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A-RINGS

BY

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Abstract. A ring R is called an E-ring if every endomorphism of R^+, the additive group of R, is multiplication on the left by an element of R. This is a well known notion in the theory of abelian groups. We want to change the "E" as in endomorphisms to an "A" as in automorphisms: We define a ring to be an A-ring if every automorphism of R^+ is multiplication on the left by some element of R. We show that many torsion-free finite rank (tffr) A-rings are actually E-rings. While we have an example of a mixed A-ring that is not an E-ring, it is still open if there are any tffr A-rings that are not E-rings. We will employ the Strong Black Box [5] to construct large integral domains that are A-rings but not E-rings.

0. Introduction. Let R be a ring and R^+ the additive group of R. For $a \in R$ define $a_l, a_r \in \mathrm{End}(R^+)$ by $a_l(x) = ax$ and $a_r(x) = xa$ for all $x \in R$. Let $R_l = \{a_l : a \in R\}$ and $R_r = \{a_r : a \in R\}$. Then R is called an *E-ring* if $\mathrm{End}(R^+) = R_l$. This notion of an E-ring is well known in abelian group theory, and we refer the reader to the survey article [8] for more information about E-rings and related literature. [1, Section 14] is a good source for E-rings R such that R^+ is torsion-free of finite rank, or *tffr* for short.

We want to modify the definition of E-rings by considering only the group $\mathrm{Aut}(R^+)$ of automorphisms of R^+ instead of the entire ring of endomorphisms of R^+.

DEFINITION. The ring R is called an *A-ring* if $\mathrm{Aut}(R^+) \subseteq R_l$. If R^+ has elements of order 2, we require $1 \in R$. (We will show $1 \in R$ if R^+ has no elements of order 2.) If an abelian group G is the additive group of some A-ring, then we call G an *A-group*.

We will see that each A-ring R has a unity $1 \in R$ and $\mathrm{Aut}(R^+) = (U(R))_l$, where $U(R)$ is the group of units of R. Moreover, we will show that $U(R) \subseteq Z(R)$, the center of R, for any A-ring R, which implies that $U(R) \approx \mathrm{Aut}(R^+)$ is commutative. In Section 1 we obtain some information on abelian groups G with $\mathrm{Aut}(G)$ commutative.

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In Section 2 we concentrate on tffr A-rings. While we have an example of a mixed A-ring that is not an E-ring, it is open if there are any tffr A-rings that are not E-rings. Our results seem to indicate that there are no such rings. We will show that each strongly indecomposable tffr A-ring is indeed an E-ring and any tffr A-ring of rank two is an E-ring. We define the notion of a strong A-ring and show that all tffr strong A-rings are E-rings. The key to this result is that quasi-summands of strong A-rings are again strong A-rings, a property that seems to be elusive for A-rings in general.

In the last section we give a construction of commutative, torsion-free A-rings of large ($> 2^{\aleph_0}$) cardinality. We start with a suitable integral domain S and construct a commutative S-algebra R such that $\mathrm{Aut}(R^+) = (U(S))_l$ but $\mathrm{End}(R^+) = R_l[\gamma] \approx R[x]$, the polynomial ring in a single variable over R. This shows that R is not an E-ring. We will use the Strong Black Box, as developed in [5], in our construction. This prediction principle is also used in [4], and the forthcoming monograph [7] will contain a detailed description of the Strong Black Box.

1. Preliminaries. Recall that a ring R is called an E-ring if $\mathrm{End}(R^+) = R_l$, as defined in the introduction. We will show that every E-ring has a unity $1 \in R$. Usually, the definition of E-rings states "$1 \in R$", which is not needed:

Since $\mathrm{id}_R \in \mathrm{End}(R^+)$, there is some $e \in R$ such that $e_l = \mathrm{id}_R$. Thus $ex = x$ for all $x \in R$ and we have $e^2 = e$. Since $e_r \in \mathrm{End}(R^+)$, there is some $a \in R$ such that $e_r = a_l$. Thus $xe = ax$ for all $x \in R$, and $e = ee = ae$ (for $x = e$) and $ae = a^2$ (for $x = a$), which implies $e = a^2$. Thus $xe = (xe)e = (ax)e = a(xe) = a(ax) = a^2x = ex = x$ for all $x \in R$. This shows that $1 = e \in R$.

Now assume that R is an A-ring as defined in the introduction. We prove that R has an identity if R has no elements of additive order 2:

Since $\mathrm{id}_R \in \mathrm{Aut}(R^+)$, there is some $e \in R$ such that $\mathrm{id}_R = e_l$ and thus $ex = x$ for all $x \in R$ and $e^2 = e$. Now we have a decomposition $R^+ = R \cdot e \oplus (\mathrm{id}_R - e_r)(R)$ and there is a $\theta \in \mathrm{Aut}(R^+)$ such that $\theta\restriction_{R\cdot e} = \mathrm{id}_{R\cdot e}$ and $\theta\restriction_{(\mathrm{id}_R - e_r)(R)} = -\mathrm{id}_{(\mathrm{id}_R - e_r)(R)}$. Then $\theta = b_l$ for some $b \in R$ and $\theta^2 = \mathrm{id}_R$. This implies, for all $x \in R$, $bxe = xe$ and $b(x - xe) = -x + xe$. Thus $bx - xe = -x + xe$ and $bx + x = 2xe$, which yields $b^2 + b = 2be$ and $b^2e + be = 2be^2 = 2be$. Now we have $b^2e = be$. Since $(b_l)^2 = \mathrm{id}_R$, we have $b^2e = e$ and we get $e = be$. Thus $x = ex = bex = bx$ for all $x \in R$, which shows that $\theta = b_l = e_l = \mathrm{id}_R$ and $2(\mathrm{id}_R - e_r)(R) = \{0\}$. If R^+ has no elements of order 2, then we conclude $\mathrm{id}_R = e_r$ and $1 = e \in R$.

From now on, we will assume that $1 \in R$ for each A-ring R.

Since $\mathrm{Aut}(R^+)$ is a group, we have $\mathrm{Aut}(R^+) = (U(R))_l$, $U(R)$ the group of units of R. If $s \in U(R)$, then $\alpha = s_r \in \mathrm{Aut}(R^+)$ and $\alpha = t_l$ for some $t \in R$. Now $s = s_r(1) = \alpha(1) = t_l(1) = t$ and we have $xs = tx = sx$ for all $x \in R$ and $s \in Z(R)$. It follows that $U(R) \subseteq Z(R)$. If $w \in R$ is some nilpotent element, then $1+w \in U(R) \subseteq Z(R)$ and it follows that $w \in Z(R)$. Thus, if $N(R)$ is the nilradical of R, then $N(R) = \mathrm{Nil}(R) := \{w \in R : w \text{ nilpotent}\}$.

Now assume that R is a ring, $1 \in R$, such that each $\alpha \in \mathrm{Aut}(R)$ commutes with all the elements in R_r. Then $\alpha(x)r = \alpha(xr)$ for all $r, x \in R$ and $\alpha = (\alpha(1))_l$. We collect what we just proved in

1.1. PROPOSITION. *Let R be a ring. Then the following hold*:

(1) *If R is an A-ring, then $1 \in R$, $\mathrm{Aut}(R^+) = (U(R))_l$ and $U(R) \subseteq Z(R)$.*

(2) *If R is an A-ring, then the nilradical $N(R)$ is contained in $Z(R)$, where $N(R) = \mathrm{Nil}(R) := \{w \in R : w \text{ nilpotent}\}$.*

(3) *If $\alpha \circ a_r = a_r \circ \alpha$ for all $\alpha \in \mathrm{Aut}(R^+)$, $a \in R$, and $1 \in R$, then R is an A-ring.*

Recall that if $G \approx R^+$ is the additive group of an A-ring, we call G an A-group.

We will exhibit an example of an A-ring R such that $(U(R))_l = \mathrm{Aut}(R^+)$ is not contained in the center of $\mathrm{End}(R^+)$. This is also our first example of an A-ring that is not an E-ring.

1.2. EXAMPLE. Let $G = \mathbb{Z}(2) \oplus \mathbb{Z}$ and define a ring R with $R^+ = G$ and with the multiplication

$$(\varepsilon, z)(\varepsilon', z') = ((\varepsilon z' + \varepsilon' z) \bmod 2, zz')$$

for $\varepsilon, \varepsilon' \in \mathbb{Z}(2) = \{0, 1\}$ and $z, z' \in \mathbb{Z}$. Let $\bmod 2 : \mathbb{Z} \to \mathbb{Z}(2)$ be the natural epimorphism. Then, letting matrices operate from the right,

$$\mathrm{End}(G) = \begin{bmatrix} \mathbb{Z}(2) & 0 \\ \mathbb{Z} \bmod 2 & \mathbb{Z} \end{bmatrix}$$

is not a commutative ring (and thus not an E-ring), but

$$\mathrm{Aut}(G) = \begin{bmatrix} \{1\} & 0 \\ \mathbb{Z} \bmod 2 & \{1, -1\} \end{bmatrix}$$

is a commutative group, as one easily verifies. Let

$$\alpha = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathrm{Aut}(G), \qquad \psi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathrm{End}(G).$$

Then

$$\alpha\psi = \begin{bmatrix} 1 & 0 \\ 2 \bmod 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{but} \quad \psi\alpha = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

which shows $\operatorname{Aut}(G) \subsetneq Z(\operatorname{End}(G))$. On the other hand,

$$(\varepsilon, z)\begin{bmatrix} 1 & 0 \\ z' \bmod 2 & 1 \end{bmatrix} = (\varepsilon + zz' \bmod 2, z) = (\varepsilon, z)(z' \bmod 2, 1),$$

$$\begin{aligned}(\varepsilon, z)\begin{bmatrix} 1 & 0 \\ z' \bmod 2 & -1 \end{bmatrix} &= (\varepsilon + zz' \bmod 2, -z) = (-\varepsilon + zz' \bmod 2, -z) \\ &= (\varepsilon, z)(z' \bmod 2, -1),\end{aligned}$$

which shows that G is an A-group. Also note that G allows two non-isomorphic ring structures.

Since A-groups have commutative automorphism groups, we want to collect some information about abelian groups with that peculiar property. It is known [3, Corollary 115.2] that an abelian torsion p-group G has $\operatorname{Aut}(G)$ commutative iff G is cocyclic or $p = 2$ and $G \approx \mathbb{Z}(2) \oplus \mathbb{Z}(2^\infty)$. It seems that the prime $p = 2$ always causes trouble for automorphisms!

Next we will look at mixed groups with commutative automorphism group.

1.3. THEOREM. *Let G be a mixed group with* $\operatorname{Aut}(G)$ *commutative. Let $P = \{p \text{ prime} : t(G)_p \neq 0\}$. Then, for each $p \in P$, there is a natural number k_p and a subgroup $H^{(p)}$ such that $G = \mathbb{Z}(p^{k_p}) \oplus H^{(p)}$. If $(p, k_p) \neq (2, 1)$, then $pH^{(p)} = H^{(p)}$ and $t(H^{(p)})_p = \{0\}$. Moreover, $H^{(p)}$ is fully invariant in G. If $(p, k_p) = (2, 1)$, then $G = \mathbb{Z}(2) \oplus H^{(2)}$, $H^{(2)}$ is 2-torsion-free and* $\operatorname{Aut}(H^{(2)})$ *induces only the identity on $H^{(2)}/2H^{(2)}$.*

Proof. Assume that $t(G)$ is not reduced, i.e. there is a prime $p \in P$ and a subgroup K of G such that $G = \mathbb{Z}(p^\infty) \oplus K$. Since G is mixed, there is an element $k \in K$ with infinite order. Let $f : k\mathbb{Z} \to \mathbb{Z}(p^\infty)$ with $o(f(k)) > 2$. Then f extends to a homomorphism $\varphi : K \to \mathbb{Z}(p^\infty)$. Consider

$$\psi^+ = \begin{bmatrix} 1 & 0 \\ \varphi & 1 \end{bmatrix}, \quad \psi^- = \begin{bmatrix} 1 & 0 \\ \varphi & -1 \end{bmatrix} \in \operatorname{Aut}(G).$$

Then

$$\psi^+\psi^- = \begin{bmatrix} 1 & 0 \\ 2\varphi & -1 \end{bmatrix}, \quad \psi^-\psi^+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which implies $2\varphi = 0$, a contradiction to $2\varphi(k) = 2f(k) \neq 0$.

Since $t(G)_p$ is reduced, $t(G)_p$ has a cyclic summand $\langle a \rangle$ with $G = \langle a \rangle \oplus H$ for some subgroup H of G. If $t(H)_p \neq 0$ we may repeat this step and find a cyclic subgroup $\langle b \rangle$ of H such that $G = \langle a \rangle \oplus \langle b \rangle \oplus L$ for some subgroup L of G. It follows that $\operatorname{Aut}(\langle a \rangle \oplus \langle b \rangle) \subseteq \operatorname{Aut}(G)$ is commutative, a contradiction to [3, 115.2].

Thus $G = \mathbb{Z}(p^{k_p}) \oplus H^{(p)}$ and $H^{(p)}$ is p-torsion-free. If $H^{(p)} \neq pH^{(p)}$, there is an element in $H^{(p)}/p^{k_p}H^{(p)}$ that generates a direct summand of order p^{k_p}. Thus there is a $\varphi \in \operatorname{Hom}(H^{(p)}, \mathbb{Z}(p^{k_p}))$ such that φ is surjective.

Now construct ψ^+ and ψ^- as above and conclude that $2\varphi = 0$. Thus $p = 2$ and $k_2 = 1$.

Now consider the case $G = \mathbb{Z}(2) \oplus H^{(2)}$. Recall that $H^{(2)}$ has no elements of order 2 and $\mathrm{Aut}(G)$ is commutative. Moreover, $\bigcap\{\mathrm{Ker}(\varphi) : \varphi \in \mathrm{Hom}(H^{(2)}, \mathbb{Z}(2))\} = 2H^{(2)}$. Let $\pi \in \mathrm{Aut}(H^{(p)})$. Then

$$\psi_1 = \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 & 0 \\ \varphi & \pi \end{bmatrix} \in \mathrm{Aut}(G)$$

commute and so $\varphi = \pi\varphi$. This means that $0 = (-1+\pi)\varphi$ and $H(-1+\pi) \subseteq \mathrm{Ker}(\varphi)$ for all $\varphi \in \mathrm{Hom}(H^{(2)}, \mathbb{Z}(2))$. Thus $H(-1+\pi) \subseteq 2H$ and $\mathrm{Aut}(H)\restriction_{H/2H} = \{\mathrm{id}_{H/2H}\}$. The converse is easy to verify. ∎

1.4. Theorem. *Let G be a group with $t(G)_2 = 0$ and $\mathrm{Aut}(G)$ commutative. If $G = A \oplus B$ then $\mathrm{Aut}(G) = \mathrm{Aut}(A) \times \mathrm{Aut}(B)$. If G is an A-group, then A and B are A-groups and both are fully invariant in G.*

Proof. Let $\varphi \in \mathrm{Hom}(B, A)$ and define $\psi^{+/-}$ as in the proof of 1.3. Then again we infer $2\varphi = 0$, which now implies that $\varphi = 0$. Thus $\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A)$ and $\mathrm{Aut}(G) = \mathrm{Aut}(A) \times \mathrm{Aut}(B)$. Moreover, if R is a ring with $R^+ = G$, then A and B are ideals and there are subrings S and T of R such that $R = S \times T$ and $S^+ = A$ and $T^+ = B$. ∎

If $t(G)_2 \neq 0$, we can say the following:

1.5. Corollary. *Let $G = \mathbb{Z}(2) \oplus H$ with H not torsion. The following are equivalent*:

(I) $\mathrm{Aut}(G)$ *is commutative.*

(II) (a) $t(H)_2 = 0$ *and* $\mathrm{Aut}(H)$ *is commutative.*

(b) *Either* $2H = H$ (*and thus H is fully invariant in G*), *or* $2H \neq H$ *and* $\mathrm{Aut}(H)$ *induces the identity on $H/2H$.*

Proof. By Theorem 1.3, $t(G)_2 = \mathbb{Z}(2)$ and thus $t(H)_2 = 0$. Thus

$$\mathrm{Aut}(G) = \begin{bmatrix} \mathrm{Aut}(\mathbb{Z}_2) & 0 \\ \mathrm{Hom}(H, \mathbb{Z}(2)) & \mathrm{Aut}(H) \end{bmatrix}.$$

The rest is easy verification. ∎

The following example shows that the case $2H \neq H$ can actually occur:

1.6. Example. Pick three distinct odd primes p, q, r and define $H = (e_1\mathbb{Z}[1/p] \oplus e_2\mathbb{Z}[1/q]) + (e_1 + e_2)\mathbb{Z}[1/r]$. Then H is 2-reduced and $\mathrm{Aut}(H) = \{1, -1\}$. Thus, for $G = \mathbb{Z}(2) \oplus H$, we see that

$$\mathrm{Aut}(G) = \begin{bmatrix} 1 & 0 \\ \mathrm{Hom}(H, \mathbb{Z}(2)) & \{1, -1\} \end{bmatrix}$$

is commutative. Note that $\mathrm{End}(H) = \mathbb{Z}$, and thus

$$\mathrm{End}(G) = \begin{bmatrix} \mathbb{Z}(2) & 0 \\ \mathrm{Hom}(H, \mathbb{Z}(2)) & \mathbb{Z} \end{bmatrix}.$$

Now let $\theta : G \to \mathrm{End}(G)$ be a homomorphism. Since $\mathrm{Hom}(H, \mathbb{Z}) = 0$ we infer $2\theta(G) = 0$ and there is no ring R, $1 \in R$, such that $R^+ = G$. Thus G is an example of a group G with $\mathrm{Aut}(G)$ commutative that is not an A-group.

We will now determine which completely decomposable groups are A-groups. We leave the easy proof to the reader.

1.7. THEOREM. *Let $G = \bigoplus_{i\in I} A_i$ with A_i a subgroup of $\mathbb{Q}$ for all $i \in I$. Then $\mathrm{Aut}(G)$ is commutative if and only if the types of the A_i's are pairwise incomparable. Moreover, G is an A-group if and only if $\mathrm{Aut}(G)$ is commutative, I is finite, and the types of all the A_i's are idempotent if and only if G is an E-group.*

1.8. THEOREM. *Let G be a mixed group with $\mathrm{Aut}(G)$ commutative. Let $P = \{p \text{ prime} : t(G)_p \neq 0\}$ and assume $2 \notin P$. Then there are natural numbers k_p such that $t(G)_p = \mathbb{Z}(p^{k_p})$ and $t(G) = \bigoplus_{p\in P} \mathbb{Z}(p^{k_p})$. If P is finite, then $G = \mathbb{Z}(\prod_{p\in P} p^{k_p}) \oplus H$, where H is torsion-free and p-divisible for all $p \in P$.*

If P is infinite, then there is a pure subgroup A of $\prod_{p\in P} \mathbb{Z}(p^{k_p})$ and a torsion-free, P-divisible group H such that $G = A \oplus H$ and both A and H are fully invariant in G. Moreover, $\mathrm{Aut}(A)$ and $\mathrm{Aut}(H)$ are commutative and $\mathrm{Aut}(G) = \mathrm{Aut}(A) \times \mathrm{Aut}(H)$.

Proof. Let P' be the set of all primes not in P. Let $S = \mathbb{Z}[1/p : p \in P']$. Consider the short exact sequence $0 \to t(G) \to G \to G/t(G) \to 0$. By Theorem 1.3, $t(G) = \bigoplus_{p\in P} \mathbb{Z}(p^{k_p})$ and $G/t(G)$ is P-divisible. Now we tensor by S and obtain $0 \to t(G) \cong t(G) \otimes S \to G \otimes S \to (G/t(G)) \otimes S \to 0$ and $(G/t(G)) \otimes S$ is divisible. There are pure subgroups A and H of G such that $A \otimes S$ is the reduced part of $G \otimes S$ and $H \otimes S$ is the divisible part of $G \otimes S$. Moreover, $t(G \otimes S) \cong t(G)$. This implies that $t(G) \subseteq A$, $A/t(G)$ is P-divisible and H is torsion-free. This shows that A is the P-adic closure of $t(G)$ in G. Since $t(G)$ is P'-divisible, so is A. Now $G \subseteq (A \otimes S) \oplus (H \otimes S) = A \oplus H$. Thus $G = A \oplus H$ and A is a pure subgroup of the P-adic closure of $t(G)$, namely $\prod_{p\in P} \mathbb{Z}(p^{k_p})$. ∎

1.9. PROPOSITION. *Let $R = (\mathbb{Z}(p^n), +, *)$ be a unital ring with additive group $\mathbb{Z}(p^n)$. Then R is isomorphic to the ring $\mathbb{Z}(p^n)$.*

Proof. It is easy to see that there is a $v \in U(\mathbb{Z}(p^n))$ such that $x * y = vxy$ for all $x, y \in \mathbb{Z}(p^n)$. The map $\theta : \mathbb{Z}(p^n) \to R$ with $\theta(x) = v^{-1}x$ is a ring isomorphism. ∎

1.10. THEOREM. *Let G be a mixed A-group and R an A-ring with $R^+ = G$. Assume that $t(G)_2 = 0$ and that G has no torsion-free summand. Then $P = \{p \text{ prime} : t(G)_p \neq 0\}$ is infinite and R is a pure subring of $\prod_{p\in P} \mathbb{Z}(p^{k_p})$, where $t(G)_p = \mathbb{Z}(p^{k_p})$ for all $p \in P$. Moreover, all such A-rings are E-rings.*

Proof. Since $t(G)_p$ is fully invariant in G, the subring $t(G)$ of R is isomorphic to the natural ring structure by 1.9. By 1.8, G is isomorphic to a pure subgroup of $\prod_{p\in P} \mathbb{Z}(p^{k_p})$, the P-adic completion of $t(G)$. By continuity we may assume that R is a subring of the natural ring $\prod_{p\in P} \mathbb{Z}(p^{k_p})$. Any such ring is an E-ring (cf. [6]). ∎

The example at the beginning shows that the hypothesis $t(G)_2 = 0$ is needed in 1.10.

2. Torsion-free finite rank (tffr) A-rings. We will now consider A-groups and A-rings that are torsion-free of finite rank (tffr). While it is known that all E-rings R have $N(R) = \{0\}$ (cf. [1, Corollary 14.7]), we can only prove something weaker for A-rings.

2.1. PROPOSITION. *Let R be a tffr A-ring and $N = N(R)$ the nilradical of R. Then $N^2 = 0$.*

Proof. By a result due to Beaumont–Pierce (see [1, Corollary 14.2]) there is a subring T of R and an integer n such that $nR \subseteq T \oplus N \subseteq R$. Let $s \in N$ and $\theta : R \to R$ be the map that is the composition of multiplication by n, followed by the natural projection onto N, followed by the multiplication by $s \in N$. Then nNs is contained in the image of θ. Moreover, by 1.1(2), s is nilpotent, and thus θ is nilpotent. Now $1 - \theta \in \mathrm{Aut}(R^+)$ and is thus a multiplication. This shows that θ is a multiplication by some element $r \in R$. Now $n = n1 \in T$ and $T \subset \mathrm{Ker}(\theta)$, which implies that $0 = \theta(n1) = n\theta(1) = nr$ and $r = 0$. Thus $\theta = 0$ and $nNs = \{0\}$ for all $s \in N$, which means $N^2 = \{0\}$. ∎

2.2. THEOREM. *If R is a tffr strongly indecomposable A-ring, then R is an E-ring.*

Proof. By a result due to J. D. Reid (cf. [3, 92.3]), $\mathbb{Q}\mathrm{End}(R^+)$ is an artinian algebra and a local ring in which all non-units are nilpotent. Let $\varphi \in \mathrm{End}(R^+)$ be such that $\varphi(1) = 0$. Then $\mathrm{Ker}(\varphi) \neq \{0\}$ and φ induces an element in $\mathbb{Q}\mathrm{End}(R^+)$ that is nilpotent. Thus φ is nilpotent and $1 - \varphi \in \mathrm{Aut}(R^+)$ is a multiplication. This implies that φ is a multiplication with $\varphi(1) = 0$. Thus $\varphi = 0$ and $\mathrm{Hom}(R^+/\langle 1\rangle, R^+) = \{0\}$, which implies that R is an E-ring. ∎

2.3. PROPOSITION. *If R is a tffr A-ring of rank 2, then R is an E-ring.*

Proof. Let $G = R^+$ and let k be the number of distinct types of elements of G. By another result due to Beaumont–Pierce (see [1, Theorem 3.2]), the following are possible:

(a) $k = 1$, G is strongly indecomposable or $G = A \oplus B$ with $A \cong B$. In the latter case, $\mathrm{Aut}(G)$ is not commutative.

(b) $k = 2$ and G is strongly indecomposable or else $G = A \oplus B$ with $\mathrm{type}(A) < \mathrm{type}(B)$. In the latter case, $\mathrm{Aut}(G)$ is not commutative.

(c) $k = 3$ and G is strongly indecomposable or there is some $k \in \mathbb{N}$ such that $kG \subseteq A \oplus B \subseteq G$ and A, B have incomparable types. In the latter case, if $G \cong A \oplus B$, then G is an E-group, otherwise G is almost completely decomposable but indecomposable and A, B are subrings of $\mathbb{Q}$ that are fully invariant in G. Thus G is an E-group.

(d) $k > 3$ and G is strongly indecomposable.

Thus we may assume that G is strongly indecomposable and G is an E-group by Theorem 2.2. ∎

We now define a class of tffr rings in terms of their quasi-automorphisms and quasi-units.

2.4. Definition. A tffr ring R is called a *strong A-ring* if $U(\mathbb{Q}\mathrm{End}(R^+)) = U(\mathbb{Q}R_l)$, i.e. R_l and $\mathrm{End}(R^+)$ have the same quasi-units.

Our goal is to show that all strong A-rings are actually E-rings. The proof is presented in a sequence of propositions. Here is the first step:

2.5. Proposition. *Let R be a tffr ring, $1 \in R$. Then R is a strong A-ring if and only if each element of $U(\mathbb{Q}\mathrm{End}(R^+))$ commutes with each element of R_r. Thus $\{a \in R : \mathrm{Ker}(a_r) = 0\}$ is contained in $Z(R)$.*

Proof. Suppose R is a strong A-ring and let $\alpha \in U(\mathbb{Q}\mathrm{End}(R^+)) = U(\mathbb{Q}R_l)$. Then there is a natural number n and $b \in R$ such that $n\alpha = b_l$. Now b_l commutes with each element of R_r. Thus α commutes with each element of R_r.

To show the converse, let $\alpha \in U(\mathbb{Q}\mathrm{End}(R^+))$ and $a \in R$. Then for all $x \in R$ we have $\alpha(xa) = \alpha(x)a$ and $\alpha(a) = \alpha(1)a$ for all $a \in R$. Let $\beta = \alpha^{-1}$. As before we have $\beta(a) = \beta(1)a$ for all $a \in R$ and it follows that $\alpha(1)\beta(1) = 1$, which shows that $\alpha(1) \in U(\mathbb{Q}R)$ and $\alpha \in U(\mathbb{Q}R_l)$, and R is a strong A-ring. ∎

2.6. Proposition. *Each tffr strong A-ring R is an A-ring.*

Proof. Let $\alpha \in \mathrm{Aut}(R^+)$. Then there is a natural number n such that $n\alpha = r_l$ for some $r \in R$. Thus $n\alpha(1) = r \in nR$ and $r = ns$ for some $s \in R$, and it follows that $\alpha = s_l$. Since $\alpha \in \mathrm{Aut}(R^+)$, there is $t \in R$ with $\alpha(t) = 1$ and we have $st = 1$ and s is a unit in R. This shows that R is an A-ring. ∎

2.7. Proposition. *Let R be a strong A-ring. Then $U(\mathbb{Q}R) \subseteq \mathbb{Q}Z(R)$ and $U(\mathbb{Q}\mathrm{End}(R^+))$ is commutative.*

Proof. Let b be a quasi-unit of R. Then $b_r \in U(\mathbb{Q}\mathrm{End}(R^+)) = U(\mathbb{Q}R_l)$ and for some $m \in \mathbb{N}$ we have $mb_r = a_l$ for some $a \in R$. Now $mb = mb_r(1) = a_l(1) = a$ and it follows that $b_r = b_l$ and b is in the center of R. ∎

Note. If R is a tffr E-ring, then R is a strong A-ring, because in this case $\mathrm{End}(R^+) = R_l$.

Next we show that strong A-rings quasi-decompose just like E-rings.

2.8. Proposition. *Let R be a strong A-ring such that $R^+ \doteq H \oplus K$ is a quasi-decomposition. Then H, K are strong A-rings and* $\mathrm{Hom}(H,K) = 0 = \mathrm{Hom}(K,H)$.

Proof. There is a natural number n such that $n(H \oplus K) \subseteq R \subseteq H \oplus K$. Let $\pi_H : H \oplus K \to H$ and $\pi_K : H \oplus K \to K$ be the natural projections. Let $H' = \pi_H(R)$ and $K' = \pi_K(R)$. Then H is quasi-equal to H' since $nH \subseteq H' \subseteq H$. The same holds for K and K'. Thus, we may assume that π_H and π_K are onto. Let $\varphi \in \mathrm{Hom}(H,K)$. With matrices operating on the right, we have elements

$$\psi^+ = \begin{bmatrix} n & n\varphi \\ 0 & n \end{bmatrix}, \qquad \psi^- = \begin{bmatrix} n & n\varphi \\ 0 & -n \end{bmatrix}$$

in $\mathrm{End}(R)$. Moreover, $\psi^+, \psi^- \in U(\mathbb{Q}\mathrm{End}(R^+))$, a commutative group by 2.7. Thus $\psi^+\psi^- = \psi^-\psi^+$ and $\varphi = 0$ follows. This shows $\mathrm{Hom}(H,K) = 0 = \mathrm{Hom}(K,H)$.

For $h \in H$, the map $(nh)_r : nH \to R$ is a homomorphism and $(nh)_r : H \to H \oplus K$. Thus $(nH)(nH) \subseteq H$ and also $(nK)(nK) \subseteq K$. Moreover $(nH)(nK) \subseteq H \cap K = 0$. Thus $(nH)(nK) = 0 = (nK)(nH)$. Now let $h_1, h_2 \in H$. Then there are elements $k_1, k_2 \in K$ with $h_1 + k_1 \in R$ and $h_2 + k_2 \in R$. It follows that $n^2(h_1+k_1)(h_2+k_2) = (nh_1+nk_1)(nh_2+nk_2) = (nh_1)(nh_2) + (nk_1)(nk_2) = h_3 + k_3 \in n^2R \subseteq n^2(H \oplus K)$. Thus $h_3 = n^2h_4$ for some $h_4 \in H$ and we can define $h_1h_2 = h_4$. This makes H into a ring and the same works for K. This shows that $n(H \oplus K) \subseteq R \subseteq H \oplus K \approx H \times K$ is an inclusion of subrings.

Let $\alpha \in U(\mathbb{Q}\mathrm{End}(H))$. Then

$$\psi = \begin{bmatrix} n\alpha & 0 \\ 0 & n \end{bmatrix} \in U(\mathbb{Q}\mathrm{End}(R^+)).$$

Thus there is a natural number m such that $m\psi = (h+k)_l$ with $h + k$ a quasi-unit of R. We get $mn\alpha = \psi\restriction_H = h_l$ and thus $\alpha \in U(\mathbb{Q}H_l)$. Thus H is a strong A-ring and the same holds for K. ∎

Now we can prove our result:

2.9. THEOREM. *Let R be a tffr strong A-ring. Then R is an E-ring.*

Proof. By 2.8 we have $n(R_1 \times \ldots \times R_k) \subseteq R \subseteq R_1 \times \ldots \times R_k$ where each R_i is a strongly indecomposable strong A-ring and $\mathrm{Hom}(R_i, R_j) = 0$ for each $1 \le i \ne j \le k$. By 2.2 and 2.6, each R_i is an E-ring, which implies that R is an E-ring (cf. [1, Corollary 14.7]). ∎

3. Large A-rings. While we have not been able to find tffr A-rings that are not E-rings, we are more successful in the infinite rank case. We will prove the following result:

3.1. THEOREM. *Let κ, μ, λ be infinite cardinals such that $\mu^\kappa = \mu$ and $\lambda = \mu^+$, the successor cardinal of μ. Let S be an integral domain such that $|S| \le \kappa$ and S^+ is torsion-free and p-reduced for the prime integer p. Moreover, assume that there is some p-adic integer π such that π is transcendental over S. Then there exists an S-algebra R such that:*

(a) $|R| = \lambda$ *and R is an integral domain.*

(b) $\mathrm{End}(R^+) = R_l[\gamma] \approx R[x]$ *and γ is an injective ring homomorphism of R but γ is not surjective.*

(c) $\mathrm{Aut}(R^+) = (U(S))_l$.

Thus R is an integral domain and an A-ring that is not an E-ring.

We could prove this theorem in almost the same way as in the construction of large E-rings in [2], but we prefer to apply a more sophisticated version of the Black Box as introduced in [5] because this new version is easier to apply and also presents a λ-filtration of our desired ring R. We will present the main steps leading to the Strong Black Box [5] without duplicating the proofs. Let S have the properties as given in 3.1. Then $\bigcap_{i<\omega} p^i S = \{0\}$ and S is Hausdorff in its p-adic topology.

Let $B = S[x_{\alpha,n} : \alpha < \lambda,\ n < \omega]$ be the commutative polynomial ring with indeterminates $x_{\alpha,n}$. Let $\mathbb{M}$ be the set of all monomials $m \in B$, i.e. $m = \prod_{i=1}^k x_{\alpha_i,n_i}^{e_i}$ with $e_i > 0$ and $\{(\alpha_i, n_i) : 1 \le i \le k\}$ a finite subset of $\lambda \times \omega$. Each $a \in B$ has a unique representation $a = \sum_{m \in A} m a_m$ where $a_m \in S$ and A a finite subset of $\mathbb{M}$. We define $\deg(m) = \sum_{i=1}^k e_i$ to be the degree of the monomial m. Note that $B = \bigoplus_{m \in \mathbb{M}} Sm$ is a free S-module. Let $\widehat{B}$ be the p-adic completion of B and let $\subseteq_*$ denote "contained as a p-pure subgroup". For any $g = \sum m a_m \in \widehat{B} \subseteq \prod_{m \in \mathbb{M}} \widehat{S} m$ we define the *support* of g to be $[g] = \{m \in M : a_m \ne 0\}$ and if M is a subset of $\widehat{B}$, then $[M] = \bigcup_{g \in M} [g]$.

We define the λ-*support* of $g \in \widehat{B}$ by $[g]_\lambda = \{\alpha < \lambda :$ there are $m \in [g]$, $n < \omega$ and $m' \in \mathbb{M}$ such that $m = x_{\alpha,n} m'\}$. Note that $[g]_\lambda$ is an at most countable set of ordinals below λ and $[g]_\lambda$ is the set of all ordinals $\alpha < \lambda$ such that some variable $x_{\alpha,n}$ actually shows up in the representation of $g \in \widehat{B}$ as a multivariate polynomial. Finally we define an S-linear ring homomorphism $\gamma : B \to B$ by $\gamma(x_{\alpha,n}) = x_{\alpha,n+1}$ for all $\alpha < \lambda$ and $n < \omega$.

Next define a norm by $\|\{\alpha\}\| = \alpha + 1$ for any $\alpha < \lambda$ and $\|M\| = \sup_{\alpha\in M} \|\alpha\|$ for any subset $M \subseteq \lambda$. Moreover $\|g\| = \|[g]_\lambda\|$ for any $g \in \widehat{B}$. Note that $\|g\| = \min\{\beta < \lambda : [g]_\lambda \subseteq \beta\}$ and $[g]_\lambda \subseteq \beta$ holds iff $g \in \widehat{B}_\beta$ where $B_\beta = S[x_{\alpha,n} : \alpha < \beta, n < \omega]$.

Fix, once and for all, bijections $h_\alpha : \mu \to \alpha$ for all $\mu \le \alpha < \lambda$ such that $h_\mu = \mathrm{id}_\mu$ and for technical reasons we define $h_\beta = \mathrm{id}_\mu$ as well for $\beta < \mu$.

3.2. Definition. Define P to be a *canonical* subalgebra of B if $P = S[x_{\alpha,n} : \alpha \in I, n < \omega]$ for some $I \subset \lambda$ with $|I| \le \kappa$ such that $h_\alpha(I \cap \mu) = I \cap h_\alpha(\mu)$ for all $\alpha \in I$.

Accordingly, an additive homomorphism $\varphi : P \to \widehat{B}$ is *canonical* if P is canonical and $\varphi(P) \subseteq \widehat{P}$. We also define $[\varphi] = [P]$, $[\varphi]_\lambda = [P]_\lambda$, and $\|\varphi\| = \|P\|$. Moreover, let E be a stationary subset of $\lambda^\circ = \{\alpha < \lambda : \alpha$ has countable cofinality$\}$ such that $\lambda^\circ - E$ is stationary in λ as well.

We are now ready to state

3.3. Strong Black Box. *Let μ, κ, λ, S, B, E be as above. Then there is a family of canonical homomorphisms φ_β, $\beta < \lambda$, such that*:

(1) $\|\varphi_\beta\| \in E$ *for all* $\beta < \lambda$.

(2) $\|\varphi_\varrho\| \le \|\varphi_\beta\|$ *for all* $\varrho \le \beta < \lambda$.

(3) $\|[\varphi_\varrho]_\lambda \cap [\varphi_\beta]_\lambda\| < \|[\varphi_\beta]_\lambda\|$ *for all* $\varrho < \beta < \lambda$.

(4) Prediction. *For any homomorphism $\psi : B \to \widehat{B}$ and for any subset I of λ with $|I| \le \kappa$, the set $\{\alpha \in E :$ there is $\beta < \lambda$ with $\|[\varphi_\beta]_\lambda\| = \alpha$ and $I \subseteq [\varphi_\beta]_\lambda\}$ is stationary in λ.*

Remark. In the older version of the Black Box some ordinal λ^* with $|\lambda^*| = \lambda$ was used to enumerate the canonical homomorphisms. In our setting it turns out that $\lambda^* = \lambda$: If there is a canonical homomorphism φ_λ then $\|\varphi_\lambda\| = \delta < \lambda$ and we have λ (distinct) canonical subalgebras of cardinality $\le \kappa$ contained in a set of cardinality μ with $\mu^\kappa = \mu$. But there are only μ such subalgebras, and not $\lambda = \mu^+$ of them.

The one thing we need to prove in detail is the (algebraic) Step Lemma which will allow us to eliminate unwanted homomorphisms.

3.4. Step Lemma. *Let S, B, γ be as above and π a p-adic integer which is transcendental over S. Moreover, the following is given*:

(1) *Let $P = S[x_{\alpha,n} : \alpha \in I^*, n < \omega]$ for some subset I^* of λ and let M be a subring of $\widehat{B}$ with $P \subseteq_* M \subseteq_* \widehat{B}$ such that π is transcendental over M and $\gamma(M) \subseteq M$.*

(2) *There is a set* $I = \{\alpha_i : i < \omega\} \subset \lambda$ *with* $\alpha_i < \alpha_j$ *for all* $i < j < \omega$ *such that* $I \subseteq I^* = [P]_\lambda$ *and* $I \cap [g]_\lambda$ *is finite for all* $g \in M$.

(3) *Let* $\psi : P \to \widehat{M}$ *be a homomorphism that is not in* $(M[\gamma])\restriction_P$.

Then there is some $y \in \widehat{P}$ *such that* $\psi(y) \notin M' = (M[\gamma^i(y) : i < \omega])_*$. *Moreover,* π *is transcendental over* M'. *The element* y *will be either* $x = \sum_{i<\omega} p^i x_{\alpha_i,0}$ *or* $y = x + b\pi$ *with a suitable element* $b \in P$. *Note that* $\gamma^k(x) = \sum_{i<\omega} p^i x_{\alpha_i,k}$ *for all* $k < \omega$. *Also*

(4) M *and* M' *have the same group of units.*

Proof. Let $x = \sum_{i<\omega} p^i x_{\alpha_i,0}$ and assume $\psi(x) \in M' = (M[\gamma^i(y) : i < \omega])_*$. Then for some $a < \omega$ we have $p^a\psi(x) \in M[\gamma^i(x) : i < \omega]$. Note that by the disjointness condition (2) the p-adic integer π is still transcendental over M'. Let

$$(*) \qquad p^a\psi(x) = \sum_{m\in T} m a_m$$

where m is a monomial in the elements $\gamma^i(x)$, $i < \omega$. Choose a representation such that $N = \max\{\deg(m) : m \in T\}$ is the least possible.

Assume $N \geq 2$. Now pick another variable $x_0 = x_{\delta,0} \in P$ such that none of the $x_{\delta,n}$ occurs in any of the finitely many a_m, $m \in T$, and define $y = x + \pi x_0$. Moreover, define $M'' = (M[\gamma^i(y) : i < \omega])_*$ and assume $\psi(y) \in M''$. Then there are some $a' < \omega$ and $b_{m'} \in M$ and a set T' of monomials in the variables $\gamma^i(y)$ such that

$$(**) \qquad p^{a'}\psi(x + \pi x_0) = \sum_{m'\in T'} m' b_{m'}.$$

We now multiply equation $(*)$ by $p^{a'}$ and equation $(**)$ by p^a and subtract the former from the latter to obtain

$$(***) \qquad p^{a+a'}\psi(x_0)\pi = \sum_{m'\in T'} p^a m' b_{m'} - \sum_{m\in T} p^{a'} m a_m \in M\pi.$$

For each monomial $m' \in T'$ we form the monomial m'' by simply erasing the $\gamma^i(x_0\pi)$ term. (In other words, we set $x_0 = 0$.) Now we expand the monomials $m' \in T'$ and collect like terms by powers of π. This turns $(***)$ into

$$(\#) \quad p^{a+a'}\psi(x_0)\pi = \pi^{N'} g_{N'} + \sum_{j=1}^{N'-1} \pi^j g_j + \Big(\sum_{m'\in T'} p^a m'' b_{m'} - \sum_{m\in T} p^{a'} m a_m\Big).$$

Note that $N' \geq 1$, all $g_j \in M$, and $(\sum_{m'\in T'} p^a m' b_{m'} - \sum_{m\in T} p^{a'} m a_m) \in M'$. Moreover, $\psi(x_0) \in M$ by hypothesis.

First of all, this implies $\sum_{m'\in T'} p^a m' b_{m'} - \sum_{m\in T} p^{a'} m a_m = 0$ and $N' = N \geq 2$ and $\{m'' : m' \in T'\} = T$. Moreover $p^a b_{m'} = p^{a'} a_m$ for all $m \in T$.

Note that $g_N = 0$ as well, because $N = N' \geq 2$. We need to have a closer look at that term. Note that

$$0 = g_N = \sum_{m \in T,\, \deg(m)=N} p^a \widetilde{m} b_{m'} = \sum_{m \in T,\, \deg(m)=N} p^{a'} \widetilde{m} a_m$$

where $\widetilde{m}$ is the monomial obtained from m (or m') by replacing $\gamma^i(x)$ (or $\gamma^i(x_0 + x)$) by $\gamma^i(x_0) = x_{\delta,i}$. Since $m \mapsto \widetilde{m}$ is injective, and all $x_{\delta,i}$ are transcendental over the a_m, by the choice of $x_{\delta,0}$, we conclude that $a_m = 0$ whenever $m \in T$ and $\deg(m) = N$. This is a contradiction to the minimality of N.

Thus we may assume that $N = 1$ and we have, by way of contradiction, for x chosen as above,

$$(+) \qquad p^a \psi(x) = \sum_{i=0}^{k} a_i \gamma^i(x) \quad \text{for some } a_i \in M.$$

We define M' as above as well.

Assume that $p^a \psi\restriction_P \neq \sum_{i=0}^k a_i \gamma^i$. Then there is some $w \in P$ with $p^a \psi(w) \neq \sum_{i=0}^k a_i \gamma^i(w)$. Let $y = w\pi + x$ and define M'' for this choice. Now assume that

$$(++) \qquad p^{a'} \psi(w\pi + x) = \sum_{m \in T} m b_m$$

where m is a monomial in the variables $\gamma^i(b\pi + x)$. As above we subtract $p^{a'}(+)$ from $p^a(++)$ and obtain $p^a \sum_{m \in T} \overline{m} b_m - p^{a'} \sum_{i=0}^k a_i \gamma^i(x) \in \pi M'$. Thus $p^a \sum_{m \in T} \overline{m} b_m = p^{a'} \sum_{i=0}^k a_i \gamma^i(x)$ and the maximal degree of polynomials in T is at most 1. Thus we have

$$(+++) \qquad p^{a'} \psi(b\pi + x) = \sum_{i=0}^{k'} b_i \gamma^i(w\pi + x).$$

Again we do our subtraction and obtain

$$p^{a+a'} \psi(w)\pi = p^a \sum_{i=0}^{k'} b_i \gamma^i(w)\pi + p^a \sum_{i=0}^{k'} b_i \gamma^i(x) - p^{a'} \sum_{i=0}^{k'} a_i \gamma^i(x).$$

The fact that π is transcendental over M' now tells us that $k = k'$ and $p^a b_i = p^{a'} a_i$ for all $0 \leq i \leq k$. Therefore,

$$p^{a+a'} \psi(w) = p^a \sum_{i=0}^{k'} b_i \gamma^i(w) = p^{a'} \sum_{i=0}^{k} a_i \gamma^i(w)$$

and it follows that $\psi(w) = \sum_{i=0}^k a_i \gamma^i(w)$, a contradiction to the choice of w.

Now we are finally down to the case where

$$p^a\psi\restriction_P = \sum_{i=0}^{k} a_i\gamma^i \quad \text{with } a_i \in M.$$

We may pick some variable $\widetilde{x}$ from P such that none of the $\gamma^j(\widetilde{x})$, $j<\omega$, occurs in any of the $a_i \in M$ and get $p^a\psi(\widetilde{x}) = \sum_{i=0}^{k} a_i\gamma^i(\widetilde{x}) \in p^aM$. Thus $a_i = p^a m_i$ with $m_i \in M$ and $\psi\restriction_P = (\sum_{i=0}^{k} m_i\gamma^i)\restriction_P$.

Now we need to prove (4). Suppose $u \in M'$ is a unit in M' such that $u \notin M$. Let k be minimal such that $u \in (M[\gamma^i(x) : 0 \le i \le k])_* = ((M[\gamma^i(x) : 0 \le i < k])[\gamma^k(x)])_*$. Note that $\gamma^k(x)$ is transcendental over $M[\gamma^i(x) : 1 \le i < k]$. If v is the inverse of u, then $v \in ((M[\gamma^i(x) : 1 \le i < k])[\gamma^k(x)])_*$ as well. This shows that $k = 0$, since in polynomial rings only constants are units and we obtain $u, v \in (M[x])_*$. Since x is transcendental over M, we infer $u \in M$. ∎

We will now construct our ring R.

Let φ_β, $\beta < \lambda$, be the sequence of canonical homomorphisms provided by the Strong Black Box 3.3. Let $P_\beta = S[x_{\alpha,n} : \alpha \in [\varphi_\beta]_\lambda, n < \omega]$ be the domain of φ_β. We will construct R as the union of a λ-filtration $R = \bigcup_{\beta<\lambda} R^\beta$ of p-pure subrings of $\widehat{B}$ with $R^0 = P_0$ such that

$$(*) \qquad R^\beta \subseteq \Big(S\Big[\bigcup_{\alpha<\beta} \widehat{P}_\alpha\Big]\Big)_* \quad \text{and} \quad \{g \in B : \|g\| < \|\varphi_\beta\|\} \subset R^\beta.$$

If β is a limit ordinal, we let $R^\beta = \bigcup_{\alpha<\beta} R^\alpha$. Now suppose we have already constructed R^β. Consider the canonical homomorphism φ_β. Since $\|\varphi_\beta\| \in \lambda^\circ$ is a limit ordinal of countable cofinality, there are ordinals $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ in $[\varphi_\beta]_\lambda$ such that $\|\varphi_\beta\| = \sup_{n<\omega}\{\alpha_n\}$. Let $I = \{\alpha_n : n < \omega\}$. Then $I \cap [g]_\lambda$ is finite by $(*)$ and condition (2) in 3.3. If φ_β maps P_β into R^β and φ_β is not induced by some map in $R^\beta[\gamma]$, then apply the Step Lemma to I, $P = P_\beta$, $M = R^\beta$, and $\psi = \varphi_\beta$. Thus there is some $y = y_\beta \in \widehat{P}_\beta$ and $R^{\beta+1} = (R^\beta[\gamma^i(y_\beta) : i < \omega])_*$ such that π is transcendental over $R^{\beta+1}$ and $\varphi_\beta(y_\beta) \notin R^{\beta+1}$. Moreover, $R^{\beta+1}$ satisfies $(*)$, because $y_\beta \in \widehat{P}_\beta$.

If $\varphi_\beta \in (R^\beta[\gamma])\restriction_{P_\beta}$, then we do not need to apply the Step Lemma and we simply define $R^{\beta+1} = (R^\beta[\gamma^i(y_\beta) : i < \omega])_*$, where $y_\beta = \sum_{i<\omega} p^i x_{\alpha_i,0}$.

3.5. Lemma. *Let R be the ring constructed above. The following hold*:

(a) $\{\gamma^i(y_\beta) : \beta < \lambda,\ i < \omega\}$ *is transcendental over* B.

(b) *If* $g \in R - B$, *then there is a finite subset* N *of* $\lambda \times \omega$ *and* $a < \omega$ *such that* $p^a g \in B[\gamma^i(y_\beta) : (\beta, i) \in N]$ *and* $[g]_\lambda \cap [\gamma^i(y_\beta)]_\lambda$ *is infinite iff* $(\beta, i) \in N$.

If $\|g\|$ *is a limit ordinal*, *then* $\|g\| = \|y_\beta\|$ *where* β *is the largest ordinal such that* $(\beta, j) \in N$ *for some* $j < \omega$. *Moreover*,

(c) $R \cap \widehat{P}_\beta \subseteq R^{\beta+1}$ *for all* $\beta < \lambda$.
(d) $U(R) = U(S)$.

The proof of 3.5 is the same as that of Lemma 2.2.4 and 2.2.5(a) in [5] and left to the reader. (For example, clause (c) follows from (b) and 3.3(3). Moreover, (d) follows from 3.4(4).)

We will now show that $\mathrm{End}(R^+) = R[\gamma]$. Again, we can almost copy the proof of 2.2.1 from [5]. We want to outline the proof anyway.

Let $R = \bigcup_{\beta<\lambda} R^\beta$ be the ring constructed above. Obviously, $R[\gamma] \subseteq \mathrm{End}(R^+)$ by the construction of R. Moreover, π is transcendental over each R^β and thus R and the R^β, $\beta < \lambda$, form a λ-filtration of R. Let $I = \{\alpha_i : i < \omega\} \subset \lambda$ be such that $\alpha_i < \alpha_j$ for all $i < j < \omega$ such that $\eta = \sup_{i<\omega}\{\alpha_i\} \in \lambda^\circ - E \neq \emptyset$ by the choice of E. Then $[g]_\lambda \cap I$ is finite for all $g \in R$.

Let $\psi \in \mathrm{End}(R^+) - R[\gamma]$. By the Step Lemma, there is some $y \in \widehat{B}$ such that $y = \sum_{i<\omega} p^i x_{\alpha_{i,0}}$ up to, possibly, a π-multiple of some element in B, and $\psi(y) \notin (R[\gamma^i(y) : i < \omega])_*$. Now we apply 3.3 and conclude that $E' = \{\alpha \in E :$ there is $\beta < \lambda$ such that $\|\varphi_\beta\| = \alpha$ and $\varphi_\beta \subset \psi$ and $[y] \subseteq [\varphi_\beta]\}$ is stationary in λ. Let $C = \{\beta : \psi(R^\beta) \subseteq R^\beta\}$. This set is a cub (closed unbounded subset) of λ. Thus $E'' = E' \cap C$ is stationary in λ and we may pick some $\eta < \alpha \in E''$. Then there is some $\beta < \lambda$ such that $\alpha = \|\varphi_\beta\|$ and $\varphi_\beta \subset \psi$ and $[y] \subseteq [P_\beta] = [\varphi_\beta]$. Recall that $\eta = \|y\|$ and $y \in \widehat{P}_\beta$. Thus $R^{\beta+1}$ was constructed such that $\psi(y_\beta) = \varphi_\beta(y_\beta) \notin R^{\beta+1}$ and $\varphi_\beta(y_\beta) \in \widehat{P}_\beta$. By 3.5(c) we have $\psi(y_\beta) \notin R$, a contradiction to $\psi \in \mathrm{End}(R^+)$. This proves 3.1(b). To show part (c), observe that for any automorphism α of R^+, α is a unit in $R_l[\gamma]$, which is isomorphic to a polynomial ring over the integral domain $R_l \approx R$. Thus $\alpha \in R$ is a unit in R and by 3.5(d) we have $\alpha \in U(R) = U(S)$. ■

Added in proof (June 2003). The first named author has a forthcoming paper answering the question stated in the introduction: All tffr A-rings are indeed E-rings.

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