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COUNTING LINEARLY ORDERED SPACES

BY

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Abstract. For a transfinite cardinal κ and $i \in \{0,1,2\}$ let $\mathcal{L}_i(\kappa)$ be the class of all linearly ordered spaces X of size κ such that X is totally disconnected when $i = 0$, the topology of X is generated by a dense linear ordering of X when $i = 1$, and X is compact when $i = 2$. Thus every space in $\mathcal{L}_1(\kappa) \cap \mathcal{L}_2(\kappa)$ is connected and hence $\mathcal{L}_1(\kappa) \cap \mathcal{L}_2(\kappa) = \emptyset$ if $\kappa < 2^{\aleph_0}$, and $\mathcal{L}_0(\kappa) \cap \mathcal{L}_1(\kappa) \cap \mathcal{L}_2(\kappa) = \emptyset$ for arbitrary κ. All spaces in $\mathcal{L}_1(\aleph_0)$ are homeomorphic, while $\mathcal{L}_2(\aleph_0)$ contains precisely $\aleph_1$ spaces up to homeomorphism. The class $\mathcal{L}_1(\kappa) \cap \mathcal{L}_2(\kappa)$ contains precisely 2^κ spaces up to homeomorphism for every $\kappa \geq 2^{\aleph_0}$. Our main results are explicit constructions which prove that both classes $\mathcal{L}_0(\kappa) \cap \mathcal{L}_1(\kappa)$ and $\mathcal{L}_0(\kappa) \cap \mathcal{L}_2(\kappa)$ contain precisely 2^κ spaces up to homeomorphism for every $\kappa > \aleph_0$. Moreover, for any κ we investigate the variety of second countable spaces in the class $\mathcal{L}_0(\kappa) \cap \mathcal{L}_1(\kappa)$ and the variety of first countable spaces of arbitrary weight in the class $\mathcal{L}_2(\kappa)$.

1. Introduction. Write $|S|$ for the cardinality (*size*) of a set S. As usual, $\aleph_0 := |\mathbb{N}|$ and $c := |\mathbb{R}|$, and 2^κ is the size of the power set of any set S with $|S| = \kappa$. Thus $2^\kappa > \kappa$ for every cardinal number κ and $c = 2^{\aleph_0} > \aleph_0$. The enigmatic region $\mathcal{K}$ of all cardinals κ with $\aleph_0 < \kappa < c$ is possibly very large. In fact, it is consistent with standard set theory that $|\mathcal{K}| = c$ (see the remark below).

A *linearly ordered space* X is a space whose topology is the order topology of some linear ordering of X. Just as metric spaces, linearly ordered spaces satisfy all separation axioms (they are completely normal). Naturally, if κ is a transfinite cardinal and $\mathcal{F}$ is any family of mutually non-homeomorphic linearly ordered spaces of size κ then $|\mathcal{F}| \leq 2^\kappa$. The following theorem, which is covered by [4, Theorem 3] and [5, 7.1], shows that the upper bound 2^κ can be achieved for every cardinal $\kappa \geq c$ and for $\kappa = \aleph_0$.

THEOREM 1.

(i) *For every cardinal $\kappa \geq c$ there exist 2^κ mutually non-homeomorphic connected and compact linearly ordered spaces of size κ.*
(ii) *There exist c mutually non-homeomorphic linearly ordered spaces of size $\aleph_0$.*

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Let us call a space X *densely ordered* when its topology is generated by a *dense* linear ordering (i.e. infinitely many points lie between any two points). Clearly, every connected linearly ordered space is densely ordered and every densely ordered space is dense in itself. A compact linearly ordered space is densely ordered if and only if it is connected (cf. [8, 39.7, 39.8]). If X is a linearly ordered space with $|X| < c$ then X is totally disconnected (cf. [2, 6.1.4]). Thus there are two natural directions to modify the combination *connected* plus *compact* in order to raise two interesting counting problems for linearly ordered spaces of any size in the region $\mathcal{K}$ and, more generally, for totally disconnected linearly ordered spaces of uncountable size. These two problems are solved by the following theorems which also complement Theorem 1(i) in interesting ways.

THEOREM 2. *For every cardinal $\kappa > \aleph_0$ there exist 2^κ mutually non-homeomorphic totally disconnected densely ordered spaces of size κ.*

THEOREM 3. *For every cardinal $\kappa > \aleph_0$ there exist 2^κ mutually non-homeomorphic scattered and compact linearly ordered spaces of size κ.*

Theorem 3 has the following important consequence.

COROLLARY 1. *For every cardinal $\kappa > \aleph_0$ there exist precisely 2^κ compact Hausdorff spaces of size κ up to homeomorphism.*

(Notice that, by the first argument in the proof of Theorem 5 below, one cannot find more than 2^κ non-homeomorphic compact Hausdorff spaces of size κ for any cardinal κ.) As a consequence of a famous classification theorem due to Mazurkiewicz and Sierpiński [6], there exist precisely $\aleph_1$ compact Hausdorff spaces of size $\aleph_0$ up to homeomorphism, and they all are linearly ordered spaces. ($\aleph_1$ is the smallest cardinal greater than $\aleph_0$, and hence $\aleph_1 = \min \mathcal{K}$ provided that $\mathcal{K} \neq \emptyset$.) Therefore, Corollary 1 and Theorem 3 would be either unprovable or false for $\kappa = \aleph_0$. Also in Theorem 2 we have to exclude the case $\kappa = \aleph_0$ because (in view of [2, 6.2.A.d]) any dense-in-itself and countable linearly ordered space is homeomorphic to the Euclidean space $\mathbb{Q}$. Recall that a Hausdorff space is *scattered* if and only if it does not contain a non-empty dense-in-itself point set. Note that any scattered and compact Hausdorff space is totally disconnected (see [8, Fig. 9] and [2, 6.2.9]). Note also that any compact Hausdorff space of size smaller than c is scattered (cf. [2, 3.12.11]).

An essential step in proving Theorem 2 is the following theorem.

THEOREM 4. *If $\aleph_0 < \kappa \leq c$ then there exist 2^κ mutually non-homeomorphic dense and totally disconnected subspaces X of $\mathbb{R}$ with $|X| = \kappa$.*

Theorem 4 (together with Theorem 1(ii)) also solves the counting problem concerning second countable linearly ordered spaces because any second countable linearly ordered space is homeomorphic to a subspace of the Euclidean space $\mathbb{R}$ (cf. [2, 4.2.9, 6.3.2.c]) and, clearly, if X is a dense subspace of $\mathbb{R}$ then the order topology of the naturally ordered set X equals the Euclidean subspace topology of X. As we will see, it is rather easy to establish the conclusion of Theorem 4 for all cardinals $\kappa \leq c$ with $2^\kappa > c$. The challenge is to prove it for all uncountable cardinals κ with $2^\kappa = c$.

REMARK. For any cardinal κ in the enigmatic region $\mathcal{K}$ it is *undecidable* whether $2^\kappa > c$ or $2^\kappa = c$. (This is a trivial consequence of Easton's Theorem [3, 15.18].) Moreover, it is consistent with standard set theory that there exist c cardinals $\kappa < c$ such that $2^\kappa = c$ for infinitely many $\kappa < c$, and $2^\kappa > c$ for infinitely many $\kappa < c$. (Because if γ is the smallest ordinal number with $\gamma = \aleph_\gamma$ and $\mathrm{cf}(\gamma) = \aleph_{\omega+2}$ then, by applying Easton's Theorem [3, 15.18], it is consistent with set theory that $2^{\aleph_0} = 2^{\aleph_{\omega+1}} = \aleph_\gamma$ and $2^{\aleph_{\omega+2}} = \aleph_{\gamma+1}$.) It is also worth mentioning that the popular hypothesis that c is a *real-valued measurable* cardinal implies that $|\mathcal{K}| = c$ (cf. [3, 10.15]) and $2^\kappa = c$ for every $\kappa \in \mathcal{K}$ (cf. [3, 22.2]).

Since the natural ordering of a dense subset of $\mathbb{R}$ is a dense linear ordering, Theorem 4 also solves the counting problem concerning second countable densely ordered spaces. The counting problem concerning second countable compact linearly ordered spaces is solved by the Mazurkiewicz–Sierpiński theorem [6] and by the following interesting theorem. (Note that if X is a first countable compact Hausdorff space then either $|X| \leq \aleph_0$ or $|X| = c$; see [2, 3.12.11.d]. Note also that the weight of a compact Hausdorff space can never be greater than its size; see [2, 3.1.21].)

THEOREM 5. *Let $\kappa \leq c$ be a transfinite cardinal. Then up to homeomorphism there exist precisely 2^κ first countable compact Hausdorff spaces X such that $|X| = c$ and κ is the weight of X. Moreover, there exist 2^κ mutually non-homeomorphic separable, dense-in-itself, first countable, compact linearly ordered spaces of size c and weight κ.*

Note that the size of a dense-in-itself compact Hausdorff space cannot be smaller than c (cf. [2, 3.12.11.a]). Furthermore, it is plain that any separable linearly ordered space is first countable. As a consequence, if X is a separable linearly ordered space then $|X| \leq c$. (Consider a linearly ordered compactification $Y \supset X$ of X constructed via Dedekind cuts as in [7, Theorem 2.32]. Then Y must also be separable, hence first countable, whence $|Y| \leq c$.) Therefore, the counting problem concerning separable linearly ordered spaces is solved by Theorems 1(ii) and 4.

REMARK. In [5] we have shown that up to homeomorphism there are precisely 2^κ metrizable spaces of size κ for every $\kappa \geq c$ and also for $\kappa = \aleph_0$. But we did not deal with metrizable spaces of sizes in the enigmatic region

$\aleph_0 < \kappa < c$. This gap in a complete solution of the fundamental counting problem concerning metric spaces is now closed by Theorem 4.

2. Preparation of the proofs. Let Ω be the canonically well-ordered class of all ordinal numbers (with $\mathbb{N} \cup \{0\} \subset \Omega$). Note that any non-empty subset of the class Ω has a well-defined supremum. If $\alpha, \beta \in \Omega$ then let $[\alpha, \beta] := \{\xi \in \Omega \mid \alpha \leq \xi \leq \beta\}$ and $[\alpha, \beta[:= [\alpha, \beta] \setminus \{\beta\}$ and $]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}$. If we speak of the *space* $[\alpha, \beta]$, $[\alpha, \beta[$ or $]\alpha, \beta[$ then we refer to the order topology of the canonical well-ordering. So we may say that for $\alpha < \beta$ the space $[\alpha, \beta]$ is compact and scattered.

Define $|\alpha| := |[0, \alpha[|$ for each $\alpha \in \Omega$. (This definition is a tautology if ordinal numbers are defined in the standard way as in [3] where $\alpha = [0, \alpha[$ for every $\alpha \in \Omega$.) As usual we consider cardinal numbers to be defined as initial ordinal numbers. So for each cardinal number κ we have $\kappa = \min\{\gamma \in \Omega \mid |\gamma| = \kappa\}$. In particular, the ordinal $\omega = \sup \mathbb{N}$ equals the cardinal $\aleph_0$ and $\omega_1 = \sup\{\alpha \in \Omega \mid |\alpha| = \aleph_0\} = \min\{\alpha \in \Omega \mid |\alpha| > \aleph_0\}$ equals the cardinal $\aleph_1$.

For any cardinal κ let (as usual) κ^+ denote the smallest cardinal greater than κ. (For example, $\aleph_1 = \aleph_0^+$.) Clearly, for cardinals κ and ordinals α we have $|\alpha| = \kappa$ if and only if $\kappa \leq \alpha < \kappa^+$.

For $\xi \in \Omega$ we write (as usual) ω^ξ for the *ordinal power* with basis ω and exponent ξ. So all spaces $[0, \omega^\xi]$ are compact and for $\xi > 0$ we have $|[0, \omega^\xi]| = \max\{\aleph_0, |\xi|\}$. In particular, $|[0, \omega^\xi]| = |\xi|$ for every ordinal $\xi \geq \omega$.

The natural way to prove our theorems is to use the powerful machinery of *Cantor derivatives*. Let X be a Hausdorff space. If A is a point set in X, then the first derivative $A' = A^{(1)}$ of A is the set of all limit points of A in X. (Note that $A' \subset A$ if and only if A is closed, whereas $A \subset A'$ if and only if A is dense in itself.) The higher derivatives are defined recursively in the following way. For $\alpha \in \Omega$ we put $A^{(\alpha+1)} := (A^{(\alpha)})'$ where (since $0 \in \Omega$) $A^{(0)} := A$. And $A^{(\lambda)} := \bigcap\{A^{(\alpha)} \mid \alpha \in \Omega \wedge \alpha < \lambda\}$ if $\lambda > 0$ is a limit ordinal. For $0 \neq \alpha \in \Omega$ the point set $A^{(\alpha)}$ is always closed. Clearly, $A^{(\alpha)} \supset A^{(\beta)}$ whenever $0 < \alpha \leq \beta$, and for $A \subset B \subset X$ we have $A^{(\alpha)} \subset B^{(\alpha)}$ for every $\alpha \in \Omega$.

The following lemma is evident. (Historically, Cantor's definition of the ordinal powers of ω is designed precisely, so that the following is true.)

LEMMA 1. *Let $0 \neq \xi \in \Omega$. In the compact space $[0, \omega^\xi]$, for every ordinal $\alpha > 0$ the point sets $[0, \omega^\xi[^{(\alpha)}$ and $[0, \omega^\xi]^{(\alpha)}$ coincide, and they contain the point ω^ξ if and only if $\alpha \leq \xi$. And $[0, \omega^\xi]^{(\xi)} = [0, \omega^\xi[^{(\xi)} = \{\omega^\xi\}$.*

REMARK. If κ is a transfinite cardinal then (by Lemma 1)

$$\mathcal{F}_\kappa = \{[0, \omega^\xi] \mid \xi \in \Omega \wedge \kappa \leq \xi < \kappa^+\}$$

is a family of mutually non-homeomorphic compact, scattered linearly ordered spaces of size κ with $|\mathcal{F}_\kappa| = \kappa^+$. But this is not sufficient to prove Theorem 3 (within standard set theory) because (in view of [3, 15.18] and [3, 5.17]) the inequality $\kappa^+ < 2^\kappa$ is consistent with standard set theory for every transfinite cardinal κ.

If X is any Hausdorff space then let

$$X^{(\Omega)} := \bigcap\{X^{(\alpha)} \mid \alpha \in \Omega\} = \bigcup\{A \subset X \mid A \subset A'\}$$

denote the *perfect kernel*, i.e. the maximal dense-in-itself point set in X. (Thus X is scattered if and only if $X^{(\Omega)} = \emptyset$.) Clearly, $X^{(\Omega)}$ is closed, and we have $X^{(\Omega)} = X^{(\alpha)}$ and $(X \setminus X^{(\Omega)})^{(\alpha)} = \emptyset$ for some $\alpha \in \Omega$ (with $\alpha < |X|^+$).

Consequently, the class

$$\Sigma(X) := \{\alpha \in \Omega \mid ((X \setminus X^{(\Omega)})^{(\alpha)} \setminus (X \setminus X^{(\Omega)})^{(\alpha+1)}) \cap X^{(\Omega)} \neq \emptyset\}$$

is a set and $0 \notin \Sigma(X)$. One may regard $\Sigma(X)$ as a sort of *signature set* of the space X since, naturally, two spaces X_1, X_2 cannot be homeomorphic if $\Sigma(X_1) \neq \Sigma(X_2)$.

As usual, a point x in a Hausdorff space is a *condensation point* if and only if every neighborhood of x contains uncountably many or, equivalently, at least $\aleph_1$ points. Let $\mathrm{cp}(X)$ denote the set of all condensation points in X and let $b(X)$ denote the boundary of the point set $\mathrm{cp}(X)$ in the space X. Clearly, $\mathrm{cp}(X)$ is closed, whence $b(X) \subset \mathrm{cp}(X)$. Thus $x \in b(X)$ if and only if every neighborhood of x contains uncountably many points amongst which there is a point y such that some neighborhood of y contains only countably many points.

3. Proof of Theorem 4

Lemma 2. *Let C be a set of size c and let $\mathcal{F}$ be a family of separable Hausdorff spaces such that the underlying sets are all contained in C. If $|\mathcal{F}| > c$ then $\mathcal{F}$ contains a family $\mathcal{G}$ with $|\mathcal{G}| = |\mathcal{F}|$ such that the spaces in $\mathcal{G}$ are mutually non-homeomorphic.*

Proof. Clearly, any homeomorphism between Hausdorff spaces is completely determined by the values at the points of a dense subset of its domain. Naturally, there are precisely c mappings from a countable non-empty set into C. Consequently, if $\mathcal{H}$ is a family of homeomorphic spaces and $\mathcal{H} \subset \mathcal{F}$ then $|\mathcal{H}| \leq c$. Hence the proof is finished by applying a straightforward counting argument.

In order to prove Theorem 4, let κ be a cardinal with $\aleph_0 < \kappa \leq c$. We distinguish two cases: $2^\kappa > c$ and $2^\kappa = c$. Assume firstly that $2^\kappa > c$. In this case it is easy to find a family $\mathcal{G}$ such that $|\mathcal{G}| = 2^\kappa$ and the members of $\mathcal{G}$

are mutually non-homeomorphic dense and totally disconnected subspaces X of $\mathbb{R}$ of size κ. Let D be a countable set of irrational numbers such that D is dense in $\mathbb{R}$, e.g., $D = \{x + \pi \mid x \in \mathbb{Q}\}$. Let

$$\mathcal{F} = \{X \mid \mathbb{Q} \subset X \subset \mathbb{R} \setminus D \wedge |X| = \kappa\}.$$

Then, of course, $|\mathcal{F}| = 2^\kappa$. Since $|\mathcal{F}| > c$, we may apply Lemma 2 for $C = \mathbb{R}$ in order to find an equipollent subfamily $\mathcal{G}$ of $\mathcal{F}$ such that distinct spaces in $\mathcal{G}$ are never homeomorphic.

This proves both the conclusions in Theorems 2 and 4 for all cardinals $\kappa \le c$ with $2^\kappa > c$. To conclude the proof of Theorem 4, we have to settle the case where $\aleph_0 < \kappa \le c$ and $2^\kappa = c$. This is done in the following theorem.

Theorem 6. *For every cardinal κ with $\aleph_0 < \kappa \le c$ there exist c mutually non-homeomorphic totally disconnected and dense subspaces X of $\mathbb{R}$ with $|X| = \kappa$.*

Remark. If $2^{\aleph_1} > c$ then in view of Lemma 2 it is easy to find $2^{\aleph_1}$ mutually non-homeomorphic dense subspaces X of $\mathbb{R}$ such that not only $|X| = \aleph_1$, but also $|X \cap [a,b]| = \aleph_1$ whenever $a < b$. In view of [1] this is not possible if $2^{\aleph_1} = c$.

4. Proof of Theorem 6. Fix $\aleph_0 < \kappa \le c$ and let $\mathbb{N}_u$ be the set of all odd natural numbers, and let L_κ be any subfield of $\mathbb{R}$ with $|L_\kappa| = \kappa$. (For example, let T be a transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$, and define L_κ by adjoining precisely κ numbers from T to $\mathbb{Q}$.) We choose a *field* only to guarantee that $\mathbb{Q} \subset L_\kappa$, and that $L_\kappa \cap [x,y]$ has size κ and is dense in $[x,y]$ whenever $x < y$. Although $2^\kappa > c$ for $\kappa = c$, we do not exclude the case $\kappa = c$ in Theorem 6. (In doing so we make sure that Theorem 6 is not a vacuous statement.) Therefore we also assume that the field L_κ is not equal to $\mathbb{R}$ (or, equivalently, that L_κ is totally disconnected).

For each $n \in \mathbb{N}_u$ choose a compact, countable subset K_n of $[n, n+1] \cap \mathbb{Q}$ with $\min K_n > n$ and $\max K_n = n+1$ so that the naturally ordered set K_n is order-isomorphic to the well-ordered set $[0, \omega^n]$. Then the kth derivative $K_n^{(k)}$ is infinite whenever $k < n$ and empty whenever $k > n$ and $K_n^{(n)} = \{n+1\}$. Let φ be an order-isomorphism from $[0, \omega^n]$ onto K_n.

Define $\mathcal{J}_n$ as the family of all intervals $[\varphi(\alpha), \varphi(\alpha+1)]$ where α runs through all even ordinals smaller than ω^n. (Recall that an ordinal α is *even* if and only if $\alpha = \lambda + n$ where λ is a limit ordinal and n is an even non-negative integer.)

Then $\mathcal{J}_n$ is a family of mutually exclusive compact intervals of positive length so that K_n is the boundary of the point set $\bigcup \mathcal{J}_n$ in the Euclidean space $\mathbb{R}$. (Notice that the only limit point of $\bigcup \mathcal{J}_n$ outside $\bigcup \mathcal{J}_n$ is $n+1$.) Therefore, for the dense subspace $V_n = [n, n+1] \cap (\mathbb{Q} \cup (L_\kappa \cap \bigcup \mathcal{J}_n))$ of the compact Euclidean space $[n, n+1]$ with $|V_n| = \kappa$, we have $\mathrm{cp}(V_n) = (L_\kappa \cap \bigcup \mathcal{J}_n) \cup \{n+1\}$ and $b(V_n) = K_n$.

Let $\mathbb{D}$ denote the classical Cantor ternary set. So $\mathbb{D} \subset [0,1]$ is a compact, nowhere dense subset of the Euclidean space $\mathbb{R}$ with $\min \mathbb{D} = 0$ and with $\max \mathbb{D} = 1$, and the space $\mathbb{D}$ is dense in itself. Hence for the Euclidean space $Y = \mathbb{D} \cup ([0,1] \cap \mathbb{Q})$ we have $b(Y) = \mathrm{cp}(Y) = \mathbb{D}$.

Let $\{I_1, I_2, \ldots\}$ be the (countable) collection of all intervals $[a,b] \subset [0,1]$ with $a, b \in \mathbb{Q}$ and $a < b$ such that $\mathbb{D} \cap \,]a,b[\, \neq \emptyset$. Clearly, we always have $|\mathbb{D} \cap I_k| = c$, and so for each $k \in \mathbb{N}$ we can choose a set $T_k \subset \mathbb{D} \cap I_k$ with $|T_k| = \aleph_1$. Put

$$D_0 := (\mathbb{Q} \cap \mathbb{D}) \cup \bigcup_{k=1}^{\infty} T_k.$$

The set D_0 is a thinned-out modification of the Cantor ternary set such that $|D_0| = \aleph_1$, and in the Euclidean space $\mathbb{R}$ the set D_0 is dense in itself and its closure is the whole set $\mathbb{D}$. Therefore, also for the Euclidean space $Y_0 = D_0 \cup ([0,1] \cap \mathbb{Q})$ (where $|Y_0| = \aleph_1$) we have the essential identities

$$b(Y_0) = \mathrm{cp}(Y_0) = D_0.$$

Now put

$$D_n := \{x + n + 1 \mid x \in D_0\}$$

for every $n \in \mathbb{N}_u$, whence D_n is a shifted version of D_0 and $n + 1 \in D_n \subset [n+1, n+2]$.

Finally, if $\emptyset \neq S \subset \mathbb{N}_u$ then put

$$X[S] := \mathbb{Q} \cup \bigcup_{n \in S} \Big(\Big(L_\kappa \cap \bigcup \mathcal{J}_n \Big) \cup D_n \Big).$$

Each $X[S]$ is a dense, totally disconnected subspace of $\mathbb{R}$ with $|X[S]| = \kappa$ and we always have

$$b(X[S]) = \bigcup_{n \in S} (K_n \cup D_n).$$

Moreover, the perfect kernel of the Euclidean space $b(X[S])$ is given by

$$b(X[S])^{(\Omega)} = b(X[S])^{(\omega)} = \bigcup_{n \in S} D_n.$$

Consequently, $\Sigma(b(X[S])) = S$ for each non-empty set $S \subset \mathbb{N}_u$. (Clearly, the signature set of the space $b(X[S])$ is a subset of $\mathbb{N}$.) So the c spaces $X[S]$ $(\emptyset \neq S \subset \mathbb{N}_u)$ are mutually non-homeomorphic, and this concludes the proof.

5. Proof of Theorem 2. We will prove Theorem 2 in two steps. Assume that $\kappa \geq c$. (This is enough since the case $\kappa \leq c$ is already settled by Theorem 4.) Put $K := \{\alpha \in \Omega \mid \alpha < \kappa\}$, whence $|K| = \kappa$. In the first step we construct for each $S \subset K \setminus \{0\}$ a totally disconnected linearly ordered space Y_S of size κ such that $\Sigma(Y_S) = S$. In the second step we expand each space Y_S to a totally disconnected densely ordered space Z_S such that $|Z_S| = \kappa$ and $b(Z_S) = Y_S$.

Consider the set $K \times (\mathbb{Z} \setminus \mathbb{N})$ equipped with the lexicographic ordering generated by the well-ordering of K and the natural ordering of the integers. (In this ordering (k_1, z_1) is smaller than (k_2, z_2) when either $k_1 = k_2$ and $z_1 < z_2$, or $k_1 < k_2$.) One can say that the linearly ordered set $K \times (\mathbb{Z} \setminus \mathbb{N})$ is built from K by replacing each $\alpha \in K$ with a copy of $\mathbb{Z} \setminus \mathbb{N}$. Naturally, the linearly ordered space $K \times (\mathbb{Z} \setminus \mathbb{N})$ is discrete and of size κ.

Expand the linearly ordered set $K \times (\mathbb{Z} \setminus \mathbb{N})$ to a linearly ordered set Y_S for every subset S of $K \setminus \{0\}$ in the following way.

(i) For each $\xi \in S$ replace the point $(\xi, 0)$ in the linearly ordered set $K \times (\mathbb{Z} \setminus \mathbb{N})$ with a copy D_ξ of the naturally ordered Cantor ternary set $\mathbb{D}$.

(ii) For each $\xi \in S$ replace the point $(\xi, -1)$ (which is the predecessor of $(\xi, 0)$ in $K \times (\mathbb{Z} \setminus \mathbb{N})$) with a copy W_ξ of the well-ordered set $[0, \omega^\xi[$.

By construction, W_ξ has no maximum and $\sup W_\xi = \min D_\xi$, and hence $W_\xi^{(\xi)} = \{\min D_\xi\}$ for each $\xi \in S$. Clearly, $|Y_S| = \kappa$. The mutually disjoint sets $W_\xi \cup D_\xi (\xi \in S)$ are closed and the subspace $Y_S \setminus \bigcup\{W_\xi \cup D_\xi \mid \xi \in S\}$ is open and discrete, and each D_ξ is dense in itself and closed. Consequently, $Y_S^{(\Omega)} = \bigcup_{\xi \in S} D_\xi$ and hence $\Sigma(Y_S) = S$ for each $S \subset K \setminus \{0\}$. (Notice that $Y_S = K \times (\mathbb{Z} \setminus \mathbb{N})$ if $S = \emptyset$.)

Now in order to conclude the proof, let $\mathbb{K} \neq \mathbb{R}$ be a subfield of $\mathbb{R}$ with $|\mathbb{K}| = c$. In the linearly ordered set Y_S we replace D_ξ with a copy of $\mathbb{D} \cup ([0,1] \cap \mathbb{Q})$ for every $\xi \in S$ and then, by using even and odd ordinals, in an alternating way we fill the vacuum between every remaining pair of consecutive points with copies of $\mathbb{Q}$ and $\mathbb{K}$, respectively, and clearly we can do this so that a totally disconnected densely ordered space Z_S of size κ is created where Y_S is a subspace of Z_S and $b(Z_S) = Y_S$.

Remark. It is essential that all building blocks $W_\xi \cup D_\xi$ in the linearly ordered set Y_S have copies of $\mathbb{Z} \setminus \mathbb{N}$ as discrete buffers on the left—otherwise it could happen that $S \neq \Sigma(Y_S)$ and also that $b(Z_S) \neq Y_S$. (For example, if in the definition of Y_S the basic set $K \times (\mathbb{Z} \setminus \mathbb{N})$ is replaced by $K \times \{-1, 0\}$ then for $S = \mathbb{N} \cup \{\omega^2\}$ we have $\omega \in \Sigma(Y_S)$ but $\omega \notin S$.)

6. Proof of Theorem 3. It is appropriate to distinguish between *regular* and *singular* cardinal numbers. Singular cardinals are those which are not regular. A cardinal κ is *regular* if and only if $\sup A < \kappa$ whenever $\emptyset \neq A \subset [0, \kappa[$ and $|A| < \kappa$. Topologically speaking, a cardinal κ is regular if and only if in the compact linearly ordered space $[0,\kappa]$ the first derivative of a point set A with $|A| < \kappa$ never contains κ. For example, $\aleph_0$ and $\aleph_1$ are regular. Note that κ^+ is regular for every cardinal κ (cf. [3, 5.3]).

Let X be a scattered Hausdorff space and $\kappa > \aleph_0$ be a regular cardinal number. Let us call a point $x \in X$ a *κ-condensation point* if and only if $|U| \geq \kappa$ for every neighborhood U of x and $|U| = \kappa$ for some neighborhood U of x. Let $C_\kappa(X)$ denote the set of all κ-condensation points. (For example, $C_\kappa([0,\kappa]) = \{\kappa\}$.) For $x \in X$ let $\Omega_\kappa(x)$ denote the class of all ordinals α such that there exists a point set $A \subset X$ with $|A| < \kappa$ and $x \in A^{(\alpha)}$. Since X is scattered, for every $x \in X$ the class $\Omega_\kappa(x)$ is a set and, moreover, $\Omega_\kappa(x) \subset [0,\kappa[$ (because $A^{(\kappa)} = \emptyset$ whenever $A \subset X$ and $|A| < \kappa$). The set $\Omega_\kappa(x)$ is never empty since, trivially, $0 \in \Omega_\kappa(x)$ for every $x \in X$. So we may define a signature set with respect to the scattered space X and the regular cardinal κ by

$$\Sigma[X,\kappa] := \{\sup \Omega_\kappa(x) \mid x \in C_\kappa(X)\}.$$

Clearly, two scattered spaces X_1, X_2 cannot be homeomorphic if $\Sigma[X_1,\kappa] \neq \Sigma[X_2,\kappa]$ for some regular cardinal κ. If $\kappa > \aleph_0$ is a regular cardinal then $\Sigma[[0,\kappa],\kappa] = \{0\}$ and, more generally in view of the following lemma, $\Sigma[[0,\beta],\kappa] \subset \{0,\kappa\}$ for every $\beta \in \Omega$. (The case $\Sigma[[0,\beta],\kappa] = \{0,\kappa\}$ may occur, for example if $\kappa = \aleph_1$ and $\beta = \omega_1 \cdot \omega$.)

LEMMA 3. *Let $\kappa > \aleph_0$ be a regular cardinal. For $\beta \in \Omega$ consider the space $X = [0,\beta]$. If $\gamma \in C_\kappa(X)$ then either $\Omega_\kappa(\gamma) = [0,\kappa[$ or $\Omega_\kappa(\gamma) = \{0\}$.*

Proof. Since $\gamma \in C_\kappa(X)$, if $\alpha_1 < \gamma$ and $\alpha_2 \in \Omega$ and $|[\alpha_1,\alpha_2]| < \kappa$ then $[\alpha_1,\alpha_2] \subset [0,\gamma[$. Clearly, if $\sup A \neq \gamma$ whenever $\emptyset \neq A \subset [0,\gamma[$ and $|A| < \kappa$ then $\Omega_\kappa(\gamma) = \{0\}$. So assume that there is a non-empty set $A \subset [0,\gamma[$ such that $|A| < \kappa$ and $\sup A = \gamma$. For $\xi \in \Omega$ put $U_\xi := \bigcup_{\alpha\in A}[\alpha,\alpha+\omega^\xi]$. If $\xi < \kappa$ then $|[\alpha,\alpha+\omega^\xi]| = |[0,\omega^\xi]| < \kappa$ for every $\alpha \in A$; hence $U_\xi \subset [0,\gamma[$ and so $\sup U_\xi = \sup A = \gamma$. Thus $|U_\xi| < \kappa$ and (by Lemma 1) $\gamma \in U_\xi^{(\xi)}$ for every $\xi < \kappa$, and hence $\Omega_\kappa(\gamma) = [0,\kappa[$, completing the proof.

If $(X,\prec)$ is a linearly ordered set then put $[a,b]_\prec := \{x \in X \mid a \preceq x \preceq b\}$ and $]a,b[_\prec = [a,b]_\prec \setminus \{a,b\}$ whenever $a,b \in X$. Furthermore, in the usual sloppy way, if A is a set of ordinals then let A^* be the set A equipped with the backwards linear ordering of the canonical well-ordering of Ω. (In other words, if α and β are elements of the linearly ordered set A^* then α is *smaller* than β if and only if for the ordinal numbers α,β in the well-ordered class Ω we have $\beta < \alpha$.)

LEMMA 4. *Let $(X,\prec)$ be a linearly ordered set equipped with the order topology and assume that the space X is scattered. Let $0 \neq \xi \in \Omega$ and let κ be a regular cardinal number with $\kappa > |\omega^\xi|$. Let x,y,z be three points in X*

with $x \prec z \prec y$ *so that* $[x,z]_\prec$ *is order-isomorphic to* $[0,\omega^\xi]$ *and* $[z,y]_\prec$ *is order-isomorphic to* $[0,\kappa]^*$. *Then* $C_\kappa(X) \cap\,]x,y[_\prec = \{z\}$ *and* $\Omega_\kappa(z) = [0,\xi]$.

Proof. Clearly, z is the only κ-condensation point of X strictly between x and y. Since κ is regular and $[z,y]_\prec$ is order-isomorphic to $[0,\kappa]^*$, there is no set $A \subset [z,y]_\prec$ with $|A| < \kappa$ and $z \in A'$. Therefore, if $0 \neq \alpha \in \Omega$ and $z \in A^{(\alpha)}$ for a point set A in the space X with $|A| < \kappa$ then we already have $z \in (A \cap [x,z]_\prec)^{(\alpha)}$. On the other hand, $([x,z]_\prec)^{(\xi)} = \{z\}$ by Lemma 1 and $|[x,z]_\prec| = |[0,\omega^\xi]| < \kappa$. Consequently, $\Omega_\kappa(z) = [0,\xi]$, completing the proof.

Now to prove Theorem 3 let $\kappa > \aleph_0$ be a cardinal and put $L = [\omega,\kappa]$. Let $\mathcal{G}$ be the family of all non-empty sets S of successor ordinals $\alpha + 1$ where α is a limit ordinal in $L \setminus \{\kappa\}$. So if $\xi \in S \in \mathcal{G}$ then $|\xi| = |[0,\omega^\xi]| < \kappa$. Clearly, $|\mathcal{G}| = 2^\kappa$. For every $S \in \mathcal{G}$ let

$$H_S := L \times \{0\} \cup \bigcup_{\xi \in S} (\{\xi\} \times [0,\omega^\xi] \cup \{\xi+1\} \times [0,\kappa[^*)$$

and

$$G_S := L \times \{0\} \cup \bigcup_{\xi \in S} (\{\xi\} \times [0,\omega^\xi] \cup \{\xi+1\} \times [0,|\xi|^+[^*)$$

be equipped with the lexicographic ordering. One can say that the linearly ordered set H_S resp. G_S is constructed from the well-ordered set L by replacing ξ with a copy of $[0,\omega^\xi]$ and $\xi+1$ with a copy of $[0,\kappa[^*$ resp. $[0,|\xi|^+[^*$ for each $\xi \in S$.

Then the corresponding linearly ordered spaces H_S and G_S are of size κ and it is evident that all these spaces are scattered. They are also compact since the ordering is complete with a maximum and a minimum (cf. [8, 39.7]). We claim that the spaces H_S $(S \in \mathcal{G})$ are mutually non-homeomorphic if κ is regular, and the spaces G_S $(S \in \mathcal{G})$ are mutually non-homeomorphic if κ is singular.

Assume firstly that κ is regular, let $S \in \mathcal{G}$ and consider the space H_S. Clearly, $(\kappa,0) \in C_\kappa(H_S)$ and $\Omega_\kappa((\kappa,0)) = \{0\}$. Obviously, $(\gamma,0) \in C_\kappa(H_S)$ if and only if $\gamma = \kappa$ or $\gamma = \sup(S \cap [0,\gamma[)$ where $S \cap [0,\gamma[\, \neq \emptyset$. If $(\kappa,0) \neq (\gamma,0) \in C_\kappa(H_S)$ then $\Omega_\kappa((\gamma,0)) = [0,\kappa[$ and hence $\sup \Omega_\kappa((\gamma,0)) = \kappa$, because if $\xi \in S \cap [0,\gamma[$ and $\alpha < \kappa$ then $\{\xi+1\} \times [0,\omega^\alpha]^* \subset \{\xi+1\} \times [0,\kappa[^*$ and

$$\Big|\bigcup\{\{\xi+1\} \times [0,\omega^\alpha]^* \mid \xi \in S \cap [0,\gamma[\}\Big| < \kappa$$

for arbitrarily large exponents $\alpha < \kappa$. In view of Lemma 4, $C_\kappa(H_S) \setminus L \times \{0\} = \{(\xi,\omega^\xi) \mid \xi \in S\}$ and $\sup \Omega_\kappa((\xi,\omega^\xi)) = \xi$ for every $\xi \in S$. Therefore we have

$$S = \Sigma[H_S,\kappa] \setminus \{0,\kappa\}$$

for every $S \in \mathcal{G}$ and this proves Theorem 3 for regular $\kappa > \aleph_0$.

Assume now that κ is a singular cardinal and let $\mathcal{R}$ denote the set of all regular uncountable cardinals smaller than κ. (Notice that $|\xi|^+ \in \mathcal{R}$ whenever $\omega \leq \xi < \kappa$.) We claim that every $S \in \mathcal{G}$ is completely determined by the topology of G_S via

$$S = \Big(\bigcup_{\lambda \in \mathcal{R}} \varSigma[G_S, \lambda] \Big) \setminus (\{0\} \cup \mathcal{R}).$$

On the one hand, if $\xi \in S$ then $\xi \neq 0$, $\xi \notin \mathcal{R}$, $|\xi|^+ \in \mathcal{R}$ and (ξ, ω^ξ) is a $|\xi|^+$-condensation point in G_S with $\sup \varOmega_{|\xi|^+}((\xi, \omega^\xi)) = \xi$ in view of Lemma 4.

On the other hand, let y be a λ-condensation point in G_S where $\lambda \in \mathcal{R}$, and assume firstly that $y \notin L \times \{0\}$. Then y lies in

$$B_\xi := \{\xi\} \times [0, \omega^\xi] \cup \{\xi + 1\} \times [0, |\xi|^+[^*$$

for some $\xi \in S$. Since the points $\min B_\xi = (\xi, 0)$ and $\max B_\xi = (\xi + 1, 0)$ are isolated in the space G_S, the point y must be a λ-condensation point in the space B_ξ, whence $\lambda \leq |B_\xi| = |\xi|^+$. In the case $\lambda = |\xi|^+$ we must have $y = (\xi, \omega^\xi)$ and hence $\sup \varOmega_\lambda(y) = \xi \in S$ by Lemma 4. In the case $\lambda < |\xi|^+$, the point y must be the maximum resp. minimum of a copy of $[0, \gamma]$ resp. $[0, \gamma]^*$ within the linearly ordered set B_ξ while γ is a λ-condensation point in the space $[0, \gamma]$, whence $\sup \varOmega_\lambda(y) \in \{0, \lambda\}$ by Lemma 3.

Assume secondly that $y = (x, 0)$ for $x \in L$. If x is a λ-condensation point in the basic space L then $\sup \varOmega_\lambda(x) \in \{0, \lambda\}$ in the space L and, clearly, $\sup \varOmega_\lambda(y) \in \{0, \lambda\}$ in the space G_S as well. If $x \notin C_\lambda(L)$ then $y \in C_\lambda(G_S)$ forces x to be the supremum of a set $\tilde{S} \subset \{\xi \in S \mid \xi < x \wedge |\xi|^+ = \lambda\}$ with $|\tilde{S}| < \lambda$, and therefore (by the same argument as for the space H_S) we must have $\varOmega_\lambda(y) = [0, \lambda[$ and hence $\sup \varOmega_\lambda(y) = \lambda$. So in any case the ordinal $\sup \varOmega_\lambda(y)$ lies in $S \cup \{0, \lambda\}$ if $y \in C_\lambda(G_S)$ for $\lambda \in \mathcal{R}$.

Remark. It is not pure chance that the size and weight of each space H_S resp. G_S coincide. Actually, if X is a scattered linearly ordered space of weight λ then $\lambda = |X|$. (Trivially, $\lambda \leq |X|$. If $\tilde{X}$ is the set of all $x \in X$ such that $|U| > \lambda$ for every neighborhood U of x then from the assumption $\lambda < |X|$ we conclude that $|X \setminus \tilde{X}| \leq \lambda$ and hence the point set $\tilde{X}$ is both non-empty and dense in itself, whence X is not scattered.)

7. Proof of Theorem 5. First of all, if $\mathcal{F}$ is a family of mutually non-homeomorphic compact Hausdorff spaces of weight κ then $|\mathcal{F}| \leq 2^\kappa$, because each space $X \in \mathcal{F}$ is homeomorphic to a closed subspace of the Hilbert cube $[0, 1]^\kappa$ (cf. [2, 3.2.5]) and, naturally, the compact space $[0, 1]^\kappa$ contains precisely 2^κ closed sets. So to prove Theorem 5 it is enough to exhibit 2^κ mutually non-homeomorphic separable, dense-in-itself, compact linearly ordered spaces of weight κ for every transfinite cardinal $\kappa \leq c$.

Again we distinguish two cases: $2^\kappa > c$ and $2^\kappa = c$. The (only) two cardinals for which we can decide which case actually occurs are $\aleph_0$ and c, since $2^{\aleph_0} = c$ and $2^c > c$. The conclusion of Theorem 5 for $\kappa = \aleph_0$ is covered by the following theorem. (Note that if X is a closed subspace of $\mathbb{R}$ then the order topology of the naturally ordered set X equals the Euclidean subspace topology of X.)

THEOREM 7. *The Euclidean space $\mathbb{R}$ contains c mutually non-homeomorphic compact subspaces which are dense in itself* (*and hence of size c*).

Proof. For $n \in \mathbb{N}_u$ let $K_n \subset [n, n+1]$ and $\mathcal{J}_n$ be as in the proof of Theorem 6. Let $h(x) = (2/\pi)\arctan x$, whence h is a strictly increasing function which maps $[0,\infty[$ onto $[0,1[$. For every infinite $S \subset \mathbb{N}_u$ consider the compact and dense-in-itself Euclidean space

$$A_S := h\Big(\bigcup_{n\in S}\Big(\{n+1\}\cup\bigcup\mathcal{J}_n\Big)\Big)\cup\{1\}.$$

Then for the c infinite subsets S of $\mathbb{N}_u$ the corresponding spaces A_S are mutually non-homeomorphic because a moment's reflection suffices to see that $\Sigma^*(A_S)\setminus\{\omega\} = S$ always holds when the signature set $\Sigma^*(X)$ of any Hausdorff space X is defined via

$$\Sigma^*(X) := \{\alpha\in\Omega \mid (\rho(X)^{(\alpha)}\setminus\rho(X)^{(\alpha+1)})\cap\delta(X)\neq\emptyset\},$$

where $\delta(X)$ is the set of all points $x \in X$ such that $\{x\}$ is a component of the space X, and $\rho(X)$ is the set of all points $x \in X$ such that $x \in Z$ for some component Z of X with $Z\setminus\{x\}$ non-empty and connected. (Notice that $\rho(A_S) = \bigcup_{n\in S} h(K_n\setminus\{n+1\})$ and $\delta(A_S) = \{h(n+1) \mid n\in S\}\cup\{1\}$.)

Now to prove Theorem 5 for uncountable weights assume firstly that $\aleph_0 < \kappa \le c$ and $2^\kappa > c$. By applying Lemma 2 it is enough to construct 2^κ separable, dense-in-itself, compact linearly ordered spaces of weight κ whose underlying sets are contained in $C = [0,1]\times\{0,1\}$. Let $\mathcal{Y}_\kappa$ denote the family of all sets $Y \subset]0,1[$ such that $|Y| = \kappa$. Clearly, $|\mathcal{Y}_\kappa| = 2^\kappa$. For each $Y \in \mathcal{Y}_\kappa$ consider the set

$$K[Y] := ([0,1]\setminus Y)\times\{0\} \;\cup\; Y\times\{0,1\}$$

equipped with the lexicographic ordering. (Let $\prec$ denote this ordering.) One can say that $K[Y]$ is constructed from the unit interval $[0,1]$ by *splitting* each point in Y *in two.*

Each non-empty subset of $K[Y]$ has a supremum and an infimum with respect to $\prec$. (Indeed, if $\emptyset \neq A \subset K[Y]$ then $\sup A$ equals $(a,0)$ or $(a,1)$, where a is the supremum of the projection of A into the number line.) Consequently (cf. [8, 39.7]), the linearly ordered space $K[Y]$ is compact. Clearly, $([0,1]\cap\mathbb{Q})\times\{0\}$ is a dense subset of $K[Y]$, whence $K[Y]$ is separable.

Since $0, 1 \notin Y$, the space $K[Y]$ has no isolated points, i.e. $K[Y]$ is dense in itself.

Finally, we claim that the weight of $K[Y]$ is $|Y| = \kappa$. Indeed, if $\mathcal{B}$ is a basis of $K[Y]$ then for every $y \in Y$ we may choose $B_y \in \mathcal{B}$ disjoint from $\{x \in K[Y] \mid (y, 1) \preceq x\}$ with $(y, 0) \in B_y$, whence $B_y \neq B_{y'}$ for distinct $y, y' \in Y$ and therefore $|\mathcal{B}| \geq \kappa$. And the rays $\{x \in K[Y] \mid x \prec (r, 0)\}$ and $\{x \in K[Y] \mid (z, 0) \prec x\}$ and $\{x \in K[Y] \mid x \prec (y, 1)\}$, where $r \in \mathbb{Q} \cap [0, 1]$ and $z \in Y \cup (\mathbb{Q} \cap [0, 1])$ and $y \in Y$, form a subbasis of $K[Y]$, and hence there exists a basis $\mathcal{B}$ with $|\mathcal{B}| = \kappa$.

Now to conclude the proof of Theorem 5 assume that $\aleph_0 < \kappa < c$ and $2^\kappa = c$. Let $\mathcal{A}$ denote the family of all spaces $A_S \subset [0, 1]$ from the proof of Theorem 7. Trivially, the corresponding subspaces $\tilde{A}_S := A_S \times \{0\}$ of the Euclidean plane $\mathbb{R}^2$ are mutually non-homeomorphic. For each $A_S \in \mathcal{A}$ choose a set $Y_S \subset]0, 1[$ with $A_S \cap Y_S = \emptyset$ and $|Y_S| = \kappa$ such that Y_S is a dense subset of the Euclidean open set $]0, 1[\setminus A_S$. For every $A_S \in \mathcal{A}$ consider the separable, dense-in-itself, compact linearly ordered space $K[Y_S]$ whose weight is $|Y_S| = \kappa$. Obviously, each $\tilde{A}_S$ is not only a subspace of $\mathbb{R} \times \{0\}$ but also a subspace of the linearly ordered space $K[Y_S]$. Since Y_S is a dense subset of $]0, 1[\setminus A_S$, the non-singleton components of $K[Y_S]$ are precisely the non-singleton components of $\tilde{A}_S$. So if U_S is the union of all non-singleton components of $K[Y_S]$ then $U_S = h(\bigcup_{n \in S}(\bigcup \mathcal{J}_n)) \times \{0\}$. Since the closure of U_S in the Euclidean plane $\mathbb{R}^2$ is $\tilde{A}_S$, it is evident that $\tilde{A}_S$ is the closure of U_S in the linearly ordered space $K[Y_S]$ as well. Thus for each $A_S \in \mathcal{A}$ the space $\tilde{A}_S$ can be recovered from $K[Y_S]$, and hence the $2^\kappa = c$ spaces $K[Y_S]$ are mutually non-homeomorphic.

Remark. Obviously, each space in the family $\mathcal{Q} := \{K[Y] \mid]0, 1[\cap \mathbb{Q} \subset Y \subset]0, 1[\}$ is totally disconnected. So $\mathcal{Q}$ contains 2^c mutually non-homeomorphic *separable and first countable*, totally disconnected, dense-in-itself compact Hausdorff spaces. On the other hand (cf. [2, 6.2.A.c, 6.2.9]), any *second countable*, totally disconnected, dense-in-itself compact Hausdorff space is homeomorphic to the Cantor ternary set $\mathbb{D}$. (For example, the c spaces $K[Y]$ in the family $\mathcal{Q}$ where Y is countable are all homeomorphic to $\mathbb{D}$.)

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