is an infinite component of Γ_A then $\mathcal{C}^{\mathrm{s}}$ is obtained from $\mathcal{C}$ by removing all projective modules in $\mathcal{C}$, and consequently $\mathcal{C}^{\mathrm{s}}$ is also connected. Moreover, $\Omega_A(\mathcal{C}^{\mathrm{s}})$ and $\Omega_A^-(\mathcal{C}^{\mathrm{s}})$ are stable parts of some components in Γ_A. Following [35] we say that a family $\mathcal{C}$ of connected components of Γ_A is *generalized standard* if $\mathrm{rad}_A^\infty(X,Y) = 0$ for all indecomposable modules X and Y from $\mathcal{C}$. Similarly, a family $\mathcal{D}$ of connected components of the stable Auslander–Reiten quiver Γ_A^{s} of A is said to be *stably generalized standard* if $\underline{\mathrm{rad}}_A^\infty(X,Y) = 0$ for all indecomposable modules X and Y from $\mathcal{D}$. Here, by $\mathrm{rad}_A^\infty(X,Y)$ (respectively, $\underline{\mathrm{rad}}_A^\infty(X,Y)$) we mean the intersection of all positive powers of the radical $\mathrm{rad}_A(X,Y)$ (respectively, stable radical $\underline{\mathrm{rad}}_A(X,Y)$).

2. Derived categories

2.1. Let $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ be a canonical algebra and $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ be the associated weighted projective line. By a *derived canonical* algebra of type Λ (or $\mathbb{X}$) we mean an algebra B whose derived category $\mathrm{D^b}(\mathrm{mod}\, B)$ of bounded complexes over $\mathrm{mod}\, B$ is equivalent, as a triangulated category, to the derived category $\mathrm{D^b}(\mathrm{mod}\, \Lambda)$ of bounded complexes over $\mathrm{mod}\, \Lambda$, or equivalently [14] to the derived category $\mathrm{D^b}(\mathrm{coh}\, \mathbb{X})$ of coherent sheaves over $\mathbb{X}$. A special case are the *concealed-canonical* (respectively, *almost concealed-canonical*) algebras of type Λ, or $\mathbb{X}$, defined as the endomorphism algebras of tilting bundles (respectively, tilting sheaves) on $\mathbb{X}$, or equivalently as the endomorphism algebras of tilting modules T over the canonical algebra Λ, where T is built from indecomposable modules of strictly positive rank (respectively, modules of nonnegative rank). We recall that $\operatorname{mod}\Lambda$, $\operatorname{coh}\mathbb{X}$ and $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ have the same Grothendieck group $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\mathbb{X}) = \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda))$. Moreover, the *rank* is the unique additive function $\mathrm{rk} : \mathrm{K}_0\mathbb{X} \to \mathbb{Z}$ which is surjective and nonnegative for (classes of) members of $\operatorname{coh}\mathbb{X}$. We also need the *degree* $\deg : \mathrm{K}_0\mathbb{X} \to \mathbb{Z}$ which is also $\mathbb{Z}$-linear, maps the class $[\mathcal{O}]$ of the structure sheaf to zero and is positive on simple sheaves; for further information on rank and degree we refer to [14, 24].

The property of being derived canonical (respectively, concealed-canonical) is preserved when passing from B to its opposite algebra, whereas the corresponding statement holds for an almost concealed-canonical algebra if and only if it is already concealed-canonical. If Λ is wild (respectively, tame) then a derived canonical algebra B of type Λ is said to be of *derived wild* (respectively, *derived tame*) *type*. The representation-infinite derived canonical algebras of derived tame type are completely described in [3], whereas information on derived canonical algebras of derived wild type is much less complete. In fact, no classification of concealed-canonical algebras of wild canonical type is known.

2.2. Following [26] by a *quasi-tilted algebra of canonical type* $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ we mean an algebra B that can be realized as the endomorphism algebra of a tilting object for a hereditary abelian K-category $\mathcal{H}$ with $\mathrm{D}^{\mathrm{b}}(\mathcal{H}) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$. It is known [17] that then B is of global dimension at most two and every indecomposable (finite-dimensional) B-module has projective dimension at most one or injective dimension at most one. Clearly, every quasi-tilted algebra of canonical type is derived canonical. Moreover, the class of quasi-tilted algebras of canonical type contains the almost concealed-canonical algebras and their opposites.

The almost concealed-canonical algebras (respectively, quasi-tilted algebras of canonical type) are those algebras which can be obtained from concealed-canonical algebras by tubular branch extensions (respectively, semiregular branch enlargements; see [23], [26]). Further, the concealed-canonical algebras (respectively, almost concealed-canonical algebras, quasi-tilted algebras of canonical type) are exactly the algebras whose module category admits a sincere separating family of stable tubes (respectively, ray tubes, semiregular tubes) by the corresponding results of [25], [23], [26]. For details and the representation theory of quasi-tilted algebras of canonical type we refer to [23], [26], [28], [36].

2.3. In our investigation of selfinjective algebras of wild canonical type we need a description of the module category of an almost concealed-canonical algebra of wild type. Let $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ be a weighted projective line of wild type, T a tilting sheaf on $\mathbb{X}$, and $B = \mathrm{End}(T)$ the associated almost concealed-canonical algebra. Let $\mathrm{coh}_+(T)$ (respectively, $\mathrm{coh}_0^+(T)$) be the full subcategory of the category $\mathrm{vect}\,\mathbb{X}$ of vector bundles on $\mathbb{X}$ (respectively, the category $\mathrm{coh}_0\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ of finite length) consisting of all F satisfying the condition $\mathrm{Ext}^1(T, F) = 0$. Similarly, we denote by $\mathrm{coh}_-(T)$ (respectively, $\mathrm{coh}_0^-(T)$) the full subcategory of $\mathrm{vect}\,\mathbb{X}$ (respectively, $\mathrm{coh}_0(T)$) consisting of all F satisfying the condition $\mathrm{Hom}(T, F) = 0$.

Furthermore, let $\mathrm{coh}_{\geq}(T)$ (respectively, $\mathrm{coh}_{\leq}(T)$) be the additive closure of $\mathrm{coh}_+(T) \cup \mathrm{coh}_0^+(T)$ (respectively, $\mathrm{coh}_-(T) \cup \mathrm{coh}_0^-(T)$). Invoking now [23, Theorem 5.1] and the identification $\mathrm{D^b}(\mathrm{mod}\,B) = \mathrm{D^b}(\mathrm{coh}\,\mathbb{X})$, we conclude that each indecomposable B-module is in one of the four parts of the module category of $\mathrm{mod}\,B$, denoted respectively by mod_+B, mod_0^+B, mod_-B, mod_0^-B, corresponding under the above identification to coh_+T, coh_0^+T, $\mathrm{coh}_-T[1]$, $\mathrm{coh}_0^-T[1]$, respectively. Moreover, for an indecomposable module M we have:

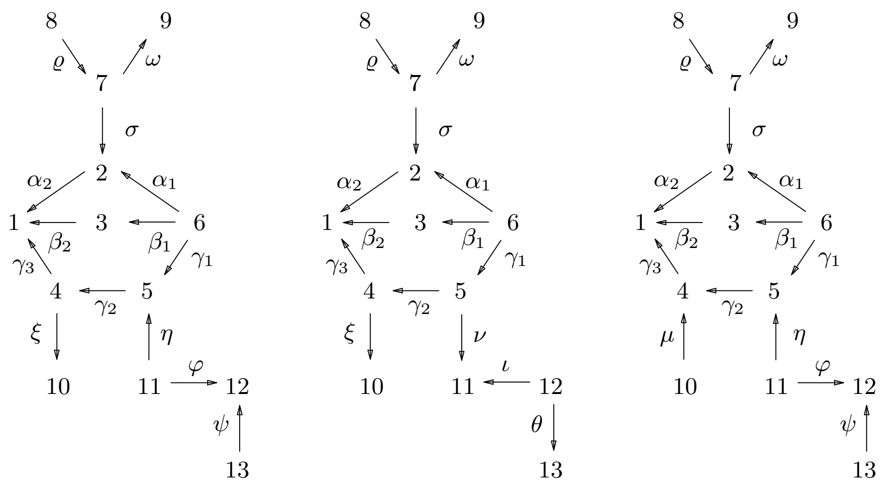
(a) $M \in \mathrm{mod}_+B \Leftrightarrow \mathrm{rk}\,M > 0$,
(b) $M \in \mathrm{mod}_0^+B \Leftrightarrow \mathrm{rk}\,M = 0$ and $\deg\,M > 0$,
(c) $M \in \mathrm{mod}_-B \Leftrightarrow \mathrm{rk}\,M < 0$,
(d) $M \in \mathrm{mod}_0^-B \Leftrightarrow \mathrm{rk}\,M = 0$ and $\deg\,M < 0$.

Further, in the ordering mod_+B, mod_0^+B, mod_-B, mod_0^-B there are no nonzero morphisms from right to left. We denote by $\mathrm{mod}_{\geq}B$ (respectively, $\mathrm{mod}_{\leq}B$) the additive closure of $\mathrm{mod}_+B \cup \mathrm{mod}_0^+B$ (respectively, $\mathrm{mod}_-B \cup \mathrm{mod}_0^-B$).

Then the indecomposable projective B-modules lie in $\mathrm{mod}_{\geq}B$, the indecomposable injective modules lie in $\mathrm{mod}_{\leq}B$, and mod_0^+B is the additive closure of a $\mathbb{P}_1(K)$-family of ray tubes, separating mod_+B from $\mathrm{mod}_{\leq}B$. Further, each component of mod_+B (respectively, $\mathrm{mod}_{\leq}B$) different from the preprojective (respectively, preinjective) component is of type $\mathbb{Z}\mathbb{A}_\infty$ or obtained from $\mathbb{Z}\mathbb{A}_\infty$ by ray (resp. coray) insertions. Finally, for each component $\mathcal{C}$ of $\mathrm{mod}_{\leq}B$ different from the preinjective component there is an indecomposable B-module Z in $\mathcal{C}$ such that the τ_B^--cone $(Z \rightarrow)$ in $\mathcal{C}$ is a full translation subquiver of a component in $(\mathrm{vect}\,\mathbb{X})[1]$, and this establishes a bijection between the set of connected components of mod_-B and the set of connected components of $\mathrm{vect}\,\mathbb{X}$ (see [28, Theorem 3.4, Corollary 6.6]). We have a dual description for the module category $\mathrm{mod}\,B^{\mathrm{op}}$ for the opposite algebra B^{op}.

2.4. We end this section with some examples of algebras discussed before. Consider the algebras given by the following bound quivers:

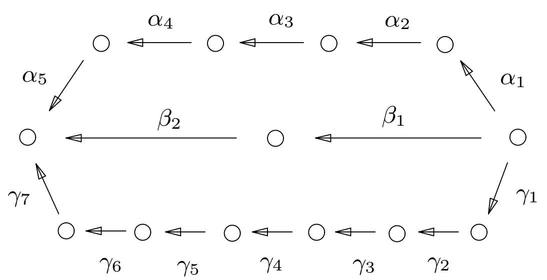
B_1 : B_2 : B_3 :



$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2\gamma_3 = 0,$ $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2\gamma_3 = 0,$ $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2\gamma_3 = 0,$

$\varrho\omega = 0,\ \gamma_2\xi = 0,\ \eta\gamma_2 = 0,$ $\varrho\omega = 0,\ \gamma_2\xi = 0,\ \gamma_1\nu = 0,$ $\varrho\omega = 0,\ \mu\gamma_3 = 0, \eta\gamma_2 = 0$

and the canonical algebra Λ of wild type $(5, 2, 7)$ given by the quiver



bound by $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + \beta_1\beta_2 + \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7 = 0$. The algebras B_1, B_2, B_3 are suitable branch enlargements of the concealed canonical algebra C of type $(2, 2, 3)$ formed by the arrows α_1, α_2, β_1, β_2, γ_1, γ_2, γ_3, and with the relation $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2\gamma_3 = 0$. According to [23], [26], B_3 is an almost concealed-canonical algebra of type Λ, B_2 is quasi-tilted of canonical type Λ but not almost concealed-canonical, B_1 is not quasi-tilted of canonical type, but is derived canonical of type Λ. In fact, as we will see in 3.2, B_1, B_2 and B_3 have isomorphic repetitive algebras.

3. Selfinjective algebras of wild canonical type

3.1. Let B be an algebra and $\{e_i \mid 1 \leq i \leq n\}$ be a complete set of primitive orthogonal idempotents of B such that $1 = e_1 + \ldots + e_n$. Then we have the associated set $\{e_{m,i} \mid m \in \mathbb{Z},\ 1 \leq i \leq n\}$ of primitive orthogonal idempotents of the repetitive algebra $\widehat{B}$ of B such that $e_{m,1} + \ldots + e_{m,n}$ is the identity of B_m, and $\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for any $m \in \mathbb{Z}$, $1 \leq i \leq n$. We may consider $\widehat{B}$ as a locally bounded K-category with the objects $e_{m,i}$ for $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$ and with morphisms given by

$$\mathrm{Hom}_{\widehat{B}}(e_{m,i}, e_{r,j}) = \mathrm{Hom}_{\widehat{B}}(e_{m,i}\widehat{B}, e_{r,j}\widehat{B}) = e_{r,j}\widehat{B}e_{m,i}.$$

Note that the modules $P_{\widehat{B}}(m,i) = e_{m,i}\widehat{B}$, $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$, form a complete set of pairwise nonisomorphic indecomposable projective $\widehat{B}$-modules. A group G of K-linear automorphisms of (the category) $\widehat{B}$ is said to be *admissible* if G acts freely on the set $\{e_{m,i} \mid (m,i) \in \mathbb{Z} \times \{1, \ldots, n\}\}$ and has finitely many orbits. Then the orbit algebra $\widehat{B}/G$ (see [13]) is a (finite-dimensional) selfinjective algebra, and we have a canonical Galois covering $F^B : \widehat{B} \to \widehat{B}/G$ with group G. In particular, the infinite cyclic group $(\nu_{\widehat{B}})$ generated by the Nakayama automorphism $\nu_{\widehat{B}}$ is admissible, and $\widehat{B}/(\nu_{\widehat{B}})$ is isomorphic to the trivial extension $B \ltimes D(B)$ of B by $D(B)$. Recall that $B \ltimes D(B)$ is the symmetric algebra whose additive structure is that of the vector space $B \oplus D(B)$ and whose multiplication is defined by $(a,f)(b,g) = (ab, ag + fb)$ for any $a, b \in B$ and $f, g \in D(B)$.

3.2. Assume B is a triangular algebra. For a sink $i \in Q_B$, the *reflection* $S_i^+ B$ of B at i is the quotient of the one-point extension $B[I_B(i)]$ by the two-sided ideal generated by e_i (see [19]). The quiver $\sigma_i^+ Q_B$ of $S_i^+ B$ is called the reflection of Q_B at i. Observe that the sink i of Q_B is replaced in $\sigma_i^+ Q_B$ by a source i'. Dually, starting from a source j of Q_B, we define the *reflection* $S_j^- B$ of B at j as the quotient of the one-point coextension $[P_B(j)]B$ by the two-sided ideal generated by e_j. The quiver $\sigma_j^- Q_B$ of $S_j^- B$ is called the reflection of Q_B at j, and the source j of Q_B is replaced in $\sigma_j^- Q_B$ by a sink j'. For a sink i (respectively, source j) of Q_B, we have $S_{i'}^- S_i^+ B \cong B$ and $\widehat{S_i^+ B} \cong \widehat{B}$ (respectively, $S_{j'}^+ S_j^- B \cong B$ and $\widehat{S_j^- B} \cong \widehat{B}$). A *reflection sequence of sinks* of Q_B is a sequence $i_1, \ldots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \ldots \sigma_{i_1}^+ Q_B$ for $1 \leq s \leq t$. Dually, a *reflection sequence of sources* of Q_B is a sequence $j_1, \ldots, j_t$ of vertices of Q_B such that j_s is a source of $\sigma_{j_s-1}^- \ldots \sigma_{j_1}^- Q_B$ for $1 \leq s \leq t$.

For example, if B_1, B_2, B_3 are the algebras considered in 2.3, then $B_1 = S_{13}^+ S_{11}^+ B_2$ and $B_3 = S_{10}^+ B_1$, and in particular we have $\widehat{B}_1 \cong \widehat{B}_2 \cong \widehat{B}_3$.

3.3. We may now define the main object of our study in this paper.

Definition. A selfinjective algebra $A = \widehat{B}/G$, where B is a derived wild derived canonical algebra and G is an admissible torsion-free group of K-linear automorphisms of $\widehat{B}$, is said to be a *selfinjective algebra of wild canonical type*.

3.4. Let $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ be a wild canonical algebra with $\boldsymbol{p} = (p_1, \ldots, p_t)$, $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ the weighted projective line attached to Λ, and B a derived canonical algebra of type Λ. Since $\underline{\mathrm{mod}}\, \widehat{B} \cong \underline{\mathrm{mod}}\, \widehat{\Lambda} \cong \mathrm{D^b}(\mathrm{coh}\, \mathbb{X})$, we infer that the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ is of the form

$$(*) \qquad \Gamma_{\widehat{B}} = \bigvee_{q \in \mathbb{Z}} (\mathcal{T}_q \vee \mathcal{R}_q),$$

where, for each $q \in \mathbb{Z}$, $\mathcal{T}_q$ is a family $\mathcal{T}_{q,\lambda}$, $\lambda \in \mathbb{X}$, of standard quasi-tubes whose stable part is a family of stable tubes of tubular type $\boldsymbol{p} = (p_1, \ldots, p_t)$, and $\mathcal{R}_q$ is a family (of cardinality $\mathrm{card}\, K$) of components whose stable parts are of the form $\mathbb{Z}\mathbb{A}_\infty$. Further, $\nu_{\widehat{B}}(\mathcal{T}_q) = \mathcal{T}_{q+2}$, $\nu_{\widehat{B}}(\mathcal{R}_q) = \mathcal{R}_{q+2}$, $\mathrm{Hom}_{\widehat{B}}(\mathcal{R}_q, \mathcal{T}_q) = 0$ for any $q \in \mathbb{Z}$, and also $\mathrm{Hom}_{\widehat{B}}(\mathcal{T}_p \vee \mathcal{R}_p, \mathcal{T}_q \vee \mathcal{R}_q) = 0$ for any $p > q$. Moreover, each quasi-tubular family $\mathcal{T}_q$ separates $\bigvee_{s<q}(\mathcal{T}_s \vee \mathcal{R}_s)$ from $\mathcal{R}_q \vee \bigvee_{t>q}(\mathcal{T}_t \vee \mathcal{R}_t)$. For each $q \in \mathbb{Z}$, we denote by $\mathcal{P}_q$ the family of all projective (equivalently, injective) $\widehat{B}$-modules from $\mathcal{T}_q$, and define $\mathcal{T}_q^+ = \mathcal{T}_q \cap \mathcal{P}_q^\perp$ and $\mathcal{T}_q^- = \mathcal{T}_q \cap {}^\perp\mathcal{P}_q$, where $\mathcal{P}_q^\perp$ and ${}^\perp\mathcal{P}_q$ denote the right (respectively, left) perpendicular category in the sense of [15]. Clearly, $\mathcal{T}_q^+ = (\mathcal{T}_{q,\lambda}^+)_{\lambda \in \mathbb{X}}$ and $\mathcal{T}_q^- = (\mathcal{T}_{q,\lambda}^-)_{\lambda \in \mathbb{X}}$, with $\mathcal{T}_{q,\lambda}^+ = \mathcal{T}_{q,\lambda} \cap \mathcal{P}_q^\perp$ and $\mathcal{T}_{q,\lambda}^- = \mathcal{T}_{q,\lambda} \cap {}^\perp\mathcal{P}_q$.

The following theorem gives a more complete information on the structure of $\mathrm{mod}\, \widehat{B}$.

THEOREM. *There exist algebras B_q^- and B_q^+, $q \in \mathbb{Z}$, such that, in the above notation, the following statements hold:*

(i) *For each $q \in \mathbb{Z}$, B_q^- is a convex almost concealed-canonical subcategory of $\widehat{B}$ of canonical type Λ, and $\mathcal{T}_q^-$ is the unique family of ray tubes in $\Gamma_{B_q^-}$.*

(ii) *For each $q \in \mathbb{Z}$, $(B_q^+)^{\mathrm{op}}$ is almost concealed-canonical of type Λ, B_q^+ is a convex subcategory of $\widehat{B}$, and $\mathcal{T}_q^+$ is the unique family of coray tubes in $\Gamma_{B_q^+}$.*

(iii) *For each $q \in \mathbb{Z}$, $\widehat{B_q^-} = \widehat{B} = \widehat{B_q^+}$, $\nu_{\widehat{B}}(B_q^-) = B_{q+2}^-$ and $\nu_{\widehat{B}}(B_q^+) = B_{q+2}^+$.*

(iv) *There exists a reflection sequence of sinks $i_1, \ldots, i_r$, $i_{r+1}, \ldots, i_s$, $i_{s+1}, \ldots, i_t$, $i_{t+1}, \ldots, i_n$ of $Q_{B_0^+}$, where n is the rank of $\mathrm{K}_0(B_0^+) \cong \mathrm{K}_0(\Lambda)$, such that $B_0^- = S_{i_r}^+ \ldots S_{i_1}^+ B_0^+$, $B_1^+ = S_{i_s}^+ \ldots S_{i_{r+1}}^+ B_0^-$, $B_1^- = S_{i_t}^+ \ldots S_{i_{s+1}}^+ B_1^+$ and $B_2^+ = S_{i_n}^+ \ldots S_{i_{t+1}}^+ B_1^+$.*

(v) *For each $q \in \mathbb{Z}$, the supports of indecomposable $\widehat{B}$-modules from $\mathcal{T}_q$ are contained in the convex subcategory D_q of $\widehat{B}$ given by the objects of B_q^+ and B_q^-.*

(vi) *For each $q \in \mathbb{Z}$, the supports of indecomposable $\widehat{B}$-modules from $\mathcal{R}_q$ are contained in the convex subcategory D'_q of $\widehat{B}$ given by the objects of B_q^- and B_{q+1}^+.*

Proof. Fix $q \in \mathbb{Z}$. Since $\mathcal{T}_q = (\mathcal{T}_{q,\lambda})_{\lambda\in\mathbb{X}}$ is a family of standard quasi-tubes, it follows from [4] that $\mathcal{C}_q = \mathcal{T}_q \cap {}^{\perp}\mathcal{P}_q \cap \mathcal{P}_q^{\perp}$ is a standard family $\mathcal{C}_{q,\lambda}$, $\lambda \in \mathbb{X}$, of stable tubes such that $\mathcal{T}_q$ is obtained from $\mathcal{C}_q$ by a sequence of admissible operations of types (ad 1), creating the standard family of ray tubes $\mathcal{T}_q^- = \mathcal{T}_q \cap {}^{\perp}\mathcal{P}_q$, and then by a sequence of admissible operations of type (ad 2*). Equivalently, $\mathcal{T}_q$ is obtained from $\mathcal{C}_q$ by a sequence of admissible operations of types (ad 1*), creating the standard family of coray tubes $\mathcal{T}_q^+ = \mathcal{T}_q \cap \mathcal{P}_q^{\perp}$, and then by a sequence of admissible operations of type (ad 2).

Further, since $\mathcal{T}_q$ separates $\bigvee_{s<q}(\mathcal{T}_s \vee \mathcal{R}_s)$ from $\mathcal{R}_q \vee \bigvee_{t>q}(\mathcal{T}_t \vee \mathcal{R}_t)$, we deduce from [25] that the support algebra Λ_q of $\mathcal{C}_q$ is a concealed-canonical algebra. Therefore, the support algebra B_q^- of $\mathcal{T}_q^-$ is a branch extension of Λ_q, and so B_q^- is an almost concealed-canonical algebra, by [23], [26]. Dually, the support algebra B_q^+ of $\mathcal{T}_q^+$ is a branch coextension of Λ_q, and hence $(B_q^+)^{\mathrm{op}}$ is almost concealed-canonical. Observe also that both B_q^+ and B_q^- are convex subcategories of $\widehat{B}$, and the support algebra D_q of $\mathcal{T}_q$ is a convex subcategory formed by the objects of B_q^+ and B_q^-. Since $\nu_{\widehat{B}}(\mathcal{T}_q) = \mathcal{T}_{q+2}$ for any $q \in \mathbb{Z}$, we may choose the algebras B_q^- and B_q^+ such that $\nu_{\widehat{B}}(B_q^-) = B_{q+2}^-$ and $\nu_{\widehat{B}}(B_q^+) = B_{q+2}^+$ for any $q \in \mathbb{Z}$.

Denote by Ω the set $\{1,\dots,n\}$ of vertices of $Q_{B_0^+}$. We may write Ω as a disjoint union $\Omega = \Omega_{\mathcal{T}_0} \cup \Omega_{\mathcal{R}_0} \cup \Omega_{\mathcal{T}_1} \cup \Omega_{\mathcal{R}_1}$, where $\Omega_{\mathcal{T}_0} = \{i \in \Omega \mid P_{\widehat{B}}(\nu_{\widehat{B}}(i)) \in \mathcal{T}_0\}$, and similarly $\Omega_{\mathcal{R}_0} = \{i \in \Omega \mid P_{\widehat{B}}(\nu_{\widehat{B}}(i)) \in \mathcal{R}_0\}$, $\Omega_{\mathcal{T}_1} = \{i \in \Omega \mid P_{\widehat{B}}(\nu_{\widehat{B}}(i)) \in \mathcal{T}_1\}$, $\Omega_{\mathcal{R}_1} = \{i \in \Omega \mid P_{\widehat{B}}(\nu_{\widehat{B}}(i)) \in \mathcal{R}_1\}$. We order the vertices of $\Omega_{\mathcal{T}_0}$, $\Omega_{\mathcal{R}_0}$, $\Omega_{\mathcal{T}_1}$ and $\Omega_{\mathcal{R}_1}$ such that $\Omega_{\mathcal{T}_0} = \{i_1,\dots,i_r\}$, $\Omega_{\mathcal{R}_0} = \{i_{r+1},\dots,i_s\}$, $\Omega_{\mathcal{T}_1} = \{i_{s+1},\dots,i_t\}$, $\Omega_{\mathcal{R}_1} = \{i_{t+1},\dots,i_n\}$, and

$$\mathrm{Hom}_{\widehat{B}}(P_{\widehat{B}}(\nu_{\widehat{B}}(i_l)), P_{\widehat{B}}(\nu_{\widehat{B}}(i_m))) = 0$$

for any $1 \le m < l \le n$. Note that this is possible because $\widehat{B}$ is triangular and $\mathrm{Hom}_{\widehat{B}}(\mathcal{R}_0, \mathcal{T}_0) = 0$, $\mathrm{Hom}_{\widehat{B}}(\mathcal{T}_1 \vee \mathcal{R}_1, \mathcal{T}_0 \vee \mathcal{R}_0) = 0$.

Then $i_1,\dots,i_r,\ i_{r+1},\dots,i_s,\ i_{s+1},\dots,i_t,\ i_{t+1},\dots,i_n$ is a reflection sequence of sinks of $Q_{B_0^+}$ satisfying the conditions of (iv). For each $q \in \mathbb{Z}$, we then also have $\widehat{B_q^+} \cong \widehat{B_q^-} \cong \widehat{B_{q+1}^+} \cong \widehat{B_{q+1}^-} \cong \widehat{B_{q+2}^+} \cong \widehat{\nu_{\widehat{B}}(B_q^+)}$. This shows that $\mathrm{mod}\,\widehat{B} \cong \mathrm{mod}\,\widehat{B_0^-}$, and consequently $\widehat{B_q^-} \cong \widehat{B} \cong \widehat{B_q^+}$ for any $q \in \mathbb{Z}$. Finally, observe that the support algebra D'_q of $\mathcal{R}_q$ is the convex subcategory of $\widehat{B}$ formed by the objects of B_q^- and B_{q+1}^+. ■

3.5. It follows from the above theorem that in our considerations of selfinjective algebras of wild canonical type, we may restrict ourselves to the selfinjective algebras given by almost concealed-canonical algebras. Moreover, we have the following direct consequence of the above theorem.

LEMMA. *Let B and D be almost concealed-canonical algebras of wild type, and let* $\mathrm{K}_0(B)$ *and* $\mathrm{K}_0(D)$ *have ranks m and n, respectively. Then the following are equivalent*:

(i) $\widehat{B} \cong \widehat{D}$.

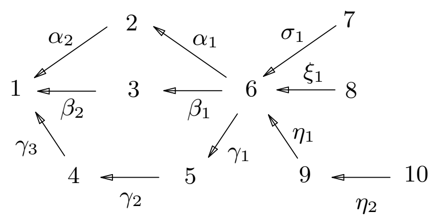
(ii) $D \cong S^+_{i_r} \dots S^+_{i_1} B$ *for a reflection sequence of sinks* $i_1, \dots, i_r$, $r \le m$, *in* Q_B.

(iii) $B \cong S^+_{j_t} \dots S^+_{j_1} D$ *for a reflection sequence of sinks* $j_1, \dots, j_t$, $t \le n$, *in* Q_D.

Clearly if $\widehat{B} \cong \widehat{D}$ then $\mathrm{K}_0(B)$ and $\mathrm{K}_0(D)$ are isomorphic, so in the above corollary we have in fact $m = n$. We note further that the above corollary also holds for almost concealed-canonical algebras of tame (domestic or tubular) type, by the corresponding results of [2], [29], [34].

3.6. Let B be an almost concealed-canonical algebra of wild type. We may identify B with the convex subcategory B_0^- of $\widehat{B}$. Following [34], B is said to be *exceptional* whenever, in the notation of Theorem 3.4, we have $B_0^- \cong B_1^-$. Otherwise B is said to be *normal*. We note that B is exceptional if and only if $B \cong S^+_{i_r} \dots S^+_{i_1} B$ for a reflection sequence of sinks $i_1, \dots, i_r$ in Q_B with r smaller than the rank of $\mathrm{K}_0(B)$. Moreover, if B is exceptional then the rank of $\mathrm{K}_0(B)$ is even.

An example of an exceptional almost concealed-canonical algebra of wild type is provided by the algebra $B = K\Delta/I$, where Δ is the quiver



and I is the ideal in $K\Delta$ generated by $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2\gamma_3$, $\sigma_1\alpha_1$, $\xi_1\beta_1$, $\eta_1\gamma_1$. Then B is an almost concealed-canonical algebra of wild type $(3,3,5)$, being a tubular extension of the canonical algebra C of tubular type $(2,2,3)$ given by the vertices $1,2,3,4,5,6$. Moreover, for $B = B_0^-$, we have $B_1^+ = S_1^+ B \cong B^{\mathrm{op}}$ and $B_1^- = S_5^+ S_4^+ S_3^+ S_2^+ B_1^+ \cong B$. Therefore, $B \cong S_5^+ S_4^+ S_3^+ S_2^+ S_1^+ B$ is an exceptional almost concealed-canonical algebra. On the other hand, the convex subcategory B' of B given by all vertices of Q_B except 10 is a normal almost concealed-canonical algebra of wild type $(3,3,4)$, because $\mathrm{K}_0(B')$ is of odd rank.

For B exceptional, we denote by $\varphi_{\widehat{B}}$ a fixed extension of $B = B_0^- \xrightarrow{\sim} B_1^- \hookrightarrow \widehat{B}$ to a K-linear automorphism of $\widehat{B}$. For B normal, we put $\varphi_{\widehat{B}} = \nu_{\widehat{B}}$.

Consider now the decomposition $(*)$

$$\Gamma_{\widehat{B}} = \bigvee_{q\in\mathbb{Z}} (\mathcal{T}_q \vee \mathcal{R}_q)$$

described above. Then the separation properties of $\mathcal{T}_q$ imply that for any K-linear automorphism f of $\widehat{B}$, there exists an integer m such that $f(\mathcal{T}_q) = \mathcal{T}_{q+m}$ and $f(\mathcal{R}_q) = \mathcal{R}_{q+m}$ for any $q \in \mathbb{Z}$. If $m \geq 0$ (respectively, $m > 0$), such an automorphism f is said to be *positive* (respectively, *strictly positive*). Finally, if $m = 0$, f is said to be *rigid*. Clearly, $\varphi_{\widehat{B}}$ and $\nu_{\widehat{B}}$ are strictly positive automorphisms of $\widehat{B}$. Observe that if B is exceptional, then $\varphi^2_{\widehat{B}} = f\nu_{\widehat{B}}$ for some rigid automorphism f of $\widehat{B}$.

LEMMA. *Let G be a torsion-free admissible group of K-linear automorphisms of $\widehat{B}$. Then G is an infinite cyclic group generated by $f\varphi^s_{\widehat{B}}$ for some $s \geq 1$ and some rigid automorphism f of $\widehat{B}$.*

Proof. For $g \in G$, let m_g be the integer such that $g(\mathcal{T}_0) = \mathcal{T}_{m_g}$. Applying induction and the separation properties of the families of quasi-tubes $\mathcal{T}_q$, $q \in \mathbb{Z}$, we deduce that then $g(\mathcal{T}_q) = \mathcal{T}_{q+m_g}$ for all $q \in \mathbb{Z}$. Similarly, $m_h = -m_g$ for $h = g^{-1}$. Suppose $m_g = 0$ for some $g \in G$. Then $g(\mathcal{T}_q) = \mathcal{T}_q$ and $g(\mathcal{R}_q) = \mathcal{R}_q$ for any $q \in \mathbb{Z}$. In particular, g acts on the finite set of projective modules contained in $\mathcal{T}_0 \vee \mathcal{R}_0$, and so some power g^r of g fixes an idempotent $e_{m,i}$ of $\widehat{B}$. Since G is admissible and torsion-free, we get $g = 1$. Choose $g \in G$ such that m_g is positive and minimal. Let $h \in G$ and $m_h = tm_g + l$ with $0 \leq l < m_g$. Then $a = hg^{-t} \in G$, $m_a = l$, and hence $l = 0$, $a = 1$. Therefore, G is infinite cyclic generated by g. If B is normal, then m_g is even, say $m = 2s$, and $f = g\varphi^{-s}_{\widehat{B}}$ is a rigid automorphism of $\widehat{B}$. Similarly, if B is exceptional and $s = m_g$, then $f = g\varphi^{-s}_{\widehat{B}}$ is rigid. Consequently, G is infinite cyclic generated by $f\varphi^s_{\widehat{B}}$ for some $s \geq 1$ and some rigid automorphism f of $\widehat{B}$. ∎

3.7. There are many normal almost concealed-canonical algebras whose Grothendieck group has even rank. Let $\boldsymbol{p} = (p_1, \ldots, p_t)$ be a weight sequence and $\delta(\boldsymbol{p})$ be the *discriminant* of $\boldsymbol{p}$ defined by

$$\delta(\boldsymbol{p}) = (t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}, \quad \text{where} \quad p = \operatorname{lcm}(p_1, \ldots, p_t).$$

Then $\boldsymbol{p}$ is said to be of *wild* (respectively, *tame*) *type* if $\delta(\boldsymbol{p}) > 0$ (respectively, $\delta(\boldsymbol{p}) \leq 0$). Note that the weight sequences $\boldsymbol{p} = (2, 4, 2m+1)$, $m \geq 2$, are wild and $\delta(\boldsymbol{p}) = 2m - 3$ exhaust all odd natural numbers. Moreover, for any almost concealed-canonical algebra B of type $(2, 4, 2m+1)$, $\mathrm{K}_0(B)$ has even rank $2(m+3)$. We have the following direct consequence of [27, Corollary 3]:

LEMMA. *Let B be an almost concealed-canonical algebra of wild type $\boldsymbol{p}$ such that $\delta(\boldsymbol{p})$ is odd. Then B is normal.*

For further examples of exceptional almost concealed-canonical algebras of wild type we refer to [27, Sections 2 and 4].

3.8. Let B be an almost concealed-canonical algebra of wild type. Let G be the infinite cyclic group of K-linear automorphisms of $\widehat{B}$ generated by a strictly positive automorphism g. Then $g(\mathcal{T}_q) = \mathcal{T}_{q+m}$ and $g(\mathcal{R}_q) = \mathcal{R}_{q+m}$ for all $q \in \mathbb{Z}$ and some fixed $m \geq 1$. We know from Theorem 3.4 that $\widehat{B}$ is locally support-finite [10], that is, for each idempotent $e_{m,i}$ of $\widehat{B}$, the set of all idempotents $e_{n,j}$ of $\widehat{B}$ with $Me_{m,i} \neq 0 \neq Me_{n,j}$ for some indecomposable finite-dimensional $\widehat{B}$-module M is finite. Applying [11, Proposition 2.5] we conclude that the push-down functor $F^B_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}/G$ associated to the Galois covering $F^B : \widehat{B} \to \widehat{B}/G$ is dense and preserves Auslander–Reiten sequences (see also [13, Theorem 3.6]).

Therefore, $\Gamma_{\widehat{B}/G}$ is obtained from $\Gamma_{\widehat{B}}$ by identifying, via F^B_λ, $\mathcal{T}_q$ with $\mathcal{T}_{q+m}$ and $\mathcal{R}_q$ with $\mathcal{R}_{q+m}$, for all $q \in \mathbb{Z}$. Thus $\Gamma_{\widehat{B}/G}$ is of the form

$$F^B_\lambda(\mathcal{T}_0 \vee \mathcal{R}_0) \vee F^B_\lambda(\mathcal{T}_1 \vee \mathcal{R}_1) \vee \ldots \vee F^B_\lambda(\mathcal{T}_{m-1} \vee \mathcal{R}_{m-1}).$$

Moreover, $F^B_\lambda : \operatorname{ind} \widehat{B} \to \operatorname{ind} \widehat{B}/G$ is a Galois covering and hence it induces the following isomorphisms (see [8], [13]):

$$\bigoplus_{g\in G} \operatorname{Hom}_{\widehat{B}}(M, g\,N) \xrightarrow{\sim} \operatorname{Hom}_{\widehat{B}/G}(F^B_\lambda(M), F^B_\lambda(N)) \xleftarrow{\sim} \bigoplus_{g\in G} \operatorname{Hom}_{\widehat{B}}(g\,M, N)$$

for any $M, N \in \operatorname{ind} \widehat{B}$. This allows one to recover all morphisms in $\operatorname{mod} \widehat{B}/G$ from the morphisms in $\operatorname{mod} \widehat{B}$.

3.9. PROPOSITION. *Let B be an almost concealed-canonical algebra of wild type, $G = (g)$ an admissible infinite cyclic group of K-linear automorphisms of $\widehat{B}$, and $g(\mathcal{T}_0) = \mathcal{T}_m$ for some $m \geq 1$. Then the following assertions are equivalent*:

(i) $m \geq 2$.

(ii) *G is generated by an element $\psi\nu_{\widehat{B}}$ for some positive automorphism ψ of $\widehat{B}$.*

(iii) *There exists r, $0 \leq r \leq m-1$, such that the family $F^B_\lambda(\mathcal{T}_r)^{\mathrm{s}}$ is stably generalized standard.*

(iv) *The families $F^B_\lambda(\mathcal{T}_r)^{\mathrm{s}}$, $0 \leq r \leq m-1$, are stably generalized standard.*

Proof. The equivalence of (i) and (ii) follows from Lemma 3.6. Moreover, $\underline{\operatorname{mod}}\, \widehat{B} \cong \mathrm{D^b}(\operatorname{coh} \mathbb{X})$ for a weighted projective line $\mathbb{X}$ of wild type. Because $\mathrm{D^b}(\operatorname{coh} \mathbb{X})$ is an abelian hereditary category, for all $p, q \in \mathbb{Z}$ we have $\underline{\mathrm{Hom}}_{\widehat{B}}(\mathcal{T}_p^{\mathrm{s}}, \mathcal{T}_q^{\mathrm{s}}) \neq 0$ if and only if $q = p$ or $q = p + 1$. Since the functor $F_\lambda^B : \mathrm{mod}\widehat{B} \to \mathrm{mod}\widehat{B}/G$ induces a Galois covering $\underline{F}_\lambda^B : \underline{\mathrm{ind}}\widehat{B} \to \underline{\mathrm{ind}}\widehat{B}/G$, the required equivalence of (i)–(iv) follows. ∎

3.10. Corollary. *Let B and G be as above. Assume that the families $F_\lambda^B(\mathcal{T}_r)$, $0 \le r \le m-1$, consist of stable tubes. Then the following conditions are equivalent*:

(i) $m \ge 2$.

(ii) *There exists r, $0 \le r \le m-1$, such that the family $F_\lambda^B(\mathcal{T}_r)$ is generalized standard.*

(iii) *The families $F_\lambda^B(\mathcal{T}_r)$, $0 \le r \le m-1$, are generalized standard.*

Proof. Observe that if $p, q \in \mathbb{Z}$, $|p - q| \ge 2$, then the tubular families $\mathcal{T}_p$ and $\mathcal{T}_q$ have different supports, and hence $\mathrm{Hom}_{\widehat{B}}(\mathcal{T}_p, \mathcal{T}_q) = 0$. Then the required equivalences follow from 3.9. ∎

For examples of wild concealed-canonical algebras B satisfying the hypothesis of 3.10 we refer to 5.4–5.6. On the other hand, each wild canonical algebra satisfies the assumptions of the next corollary.

3.11. Corollary. *Let B and G be as in the above proposition. Assume that one of the families of quasi-tubes $F_\lambda^B(\mathcal{T}_r)$, $0 \le r \le m-1$, contains a projective module. Then the following conditions are equivalent*:

(i) $m \ge 3$.

(ii) *G is generated by an element $\psi\nu_{\widehat{B}}$ for some strictly positive automorphism ψ of $\widehat{B}$.*

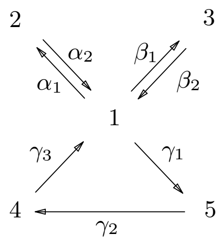
(iii) *A nonstable quasi-tube in $F_\lambda^B(\mathcal{T}_r)$ is generalized standard.*

(iv) *All quasi-tubes in $\Gamma_{\widehat{B}/G}$ are generalized standard.*

Proof. It follows from our assumptions that $\mathcal{T}_r$ contains an indecomposable projective $\widehat{B}$-module $e_{q,i}\widehat{B}$. Since $\nu_{\widehat{B}}(e_{q,i}) = e_{q+1,i}$, we have $\mathrm{top}(e_{q,i}\widehat{B}) \cong \mathrm{soc}(e_{q+1,i}\widehat{B})$, and hence $\mathrm{Hom}_{\widehat{B}}(e_{q,i}\widehat{B}, e_{q+1,i}\widehat{B}) \neq 0$. This implies that $\mathrm{Hom}_{\widehat{B}}(\mathcal{T}_r, \mathcal{T}_{r+2}) \neq 0$, because $e_{q+1,i}\widehat{B} = \nu_{\widehat{B}}(e_{q,i}\widehat{B})$ belongs to $\mathcal{T}_{r+2}$. Further, we know from 3.4 that $\mathrm{Hom}_{\widehat{B}}(\mathcal{T}_p, \mathcal{T}_q) = 0$ for any $q \ge p + 3$. Finally, observe that, in the above notation, $n \ge 3$ if and only if g is of the form $g = \psi\nu_{\widehat{B}}$ for some strictly positive automorphism ψ of $\widehat{B}$. Therefore, since $F_\lambda^B : \mathrm{ind}\,\widehat{B} \to \mathrm{ind}\,\widehat{B}/G$ is a Galois covering, the required equivalence of (i)–(iv) follows. ∎

3.12. Let B be an almost concealed-canonical algebra of wild type, $G = (g)$ an admissible infinite cyclic group of K-linear automorphisms of $\widehat{B}$, and assume $g(\mathcal{T}_0) = \mathcal{T}_m$ for some $m \ge 2$. Then it follows from Lemma 3.6 that $g = f\varphi_B^m = \psi\nu_{\widehat{B}}$ for some rigid automorphism f of $\widehat{B}$ and some positive automorphism ψ of $\widehat{B}$. Then the Auslander–Reiten quiver Γ_A of the selfinjective algebra $A = \widehat{B}/G$ consists of $m \geq 2$ $\mathbb{P}_1(K)$-families $F_\lambda^B(\mathcal{T}_0), \ldots, F_\lambda^B(\mathcal{T}_{m-1})$ of quasi-tubes and infinitely many components with stable part $\mathbb{Z}\mathbb{A}_\infty$, distributed in the families $F_\lambda^B(\mathcal{R}_0), \ldots, F_\lambda^B(\mathcal{R}_{m-1})$. Moreover, it follows from Proposition 3.9 that $F_\lambda^B(\mathcal{T}_q^-)$ and $F_\lambda^B(\mathcal{T}_q^+)$, $0 \leq q \leq m-1$, are generalized standard families of modules in $\operatorname{mod} A$. Consider the two-sided ideals $I_q^+ = \operatorname{ann}_A(F_\lambda^B(\mathcal{T}_q^+))$ and $I_q^- = \operatorname{ann}_A(F_\lambda^B(\mathcal{T}_q^-))$, $0 \leq q \leq m-1$, of A. Since $\mathcal{T}_q^-$ (respectively, $\mathcal{T}_q^+$) is a faithful family of ray (respectively, coray) tubes of the almost concealed-canonical algebra B_q^- (respectively, of the dual B_q^+ of an almost concealed-canonical algebra) we infer that $B_q^+ = A/I_q^+$ and $B_q^- = A/I_q^-$ for any $0 \leq q \leq m-1$. Therefore, the algebras B_q^- and B_q^+, $0 \leq q \leq m-1$, are natural factors of the selfinjective algebra $A = \widehat{B}/G$.

We note that the corresponding claim is not true for an exceptional almost concealed-canonical algebra B and $A = \widehat{B}/(\varphi_{\widehat{B}})$. Indeed, let B be the exceptional almost concealed-canonical algebra of type $(3,3,5)$ considered in 3.6. Then $A = \widehat{B}/(\varphi_{\widehat{B}})$ is the bound quiver algebra KQ/I, where Q is the quiver



and I is the ideal of KQ generated by $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2\gamma_3$, $\alpha_2\alpha_1$, $\beta_2\beta_1$, $\gamma_2\gamma_3\alpha_1\alpha_2\gamma_1\gamma_2$. Observe that the factor algebras of A are not quasi-tilted algebras of canonical type. This motivates the following definition.

Definition. A selfinjective algebra $A = \widehat{B}/(\psi\nu_{\widehat{B}})$, where B is almost concealed-canonical of wild type and ψ is a positive automorphism of $\widehat{B}$, is said to be a *proper selfinjective algebra of wild canonical type.*

Clearly, each selfinjective algebra of wild canonical type given by a normal almost concealed-canonical algebra is proper.

3.13. We end this section with the following characterization of proper symmetric algebras of wild canonical type.

Theorem. *An algebra A is a proper symmetric algebra of wild canonical type if and only if A is isomorphic to the trivial extension $B \ltimes D(B)$ of an almost concealed-canonical algebra B of wild type.*

Proof. This follows from the above definitions and [30, Theorem 2]. ∎

We note that the selfinjective algebra $A = \widehat{B}/(\varphi_{\widehat{B}})$ of wild type $(3,3,5)$ constructed in 3.12 is symmetric but clearly is not a trivial extension of a quasi-tilted algebra of canonical type.

4. Cones of $\mathbb{Z}\mathbb{A}_\infty$-components

4.1. Let B be an almost concealed-canonical algebra of wild type $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$, and $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ the associated weighted projective line. We use the notation introduced in Sections 2 and 3 and identify B with the convex subcategory B_0^- of $\widehat{B}$. The following theorem describes the structure of components in $\Gamma_{\widehat{B}}$ with stable part $\mathbb{Z}\mathbb{A}_\infty$.

THEOREM. *For a fixed integer q, the following statements hold*:

(i) *For each component $\mathcal{C}$ from $\mathcal{R}_q$, there exist indecomposable modules M^- and M^+ in $\mathcal{C}$ such that $\mathcal{C}^- = (M^- \to)$ is a right stable cone of $\mathcal{C}$ consisting of modules from* $\operatorname{mod}_- B_q^-$, $\mathcal{C}^+ = (\to M^+)$ *is a left stable cone of $\mathcal{C}$ consisting of modules from* $\operatorname{mod}_+ B_{q+1}^+$, *and for $X \in \mathcal{C}^-$ and $Y \in \mathcal{C}^+$ we have $\tau_{\widehat{B}}^- X = \tau_{B_q^-}^- X$ and $\tau_{\widehat{B}} Y = \tau_{B_{q+1}^+} Y$.*

(ii) *For each component $\mathcal{D}$ in* $\operatorname{mod}_- B_q^-$ *different from the preinjective component, there exists an indecomposable module Y in $\mathcal{D}$ and a component $\mathcal{C}$ in $\mathcal{R}_q$ such that the cone $\mathcal{D}^- = (Y \to)$ of $\mathcal{D}$ is a right stable full translation subquiver of $\mathcal{C}$ which is closed under successors.*

(iii) *For each component $\mathcal{E}$ of* $\operatorname{mod}_+ B_{q+1}^+$ *different from the preprojective component, there exists an indecomposable module Z in $\mathcal{E}$ and a component $\mathcal{C}$ in $\mathcal{R}_q$ such that the cone $\mathcal{E}^+ = (\to Z)$ of $\mathcal{E}$ is a left stable full translation subquiver of $\mathcal{C}$ which is closed under predecessors.*

Proof. First we note that B_q^- and B_{q+1}^+ are factor algebras of $\widehat{B}$. From Theorem 3.4(vi) we know that the support of any indecomposable $\widehat{B}$-module in $\mathcal{R}_q$ is contained in the convex subcategory D_q' of $\widehat{B}$ given by the objects of B_q^- and B_{q+1}^+. Moreover, we have $B_{q+1}^+ = S_{j_t}^+ \ldots S_{j_1}^+ B_q^-$ for a reflection sequence $j_1, \ldots, j_t$ of sinks in $Q_{B_q^-}$. Let $P_1, \ldots, P_t$ be all indecomposable projective $\widehat{B}$-modules in $\mathcal{R}_q$. Then the socles of $P_1, \ldots, P_t$ are the simple $\widehat{B}$-modules given by those vertices $j_1, \ldots, j_t$, and the tops of $P_1, \ldots, P_t$ are the simple $\widehat{B}$-modules given by the vertices of $\sigma_{j_t}^+ \ldots \sigma_{j_1}^+ Q_{B_q^-} = Q_{B_{q+1}^+}$ which are not vertices of $Q_{B_q^-}$.

Let $\mathcal{C}$ be a component from $\mathcal{R}_q$. In order to prove claim (i) for $\mathcal{C}$ it is sufficient to find indecomposable $\widehat{B}$-modules M and N in $\mathcal{C}$ such that $\mathcal{C}^- = (M \to)$ is right stable, $\mathcal{C}^+ = (\to N)$ is left stable, and $\operatorname{Hom}_{\widehat{B}}(U, P_i) = 0$, $\operatorname{Hom}_{\widehat{B}}(P_i, V) = 0$ for all modules $U \in \mathcal{C}^+$, $V \in \mathcal{C}^-$, and any $1 \le i \le t$.

Observe also that since $\mathcal{C}^+ \cong \mathbb{N}\Delta$ for a quiver Δ of the form

$$0 \to 1 \to \ldots \to r \leftarrow r+1 \leftarrow r+2 \leftarrow \ldots$$

for some $r \geq 0$, we have $\operatorname{Hom}_{\widehat{B}}(U, P_i) = 0$ for any $U \in \mathcal{C}^+$ if and only if $\operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^l N, P_i) = 0$ for any $l \in \mathbb{N}$. Similarly, $\mathcal{C}^- \cong (-\mathbb{N})\Sigma$ for a quiver Σ of the form

$$0 \leftarrow 1 \leftarrow \ldots \leftarrow s \to s+1 \to s+2 \to \ldots$$

for some $s \geq 0$, and hence $\operatorname{Hom}_{\widehat{B}}(P_i, V) = 0$ for any $V \in \mathcal{C}^-$ if and only if $\operatorname{Hom}_{\widehat{B}}(P_i, \tau_{\widehat{B}}^{-l} M) = 0$ for any $l \in \mathbb{N}$.

We know that $\underline{\operatorname{mod}}\, \widehat{B} \cong \underline{\operatorname{mod}}\, \widehat{\Lambda} \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathbb{X})$, and the full subcategory of $\underline{\operatorname{mod}}\, \widehat{B}$ given by all nonprojective objects of $\mathcal{R}_q$ is equivalent to the category of indecomposable vector bundles over $\mathbb{X}$. Applying now [24, Proposition 10.1] we conclude that there are indecomposable modules M and N in $\mathcal{C}$ such that $\underline{\operatorname{Hom}}_{\widehat{B}}(\tau_{\widehat{B}}^l N, \operatorname{rad} P_i) = 0$ and $\underline{\operatorname{Hom}}_{\widehat{B}}(P_i/\operatorname{soc} P_i, \tau_{\widehat{B}}^{-l} M) = 0$ for any $l \in \mathbb{N}$ and $1 \leq i \leq t$. Moreover, since the stable part $\mathcal{C}^{\mathrm{s}}$ of $\mathcal{C}$ is of the form $\mathbb{Z}\mathbb{A}_\infty$, we may choose M and N such that $\mathcal{C}^+ = (\to N)$ is left stable and $\mathcal{C}^- = (M \to)$ is right stable. Further, since $\tau_{\widehat{B}}^l N$, $l \in \mathbb{N}$, are nonprojective and $\tau_{\widehat{B}}^l M$, $l \in \mathbb{N}$, are noninjective, we have $\operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^l N, P_i) = \operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^l N, \operatorname{rad} P_i)$ and $\operatorname{Hom}_{\widehat{B}}(P_i, \tau_{\widehat{B}}^{-l} M) = \operatorname{Hom}_{\widehat{B}}(P_i/\operatorname{soc} P_i, \tau_{\widehat{B}}^{-l} M)$.

Suppose now that there is a nonzero morphism $f : \tau_{\widehat{B}}^l N \to P_i$ for some $l \in \mathbb{N}$ and some $1 \leq i \leq t$. Since $\underline{\operatorname{Hom}}_{\widehat{B}}(\tau_{\widehat{B}}^l N, \operatorname{rad} P_i) = 0$, f factors through a projective $\widehat{B}$-module P. But $\tau_{\widehat{B}}^l N$ and $\operatorname{rad} P_i$ belong to $\mathcal{R}_q$, and hence P is a direct sum of modules $P_1, \ldots, P_t$. Observe that if $f = f''f'$ with $f' : \tau_{\widehat{B}}^l N \to P$ and $f'' : P \to \operatorname{rad} P_i$, then the image of f' is contained in $\operatorname{rad} P$. Since $\underline{\operatorname{Hom}}_{\widehat{B}}(\tau_{\widehat{B}}^l N, \operatorname{rad} P) = 0$, we conclude that f' factors through a direct sum of modules from the family $P_1, \ldots, P_t$.

Therefore, for any positive number r, we may find a sequence of morphisms

$$P_{i_1} \xrightarrow{h_1} P_{i_2} \to \ldots \to P_{i_r} \xrightarrow{h_r} P_{i_{r+1}}$$

such that $i_1, \ldots, i_{r+1} \in \{1, \ldots, t\}$, $h_1, \ldots, h_r$ are nonisomorphisms and $h_r \ldots h_1 \neq 0$. This leads to a contradiction because the radical of the algebra $\operatorname{End}_{\widehat{B}}(P_1 \oplus \ldots \oplus P_t)$ is nilpotent. Consequently, we have proved that $\operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^l N, P_i) = 0$ for any $l \in \mathbb{N}$ and $1 \leq i \leq t$. The proof that $\operatorname{Hom}_{\widehat{B}}(P_i, \tau_{\widehat{B}}^{-l} N) = 0$ for any $l \in \mathbb{N}$ and $1 \leq i \leq t$ is similar. Hence, for $M^+ = N$, $M^- = M$, claim (i) follows.

For (ii), take a component $\mathcal{D}$ of $\Gamma_{B_q^-}$ contained in $\operatorname{mod}_- B_q^-$ and different from the preinjective component. Let $I_1, \ldots, I_t$ be the largest B_q^--submodules of the indecomposable projective-injective $\widehat{B}$-modules $P_1, \ldots, P_r$

of $\mathcal{R}_q$, respectively. Note that $I_1, \dots, I_t$ are indecomposable injective B_q^--modules lying in $\operatorname{mod}_- B_q^-$. The support algebra D'_q of $\mathcal{R}_q$, given by the objects of B_q^- and B_{q+1}^+, is an iterated one-point extension of B_q^- using modules whose largest B_q^--submodules are $I_1, \dots, I_t$. Moreover, under the identification $\underline{\operatorname{mod}}\,\widehat{B_q^-} = \mathrm{D^b}(\operatorname{coh}\mathbb{X})$, $\operatorname{mod} B_q^-$ is a full subcategory of $\operatorname{vect}\mathbb{X}[1]$, and therefore, for all modules U, V from $\operatorname{mod}_- B_q^-$, we have $\operatorname{Hom}(U, \tau_{\mathbb{X}}^{-r} V) = 0$ for $r \gg 0$, again by [24, Proposition 10.1].

Further, according to 2.3, the component $\mathcal{D}$ contains an indecomposable module Z such that the cone $(Z \to)$ of $\mathcal{D}$ is a full translation subquiver of a component of $\operatorname{vect}\mathbb{X}[1]$. Hence for any module W in $\mathcal{D}$ there is $l \in \mathbb{N}$ such that $\tau_{B_q^-}^{-l} W \in (Z \to)$, and consequently $\tau_{B_q^-}^{-s}(\tau_{B_q^-}^{-l} W) = \tau_{\mathbb{X}}^{-s}(\tau_{B_q^-}^{-l} W)$ for any $s \geq 0$. Combining the above information we conclude that there is an indecomposable B_q^--module Y in $\mathcal{D}$ such that $\mathcal{D}^+ = (Y \to)$ is a right stable subquiver of $\mathcal{D}$ (even contained in $\operatorname{vect}\mathbb{X}[1]$) such that $\operatorname{Hom}_{B_q^-}(I_j, X) = 0$ for any $X \in \mathcal{D}^+$ and any $1 \leq j \leq t$. Therefore, applying [33, (2.5.6)] we conclude that $\mathcal{D}^+$ is a full translation subquiver of a connected component $\mathcal{C}$ in $\mathcal{R}_q$.

The proof of (iii) is dual to the proof of (ii). ■

4.2. Let B be an almost concealed-canonical algebra of wild type and G an infinite cyclic group of K-linear automorphisms of $\widehat{B}$ such that $A = \widehat{B}/G$ is a proper selfinjective algebra of wild canonical type. Then Γ_A has a canonical decomposition

$$\Gamma_A = F_\lambda^B(\mathcal{T}_0) \vee F_\lambda^B(\mathcal{R}_0) \vee \dots \vee F_\lambda^B(\mathcal{T}_{m-1}) \vee F_\lambda^B(\mathcal{R}_{m-1})$$

with $m \geq 2$, induced by the decomposition of $\Gamma_{\widehat{B}}$ (see 3.4 and 3.8). We also note that then B_q^- and B_{q+1}^+, $0 \leq q \leq m-1$, with $B_0^- = B$ and $B_m^+ = B_0^+$, are factor algebras of A.

THEOREM. *For a fixed q, $0 \leq q \leq m-1$, the following statements hold*:

(i) *For each component $\mathcal{C}$ of $F_\lambda^B(\mathcal{R}_q)$, there exist indecomposable modules X^- and X^+ in $\mathcal{C}$ such that $\mathcal{C}^- = (X^- \to)$ is a right stable cone of $\mathcal{C}$ consisting of modules from* $\operatorname{mod}_- B_q^-$, *$\mathcal{C}^+ = (\to X^+)$ is a left stable cone of $\mathcal{C}$ consisting of modules from* $\operatorname{mod}_+ B_{q+1}^+$, *and for $U \in \mathcal{C}^-$ and $V \in \mathcal{C}^+$ we have $\tau_A^- U = \tau_{B_q^-}^- U$ and $\tau_A V = \tau_{B_{q+1}^+} V$.*

(ii) *For each component $\mathcal{D}$ of* $\operatorname{mod}_- B_q^-$ *different from the preinjective component, there exists an indecomposable module Y in $\mathcal{D}$ and a component $\mathcal{C}$ in $F_\lambda^B(\mathcal{R}_q)$ such that the cone $\mathcal{D}^- = (Y \to)$ of $\mathcal{D}$ is a right stable full translation subquiver of $\mathcal{C}$ which is closed under successors.*

(iii) *For each component* $\mathcal{E}$ *of* $\operatorname{mod}_+ B^+_{q+1}$ *different from the preprojective component, there exists an indecomposable module* Z *in* $\mathcal{E}$ *and a component* $\mathcal{C}$ *in* $F^B_\lambda(\mathcal{R}_q)$ *such that the cone* $\mathcal{E}^+ = (\to Z)$ *of* $\mathcal{E}$ *is a left stable full translation subquiver of* $\mathcal{C}$ *which is closed under predecessors.*

Proof. This follows from the above theorem and the facts that $F^B_\lambda : \operatorname{ind}\widehat{B} \to \operatorname{ind} A$ is a Galois covering (see 3.8) and B^-_q, B^+_{q+1}, $0 \le q \le m-1$, are factor algebras of A. ∎

4.3. Corollary. *For a fixed* q, $0 \le q \le m-1$, *the following statements hold.*

(i) *If* M *and* N *are indecomposable modules in* $F^B_\lambda(\mathcal{R}_q)$ *and* N *is nonprojective, then*

$$\operatorname{Hom}_A(M, \tau^r_A N) \neq 0 \quad \text{for } r \gg 0.$$

(ii) *If* M *and* N *are indecomposable modules in* $F^B_\lambda(\mathcal{R}_q)$, *then*

$$\operatorname{Hom}_A(M, \tau^{-r}_A N) = 0 \quad \text{for } r \gg 0.$$

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be components in $F^B_\lambda(\mathcal{R}_q)$ containing M and N, respectively. Observe that for any projective A-module P and an indecomposable nonprojective A-module X we have $\operatorname{Hom}_A(P, X) = \operatorname{Hom}_A(P/\operatorname{soc} P, X)$ and $P/\operatorname{soc} P$ is not projective. Therefore, we may assume that M is not projective.

(i) Assume N is not projective. Theorem 4.2(i) implies that there exist indecomposable A-modules $X^+_1 \in \mathcal{C}_1$ and $X^+_2 \in \mathcal{C}_2$ such that $\mathcal{C}^+_1 = (\to X^+_1)$ is a left stable cone of $\mathcal{C}_1$ consisting of modules from $\operatorname{mod}_+ B^+_{q+1}$, $\mathcal{C}^+_2 = (\to X^+_2)$ is a left stable cone of $\mathcal{C}_2$ consisting of modules from $\operatorname{mod}_+ B^+_{q+1}$, and $\tau_{B^+_{q+1}} V_1 = \tau_A V_1$, $\tau_{B^+_{q+1}} V_2 = \tau_A V_2$ for all modules $V_1 \in \mathcal{C}^+_1$ and $V_2 \in \mathcal{C}^+_2$. Moreover, by the dual to Theorem 3.4 of [28], there are indecomposable modules $Y_1 \in \mathcal{C}^+_1$ and $Y_2 \in \mathcal{C}^+_2$ such that $\mathcal{D}^+_1 = (\to Y_1)$ is a full translation subquiver of a component of $\operatorname{vect}\mathbb{X}$ and $\mathcal{D}^+_2 = (\to Y_2)$ is a full translation subquiver of a component of $\operatorname{vect}\mathbb{X}$. Clearly, there is a positive integer s such that $\tau^t_A M \in \mathcal{D}^+_1$ and $\tau^t_A N \in \mathcal{D}^+_2$ for all $t \ge s$. Applying now [24, Proposition 1.10] we obtain $\underline{\operatorname{Hom}}_A(M, \tau^r_A N) \cong \underline{\operatorname{Hom}}_A(\tau^s_A M, \tau^r_A(\tau^s_A N)) = \operatorname{Hom}(\tau^s_A M, \tau^r_{\mathbb{X}}(\tau^s_A N)) \neq 0$ for $r \gg 0$.

(ii) We may assume that N is not injective. It follows from Theorem 4.2(i) that there exists an indecomposable A-module $X^-_2 \in \mathcal{C}_2$ such that $\mathcal{C}^-_2 = (X^-_2 \to)$ is a right stable cone of $\mathcal{C}_2$ consisting of modules from $\operatorname{mod}_- B^-_q$, and $\tau^-_{B^-_q} U = \tau^-_A U$ for any module $U \in \mathcal{C}^-_2$. Applying again [28, Theorem 4.3] we conclude that there is an indecomposable module $Y_2 \in \mathcal{C}^-_2$ such that $\mathcal{D}^-_2 = (Y_2 \to)$ is a full translation subquiver of a component of $\operatorname{vect}\mathbb{X}[1]$, and $\tau^-_A W = \tau^-_{B^-_q} W = \tau^-_{\mathbb{X}} W$ for any module $W \in \mathcal{D}^-_2$. Take a positive integer s such that $\tau_A^{-t} W \in \mathcal{D}_2^-$ for all $t \geq s$. Invoking [24, Proposition 1.10] again, we infer that $\operatorname{Hom}_A(M, \tau_A^{-t}(\tau_A^{-s} N)) = \operatorname{Hom}(M, \tau_{\mathbb{X}}^{-t}(\tau_A^{-s} N)) = 0$ for $t \gg 0$. ∎

5. Distribution of simple and projective modules

5.1. In this section we are interested in the distribution of simple and projective modules in the Auslander–Reiten components of selfinjective algebras of wild canonical type. We shall use the notation introduced in Section 3. For an almost concealed-canonical algebra B and an admissible infinite cyclic group G of K-linear automorphisms of $\widehat{B}$, the Auslander–Reiten quiver Γ_A of the selfinjective algebra $A = \widehat{B}/G$ is the orbit quiver $\Gamma_{\widehat{B}}/G$ of the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$, and therefore it is sufficient to investigate the distribution of simple and projective modules in the components of $\Gamma_{\widehat{B}}$.

Proposition. *Let B be an almost concealed-canonical algebra. Then*:

(i) *For each $q \in \mathbb{Z}$, $\mathcal{R}_q$ contains at least one simple and at least one projective module.*

(ii) *$\mathcal{T}_q$ contains a simple module if and only if $\mathcal{T}_{q-1}$ and $\mathcal{T}_{q+1}$ contain projective modules.*

(iii) *$\mathcal{T}_q$ contains a projective module if and only if $\mathcal{T}_{q-1}$ and $\mathcal{T}_{q+1}$ contain simple modules.*

Proof. Since the trivial extension $\widehat{B}/(\nu_{\widehat{B}}) = B \ltimes D(B) = A$ is symmetric, $\tau_A = \Omega_A^2$, and for the push-down functor $F_\lambda^B : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ associated to the Galois covering $F^B : \widehat{B} \to \widehat{B}/(\nu_{\widehat{B}})$ we have $\Omega_A F_\lambda^B \cong F_\lambda^B \Omega_{\widehat{B}}$, $\Omega_A^- F_\lambda^B \cong F_\lambda^B \Omega_{\widehat{B}}^-$, we deduce that $\Omega_{\widehat{B}}(\mathcal{T}_q^{\mathrm{s}}) = \mathcal{T}_{q-1}^{\mathrm{s}}$, $\Omega_{\widehat{B}}(\mathcal{R}_q^{\mathrm{s}}) = \mathcal{R}_{q-1}^{\mathrm{s}}$, $\Omega_{\widehat{B}}^-(\mathcal{T}_q^{\mathrm{s}}) = \mathcal{T}_{q+1}^{\mathrm{s}}$, and $\Omega_{\widehat{B}}^{\mathrm{s}}(\mathcal{R}_{q+1}^{\mathrm{s}})$ for any $q \in \mathbb{Z}$. Moreover, for any indecomposable projective $\widehat{B}$-module P we have an Auslander–Reiten sequence

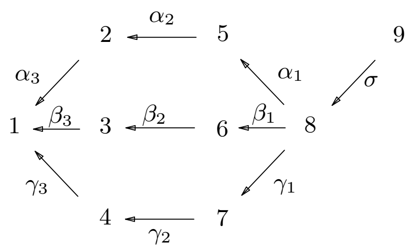
$$0 \to \operatorname{rad} P \to P \oplus \operatorname{rad} P/\operatorname{soc} P \to P/\operatorname{soc} P \to 0$$

and $\Omega_{\widehat{B}}(P/\operatorname{soc} P) = \operatorname{soc} P$, $\Omega_{\widehat{B}}^-(\operatorname{rad} P) = \operatorname{top} P$. Then (ii) and (iii) follow.

For (i) it is enough to show that each $\mathcal{R}_q$ contains a projective module. Suppose there exists $p \in \mathbb{Z}$ such that $\mathcal{R}_p$ does not contain a projective module, and so $\mathcal{R}_p$ consists of regular components of type $\mathbb{Z}\mathbb{A}_\infty$. Then it follows from 3.4 that all components of $\operatorname{mod}_- B_q^-$ different from the preinjective component are regular. Therefore, all injective modules of the almost concealed-canonical algebra B_q^- lie in its unique preinjective component. But then B_p^- is a tilted algebra $\operatorname{End}_H(T)$ where H is a hereditary algebra and T is a tilting H-module without nonzero preinjective direct summands. This leads to a contradiction because B_p^- is of wild canonical type. This finishes the proof. ■

5.2. Let $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ be a canonical algebra of wild type; we identify Λ with the convex subcategory Λ_0^- of $\widehat{\Lambda}$. Then we have $\Lambda_0^+ = \Lambda = \Lambda_0^-$. Let n be the rank of $\mathrm{K}_0(\Lambda)$. Then it follows that each $\mathcal{T}_{2m}$, $m \in \mathbb{Z}$, does not contain a projective module but contains exactly $n-2$ simple modules, and consequently each of $\mathcal{T}_{2m+1}$, $m \in \mathbb{Z}$, does not contain simple modules but contains $n-2$ projective modules. Moreover, each of the families $\mathcal{R}_q$, $q \in \mathbb{Z}$, contains exactly one simple and one projective module.

5.3. We shall now exhibit an almost concealed-canonical algebra B such that all parts $\mathcal{T}_q$ and $\mathcal{R}_q$ of $\Gamma_{\widehat{B}}$ contain both a simple and a projective module. Consider the algebra B given by the quiver



bound by $\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 + \gamma_1\gamma_2\gamma_3 = 0$ and $\sigma\gamma_1 = 0$. Then B is the one-point extension of the canonical algebra Λ of tubular type $(3,3,3)$, given by the vertices $1, 2, \ldots, 8$, by an indecomposable Λ-module lying on the mouth of a stable tube of rank 3, and consequently B is almost concealed-canonical of wild type $(3,3,4)$. Identifying B with B_0^- inside $\widehat{B}$, we conclude that the family $\mathcal{T}_0$ contains exactly one projective $\widehat{B}$-module, namely $P_{\widehat{B}}(9)$. Hence, each of the families $\mathcal{T}_{2m}$, $m \in \mathbb{Z}$, contains exactly one projective module, and consequently each of the families $\mathcal{T}_{2m+1}$, $m \in \mathbb{Z}$, contains exactly one simple module. Moreover, $P_{\widehat{B}}(8)$ lies in $\mathcal{R}_0$, $P_{\widehat{B}}(1)$ lies in $\mathcal{R}_{-1}$, and $P_{\widehat{B}}(2)$, $P_{\widehat{B}}(3), \ldots, P_{\widehat{B}}(7)$ lie in $\mathcal{T}_{-1}$. Clearly, all parts $\mathcal{T}_q$ and $\mathcal{R}_q$, $q \in \mathbb{Z}$, of $\Gamma_{\widehat{B}}$ contain both simple and projective modules.

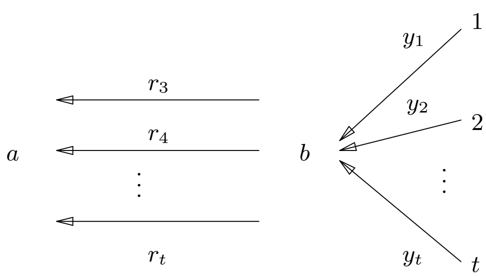
5.4. We now discuss the existence of almost concealed-canonical algebras of wild type such that all families $\mathcal{T}_q$, $q \in \mathbb{Z}$, of $\Gamma_{\widehat{B}}$ are without projective (equivalently, simple) modules. Obviously, this never happens for an almost concealed-canonical but non-concealed-canonical algebra. An equivalent problem is to find a concealed-canonical algebra of wild type whose unique family of stable tubes does not contain a simple module. The following facts proved in [20, Theorem 3, Corollary 4] show that we have plenty of such algebras.

THEOREM. *Let $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ be a wild canonical algebra and m a positive integer. Then there exist infinitely many pairwise nonisomorphic connected* *wild hereditary algebras C and quasi-simple regular C-modules M such that the one-point extensions $C[M]$ are concealed-canonical algebras of type Λ whose family $\mathcal{T}$ of stable tubes has this property: for any indecomposable module X in $\mathcal{T}$, each simple $C[M]$-module occurs with multiplicity at least m as a composition factor of X.*

5.5. Corollary. *For each wild canonical algebra $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ there exist infinitely many pairwise nonisomorphic concealed-canonical algebras of type Λ without simple modules in the tubes.*

Invoking 3.5 we conclude that for each wild canonical algebra $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ there exist infinitely many pairwise nonisomorphic repetitive algebras $\widehat{B}$ of canonical type Λ without simple modules in the families $\mathcal{T}_q$, $q \in \mathbb{Z}$, of $\Gamma_{\widehat{B}}$.

5.6. We now present a concrete example of a concealed-canonical algebra of wild type without simple modules in the tubes. Let $t \geq 3$ and let $1 = \lambda_3, \lambda_4, \ldots, \lambda_t$ be pairwise distinct nonzero elements of k. Then the algebra B given by the quiver



and bound by the relations

$$\begin{aligned} y_j r_i &= 0 && \text{for } j \neq 1, 2, i, \\ y_1 r_i &= \lambda_i y_1\, r_1 && \text{for } i = 3, \ldots, t, \\ y_2 r_i &= y_2\, r_j && \text{for } i, j = 3, \ldots, t, \end{aligned}$$

is concealed-canonical. In more detail, B is isomorphic to the endomorphism algebra of a tilting bundle T on the weighted projective line $\mathbb{X} = \mathbb{X}(2, 2, \ldots, 2; \underline{\lambda})$ with t points of weight two and the parameter sequence $\underline{\lambda} = (\lambda_3, \ldots, \lambda_t)$. Moreover, the ranks of the simple B-modules $S(a)$, $S(b)$, $S(1), \ldots, S(t)$ are given by the sequence $1, 1, -1, \ldots, -1$. In particular, $\widehat{B}$ has a family of standard stable tubes.

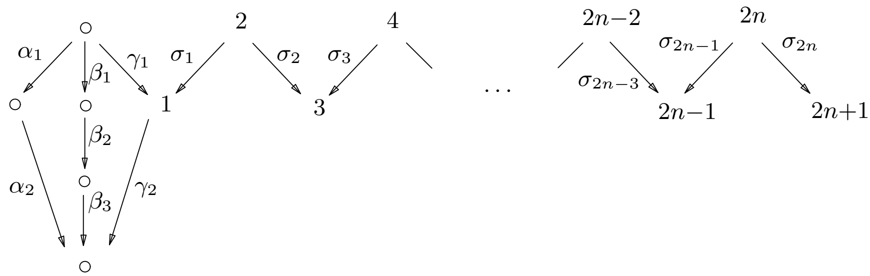
With the notations from [14] or [24] we define K as the kernel term of the exact sequence

$$0 \to K \xrightarrow{[y_1,\ldots,y_t]^{\mathrm{tr}}} \bigoplus_{i=1}^{t} \mathcal{O}(\vec{x}_i) \xrightarrow{[x_1,\ldots,x_t]} \mathcal{O}(\vec{c}) \to 0$$

in coh $\mathbb{X}$. Then it is not difficult to check that K is an exceptional bundle, and moreover the direct sum of $\mathcal{O}$, K, $\mathcal{O}(\vec{x}_1), \dots, \mathcal{O}(\vec{x}_t)$ is a tilting bundle on $\mathbb{X}$ with endomorphism ring isomorphic to B. The remaining claims now follow easily.

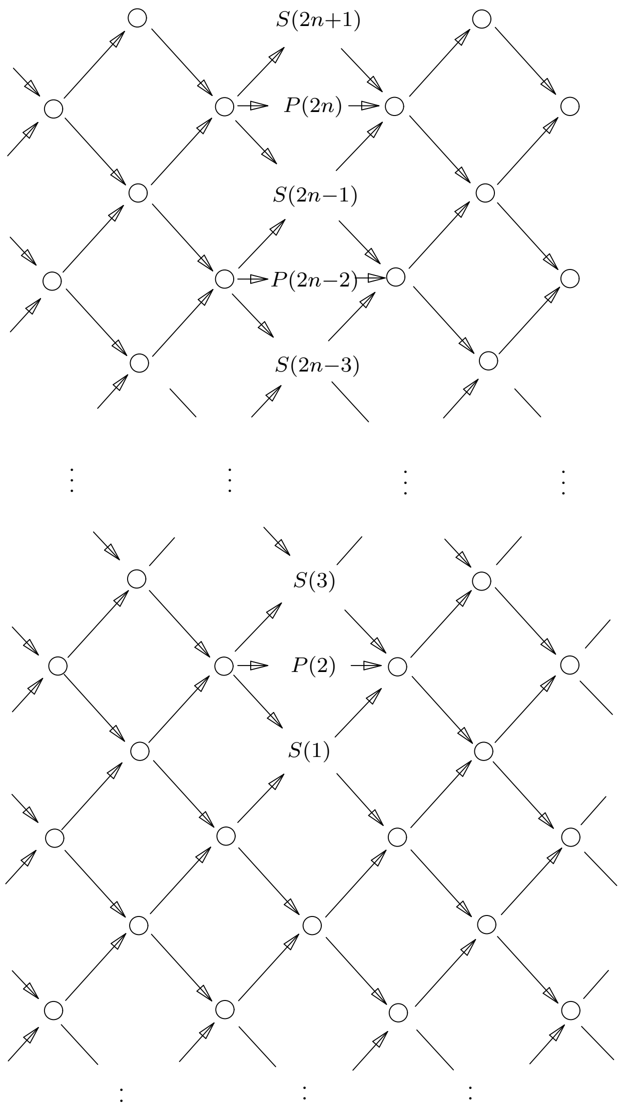
For the construction of further examples of this kind, a special type of reflections, introduced by Hübner [18] for the class of concealed-canonical algebras, is very useful. These reflections allow one, for instance, to construct such examples for each canonical algebra which is minimal wild. In particular, the preceding example is obtained from the canonical algebra by just one Hübner reflection.

5.7. Observe that for a canonical algebra Λ of wild type, the simple periodic Λ-modules lie on the mouth of the stable tubes. This implies that for any indecomposable projective $\widehat{B}$-module P with $\operatorname{rad} P$ (equivalently, $P/\operatorname{soc} P$) periodic, $\operatorname{rad} P$ and $P/\operatorname{soc} P$ lie on the mouth of a stable tube of $\Gamma^{\mathrm{s}}_{\widehat{\Lambda}}$. We shall now exhibit almost concealed-canonical algebras B for which there are simple periodic modules in $\widehat{B}$ of large stable quasi-length, that is, lying far from the mouth of the stable tubes of $\Gamma^{\mathrm{s}}_{\widehat{B}}$. Let R_n, $n \geq 0$, be the algebra given by the quiver



bound by $\alpha_1\alpha_2 + \beta_1\beta_2\beta_3 + \gamma_1\gamma_2 = 0$ and $\sigma_1\gamma_2 = 0$. Then, for $n \geq 3$, R_n is an almost concealed-canonical algebra of wild type $(2, 3, 2n + 2)$, being the corresponding tubular extension of the canonical algebra R_0. Fix $n \geq 3$ and put $B = R_n$. Then, identifying B with B_0^- inside $\widehat{B}$, we conclude that the family $\mathcal{T}_0$ contains a quasi-tube depicted on the next page and hence $\operatorname{sql}(S(2r + 1)) = (2n + 2) - (2r + 1)$ and $\operatorname{sql}(\operatorname{rad} P(2r)) = (2n + 2) - 2r$ for $r = 0, 1, \dots, n$. In particular, $\operatorname{sql} S(1) = 2n + 1$ and $\operatorname{sql}(\operatorname{rad} P(2)) = 2n$. We also note that $\mathrm{K}_0(B)$ is of rank $2n + 6$.

5.8. Finally, we shall discuss the distribution of simple modules and projective modules in the components of $\Gamma_{\widehat{B}}$ with stable part $\mathbb{Z}\mathbb{A}_\infty$. In order to state the result we need an invariant introduced in [24, Section 10]. We say that B, or the corresponding canonical algebra $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$, $\boldsymbol{p} = (p_1, \dots, p_t)$, has *Dynkin label* $\Delta \in \{\mathbb{D}_4, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ if the extended Dynkin diagram $\widetilde{\Delta}$ is



a subtree of the star $[p_1, \dots, p_t]$ with t arms of length p_i $(1 \le i \le t)$, and moreover, the number of vertices of Δ is minimal. For instance, the weight types $(2,2,2,3)$, $(3,3,4)$, $(2,4,5)$ and $(2,3,7)$ lead to Dynkin labels $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$, respectively.

PROPOSITION. *Let A be a selfinjective algebra of wild canonical type $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$, S a simple A-module lying in a component of Γ_A whose stable part is $\mathbb{Z}\mathbb{A}_\infty$, and P the projective cover of S. Then* $\mathrm{sql}(S) = \mathrm{sql}(\mathrm{rad}\, P)$ *is bounded by* $2, 3, 4$ *or* 6 *according as the Dynkin label of A (or Λ) equals* $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_7$ *or* $\mathbb{E}_8$, *respectively.*

Proof. Let $n = \mathrm{sql}(S)$. Note that S corresponds to an object from a $\mathbb{Z}\mathbb{A}_\infty$-component $\mathcal{C}$ of $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ having quasi-length n and trivial endomorphism ring. An argument of [40] yields an exceptional object in $\mathcal{C}$ of quasi-length $n-1$. The claim now follows from [24, Corollary 10.5]. ∎

We note that for any positive integer $n \geq 3$ there exists a selfinjective algebra A of wild tilted type whose Auslander–Reiten quiver Γ_A admits a component $\mathcal{C}$ with stable part $\mathbb{Z}\mathbb{A}_\infty$ and containing a simple module of stable quasi-length $n-2$ (see [12, 5.7]).

6. Growth numbers of modules

6.1. Let A be an algebra and $P_1, \ldots, P_n$ a complete set of pairwise nonisomorphic indecomposable projective A-modules. The *Cartan matrix* C_A of A is the $n \times n$ integral matrix whose (i,j)-entry is $\dim_K \mathrm{Hom}_A(P_i, P_j)$. Assume now that $\mathrm{gl.dim}\,A < \infty$. Then C_A is invertible over $\mathbb{Z}$ and C_A^{-t} defines a bilinear form $\langle -,- \rangle_A$ on $\mathrm{K}_0(A)$ given by $\langle \underline{x}, \underline{y} \rangle = \underline{x} C_A^{-t} \underline{y}^t$ for $\underline{x}, \underline{y} \in \mathrm{K}_0(A)$. Then we have an integral quadratic form χ_A on $\mathrm{K}_0(A)$, called the *Euler form* of A, given by $\chi_A(\underline{x}) = \langle \underline{x}, \underline{x} \rangle_A$ for any $\underline{x} \in \mathrm{K}_0(A)$. The bilinear form $\langle -,- \rangle_A$ has a well-known homological interpretation [33]:

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle = \sum_{i \geq 0} (-1)^i \dim_K \mathrm{Ext}_A^i(X, Y)$$

for all modules X and Y from $\mathrm{mod}\,A$. The matrix $\Phi_A = -C_A^{-t} C_A$ is called the *Coxeter matrix* of A. If $I_1, \ldots, I_n$ are the injective envelopes of the tops $S_1, \ldots, S_n$ of the projective modules $P_1, \ldots, P_n$, respectively, then $(\underline{\dim}\, P_i)\Phi_A = -\underline{\dim}\, I_i$ for any $1 \leq i \leq n$. The characteristic polynomial $P_A(T) = \det(\Phi_A - TI)$ of A is called the *Coxeter polynomial* of A. The roots of $P_A(T)$ form the set $\mathrm{Spec}(\Phi_A)$ of eigenvalues of Φ_A, called the *spectrum* of Φ_A. Moreover, $\varrho_A = \max\{|\lambda| \mid \lambda \in \mathrm{Spec}(\Phi_A)\}$ is called the *spectral radius* of A. The *radical* $\mathrm{rad}\,\chi_A$ of χ_A is equal to $\{x \in \mathrm{K}_0(A) \mid \underline{x}\Phi_A = \underline{x}\}$ (see [33]).

6.2. Let $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ be a wild canonical algebra with the weight sequence $\boldsymbol{p} = (p_1, \ldots, p_t)$ and parameter sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$. Then it is well known [24] that the Coxeter polynomial of Λ is of the form

$$P_\Lambda(T) = (T-1)^2 \prod_{i=1}^{t} \frac{T^{p_i} - 1}{T - 1},$$

and consequently $\varrho_\Lambda = 1$. Moreover, the radical of χ_Λ has rank one.

6.3. For an algebra A and an indecomposable A-module M we may define the *left growth number* $\varrho_A^+(M) = \limsup_{n\to\infty} \sqrt[n]{\dim_K \tau_A^n M}$ and similarly the *right growth number* $\varrho_A^-(M) = \limsup_{n\to\infty} \sqrt{\dim_K \tau^{-n} M}$. If

$\varrho_A^-(M) = \varrho_A^+(M)$ we denote this number by $\varrho_A(M)$ and call it the *growth number* of the module M. Observe that if M is τ_A-periodic then $\varrho_A(M) = 1$. We have the following facts on the growth number of indecomposable modules over wild canonical algebras, proved in [24, Theorem 6.1].

THEOREM. *Let $\Lambda = \Lambda(\boldsymbol{p}, \underline{\lambda})$ be a wild canonical algebra and Λ_0 be the path algebra of the wild star obtained by removing the unique source from the quiver Q_Λ. Then*:

(i) *For each stable nonperiodic indecomposable Λ-module M from* $\operatorname{mod}_+ \Lambda$, *there exist positive integers a_M^+ and a_M^- such that*

$$\lim_{n\to\infty} \frac{\dim_K \tau_\Lambda^n M}{n} = a_M^+, \qquad \lim_{n\to\infty} \frac{\dim_K \tau_\Lambda^{-n} M}{\varrho_{\Lambda_0}^n} = a_M^-.$$

In particular, $\varrho_\Lambda^+(M) = 1 < \varrho_{\Lambda_0} = \varrho_\Lambda^-(M)$.

(ii) *For each stable nonperiodic indecomposable Λ-module M from* $\operatorname{mod}_- \Lambda$, *there exist positive integers b_M^+ and b_M^- such that*

$$\lim_{n\to\infty} \frac{\dim_K \tau_\Lambda^n M}{\varrho_{\Lambda_0}^n} = b_M^+, \qquad \lim_{n\to\infty} \frac{\dim_K \tau_\Lambda^{-n} M}{n} = b_M^-.$$

In particular, $\varrho_\Lambda^+(M) = \varrho_{\Lambda_0} > 1 = \varrho_\Lambda^-(M)$.

6.4. It follows from the above theorem that for any stable nonperiodic indecomposable module M over a wild canonical algebra Λ, the growth number does not exist. The following theorem shows that the situation is different for selfinjective algebras of wild canonical type.

THEOREM. *Let A be a selfinjective algebra of wild canonical type and let M be a nonprojective indecomposable A-module. Then $\varrho_A(M) = 1$.*

Proof. Let $A = \widehat{B}/G$ for an almost concealed-canonical algebra B of wild type and G a torsion-free admissible group of K-linear automorphisms of $\widehat{B}$. Since the push-down functor $F_\lambda^B : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}/G = \operatorname{mod} A$ is dense and preserves projective modules, we have $M = F_\lambda^B(N)$ for some nonprojective indecomposable $\widehat{B}$-module N. Moreover, $F_\lambda^B(\tau_{\widehat{B}} X) \cong \tau_A F_\lambda^B(X)$, $F_\lambda^B(\tau_{\widehat{B}}^- X) = \tau_A^- F_\lambda^B(X)$, and $\dim_K F_\lambda^B(X) = \dim_K X$ for any indecomposable $\widehat{B}$-module X. Therefore, it is enough to prove that

$$\limsup_{n\to\infty} \sqrt[n]{\dim_K \tau_{\widehat{B}}^n N} = 1 = \limsup_{n\to\infty} \sqrt[n]{\dim_K \tau_{\widehat{B}}^{-n} N}.$$

Clearly, we may assume that N is nonperiodic, say N belongs to a component $\mathcal{C}$ of $\mathcal{R}_q$ for some $q \in \mathbb{Z}$.

It follows from Theorem 4.1 that there exists a positive integer s such that the cone $\mathcal{C}^+ = (\to \tau_{\widehat{B}}^s N)$ of $\mathcal{C}$ is left stable and consists of modules from $\operatorname{mod} B_{q+1}^{+}$, the cone $\mathcal{C}^{-} = (\tau_{\widehat{B}}^{-s} N \to)$ of $\mathcal{C}$ is right stable and consists of modules from $\operatorname{mod}_{-} B_{q}^{-}$, and $\tau_A U = \tau_{B_{q+1}^{+}} U$, $\tau_A^{-} V = \tau_{B_q^{-}}^{-} V$ for all modules $U \in \mathcal{C}^{+}$ and $V \in \mathcal{C}^{-}$. Further, there exist indecomposable modules $Y \in \mathcal{C}^{+}$ and $Z \in \mathcal{C}^{-}$ such that the left stable cone $\mathcal{D}^{+} = (\to Y)$ of $\mathcal{C}^{+}$ is a full translation subquiver of a component of $\operatorname{vect} \mathbb{X}$, the right stable cone $\mathcal{D}^{-} = (Z \to)$ of $\mathcal{C}$ is a full translation subquiver of a component of $\operatorname{vect} \mathbb{X}[1]$, and $\tau_A W = \tau_{\mathbb{X}} W$, $\tau_A^{-} X = \tau_{\mathbb{X}}^{-} X$ for any $W \in \mathcal{D}^{+}$ and $X \in \mathcal{D}^{-}$, where $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \underline{\lambda})$ is the corresponding weighted projective line. Finally, applying [24, Theorem 6.3] and its dual, we conclude that there exist indecomposable modules $M^{+} = \tau_{\widehat{B}}^{r} N \in \mathcal{D}^{+}$ and $M^{-} = \tau_{\widehat{B}}^{-r} N \in \mathcal{D}^{-}$, for some positive integer $r \geq s$, such that the left stable cone $\mathcal{E}^{+} = (\to M^{+})$ is a full translation subquiver of a component of $\operatorname{mod}_{+} \Lambda(\boldsymbol{p}, \underline{\lambda})$, the right stable cone $\mathcal{E}^{-} = (M^{-} \to)$ is a full translation subquiver of a component of $\operatorname{mod}_{-} \Lambda(\boldsymbol{p}, \underline{\lambda})$, and $\tau_A V = \tau_{\Lambda(\boldsymbol{p},\underline{\lambda})} V$, $\tau_A^{-} U = \tau_{\Lambda(\boldsymbol{p},\underline{\lambda})}^{-} U$ for any $V \in \mathcal{E}^{+}$ and $U \in \mathcal{E}^{-}$. Therefore, it follows from 6.3 that there is a positive integer a such that $\dim_K \tau_A^{m} N \leq am$ and $\dim_K \tau_A^{-m} N \leq am$ for $m \gg 0$. Clearly, this implies $\varrho_A^{+}(N) = 1 = \varrho_A^{-}(N)$, and hence $\varrho_A(M) = 1$. ■

We note that for any indecomposable nonprojective module M over a selfinjective algebra A of wild tilted type Δ we have $\varrho_A(M) = \varrho_H > 1$, where ϱ_H is the spectral radius of $H = K\Delta$ (see [12, Theorem 7.3]).

7. Complexity of modules and Ext-algebras. The aim of this section is to discuss the complexity of indecomposable modules and the Ext-algebras of indecomposable modules over symmetric algebras of wild canonical type.

7.1. We say that an $\mathbb{N}$-graded K-vector space $V = \bigoplus_{n \in \mathbb{N}} V_n$ has *polynomial growth* if there are a nonnegative integer c and a nonzero constant μ such that $\dim_K V_n \leq \mu n^{c-1}$ for $n \gg 0$. If these exist, then the smallest such c is denoted by $\gamma(V)$ and called the *rate of growth* of V. If V is not of polynomial growth we set $\gamma(V) = \infty$.

7.2. Let A be an algebra, M an A-module, and consider a minimal projective resolution

$$(*) \qquad \ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

of M in $\operatorname{mod} A$. If $\gamma(\bigoplus_{n \in \mathbb{N}} P_n) < \infty$ then following [1] we set $c_A(M) = \gamma(\bigoplus_{n \in \mathbb{N}} P_n)$ and call it the *complexity* of M.

7.3. Let A be a selfinjective algebra, M an A-module, and let

$$\operatorname{Ext}_A^{*}(M, M) = \bigoplus_{n \in \mathbb{N}} \operatorname{Ext}_A^{n}(M, M)$$

be the Ext-algebra of M endowed with the Yoneda multiplication. Assume $c_A(M)$ exists. Then $\gamma(\mathrm{Ext}_A^*(M,M))$ exists as well and is bounded by $c_A(M)$ (see [12, 9.3]).

Indeed, observe that $\mathrm{Ext}_A^n(M,M) \cong \underline{\mathrm{Hom}}_A(\Omega_A^n(M),M)$. Applying $\mathrm{Hom}_A(-,M)$ to the minimal projective resolution $(*)$ of M we get the inequalities

$$\dim_K \mathrm{Ext}_A^n(M,M) \le \dim_K \underline{\mathrm{Hom}}_A(\Omega_A^n(M),M) \le \dim_K \mathrm{Hom}_A(P_n,M).$$

Further,

$$\dim_K \mathrm{Hom}_A(P_n,M) \le (\dim_K P_n)(\dim_K M) \le (\dim_K M)\mu n^{c-1}$$

for $c = c_A(M)$, a constant μ, and $n \gg 0$, and hence $\gamma(\mathrm{Ext}_A^*(M,M)) \le c_A(M)$. Note also that if M is an A-module with $\Omega_A^r M \cong M$ for some $r \ge 1$, then $c_A(M) = 1$. Obviously, $c_A(M) = 0$ if and only if M is projective.

Theorem. *Let A be a symmetric algebra of wild canonical type, and M an indecomposable A-module. Then $\gamma(\mathrm{Ext}_A^*(M,M)) = c_A(M) \le 2$. Moreover, $c_A(M) = 2$ if M is nonperiodic and nonprojective.*

Proof. It follows from the above remarks that $\gamma(\mathrm{Ext}_A^*(M,M)) \le c_A(M)$. Moreover, it is clear that $\gamma(\mathrm{Ext}_A^*(M,M)) = c_A(M) = 0$ if M is projective and $\gamma(\mathrm{Ext}_A^*(M,M)) = c_A(M) = 1$ if M is periodic. Assume M is nonperiodic and nonprojective. We shall prove that $c_A(M) \le 2$. Since A is symmetric we have $\tau_A = \Omega_A^2$. Consider a minimal projective resolution

$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

of M in $\mathrm{mod}\, A$. Then $\dim_K P_n = \dim_K \Omega_A^n M + \dim_K \Omega_A^{n+1} M$. Put $N = \Omega_A^n M$. For $n = 2m$, we get $\dim_K P_{2m+1} = \dim_K \tau_A^m M + \dim_K \tau_A^{m+1} N$, and for $n = 2m+1$, $\dim_K P_{2m+1} = \dim_K \tau_A^m N + \dim_K \tau_A^m M$. Further, it follows from 6.4 and its proof that there is a positive integer a such that $\dim_K \tau_A^m M \le am$ and $\dim_K \tau_A^m N \le am$ for $m \gg 0$. This implies that $c_A(M) \le 2$. It remains to show that $\gamma(\mathrm{Ext}_A^*(M,M)) > 1$. Observe that

$$\begin{aligned}\mathrm{Ext}_A^{2m+1}(M,M) &\cong \mathrm{Ext}_A^1(\Omega_A^{2m},M) \cong \mathrm{Ext}_A^1(\tau_A^m M,M)\\ &\cong D\,\underline{\mathrm{Hom}}_A(M,\tau_A^{m+1} M)\end{aligned}$$

and $\dim_K D\,\underline{\mathrm{Hom}}_A(M,\tau_A^{m+1}M) = \dim_K \underline{\mathrm{Hom}}(M,\tau_A^{m+1}M)$ is unbounded by [24, Theorem 4.1]. Therefore, $\gamma(\mathrm{Ext}_A^*(M,M)) > 1$, and this finishes the proof. ■

We note that if A is a symmetric algebra of wild tilted type and M a nonprojective indecomposable A-module then the complexity $c_A(M)$ does not exist (see [12, Theorem 9.4]).

7.4. We are now interested in the structure of algebras $\mathrm{Ext}_A^*(M,M)$ for indecomposable modules over a symmetric algebra A of wild canonical type.

Let A be a selfinjective algebra and M an indecomposable nonprojective A-module. For each $n \geq 1$ we have isomorphisms

$$\underline{\mathrm{Hom}}_A(\Omega_A^n M, M) \cong \mathrm{Ext}_A^n(M, M) \cong \underline{\mathrm{Hom}}_A(M, \Omega_A^{-n} M).$$

We shall also consider the subalgebra

$$\mathrm{Ext}_A^{\mathrm{ev}}(M, M) = \bigoplus_{m \in \mathbb{N}} \mathrm{Ext}_A^{2m}(M, M)$$

of $\mathrm{Ext}_A^*(M, M)$ called the even part of $\mathrm{Ext}_A^*(M, M)$. We also recall from 3.12 that the class of proper symmetric algebras of wild canonical type coincides with the class of trivial extensions $B \ltimes D(B)$ of almost concealed-canonical algebras of wild type.

THEOREM. *Let A be a proper symmetric algebra of wild canonical type, and M an indecomposable nonprojective nonperiodic A-module. Then*:

(i) $\mathrm{Ext}_A^{\mathrm{ev}}(M, M)$ *is a finite-dimensional K-algebra.*
(ii) $\mathrm{Ext}_A^{2m+1}(M, M) \cdot \mathrm{Ext}_A^{2r+1}(M, M) = 0$ *for $m, r \gg 0$.*

In particular, $\mathrm{rad}\,\mathrm{Ext}_A^*(M, M)$ *is nilpotent.*

Proof. It follows from Corollary 4.3 that

$$\mathrm{Ext}_A^{2m}(M, M) \cong \underline{\mathrm{Hom}}_A(M, \Omega_A^{-2m} M) = \underline{\mathrm{Hom}}_A(M, \tau_A^{-m} M) = 0$$

for $m \gg 0$, and then the claim follows. ■

Acknowledgements. The research was initiated during the visit of the second named author at Paderborn University. He gratefully acknowledges support from the Alexander von Humboldt Foundation, Foundation for Polish Science and the Polish Scientific Grant KBN No. 5 P03A 008 21.

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Received 5 February 2003 (4310)