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TIME ANALYTICITY AND BACKWARD UNIQUENESS FOR THE BOUSSINESQ EQUATIONS

BY

QIANQIAN HOU and XIAOJING XU (Beijing)

Abstract. We prove that strong solutions of the Boussinesq equations in 2D and 3D can be extended as analytic functions of complex time. As a consequence we obtain the backward uniqueness of solutions.

1. Introduction. We consider the Boussinesq equations

$$(B)\begin{cases}\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\nu\Delta v+\theta e_n, & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial\theta}{\partial t}+(v\cdot\nabla)\theta=\kappa\Delta\theta, \\ \operatorname{div} v=0, \\ v(x,0)=v_0(x),\quad \theta(x,0)=\theta_0(x), \\ v(\cdot,t)|_{\partial\Omega}=0,\quad \theta(\cdot,t)|_{\partial\Omega}=0,\end{cases}$$

where $v:\mathbb{R}^n\to\mathbb{R}^n$ $(n=2,3)$ is a vector field corresponding to the velocity, $\theta:\mathbb{R}^n\to\mathbb{R}$ is a scalar function denoting the temperature in the context of thermal convection and the density in modeling geophysical fluids. We assume that $\Omega\subset\mathbb{R}^n$ is an open bounded domain with $\partial\Omega$ of class C^2. In problem (B), the viscosity ν and the diffusion coefficients κ are both positive constants and $e_n=(0,\dots,1)$ denotes the unit vector in $\mathbb{R}^n$.

The Boussinesq equations concerned here model large-scale atmospheric and oceanic flows, and also play important roles in the study of Rayleigh–Bénard convection (see, e.g., [6]). These equations retain some key features of the 3D Navier–Stokes equations and the Euler equations such as the vortex stretching mechanism. As pointed out in [4], the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows.

In the 2D case, the global in time regularity of solutions to problem (B) with $\nu>0$ and $\kappa>0$ is well-known (see [1], [5], [9]); in the 3D case, the local in time regularity of this problem is also known (see [7]). In this pa-

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per, we contribute to these theories by considering the analyticity in time of solutions to the Boussinesq problem (B) and, to do it, we use the method developed by Foiaş and Temam [3] who dealt with the Navier–Stokes equations. We prove that strong solutions of the Boussinesq problem (B) in 2D and 3D can be extended as analytic functions of complex time; as a consequence we obtain the backward uniqueness of solutions. Compared with the Navier–Stokes system, in our case, we have to deal with an additional difficulty: we need uniform estimates on both velocity and temperature at the same time.

In order to study our problem, we first apply the Leray projector $\mathbb{P}$ to the first equation of (B), and obtain the following system:

$$(B_1)\begin{cases} \dfrac{\partial v}{\partial t} + \nu A v + B(v,v) = \mathbb{P}\theta e_n, \\ \dfrac{\partial \theta}{\partial t} + (v\cdot\nabla)\theta = \kappa\Delta\theta, \\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \\ v(\cdot,t)|_{\partial\Omega} = 0, \quad \theta(\cdot,t)|_{\partial\Omega} = 0, \end{cases}$$

with the bilinear operator

$$B(u,v) = \mathbb{P}(u\cdot\nabla)v,$$

the Stokes operator $A = \mathbb{P}(-\Delta) : \mathcal{D}(A) \to H$, and the spaces

$$H = \mathbb{P}L^2(\Omega), \qquad \mathcal{D}(A) = H^2(\Omega)\cap V(\Omega),$$

where

$$V(\Omega) = \{u \in H_0^1(\Omega)^n \mid \operatorname{div} u = 0\}.$$

We refer the readers to the book by Constantin and Foiaş [2] for more details.

We are now in a position to formulate the main results of this work. First, in Theorem 1.1, we show that strong solutions in the 2D case are analytic in a complex neighborhood of the real half-line $(0,\infty)$, and solutions in the 3D case are analytic in a neighborhood of $(0,T_0)$ for some $T_0\in(0,\infty)$. As a consequence, in Theorem 1.2 we derive the backward uniqueness of solution.

Theorem 1.1. *Let $v_0\in V(\Omega)$, $\theta_0\in H_0^1(\Omega)$, and $\nu,\kappa>0$.*

(i) *If $n=2$, there exists an open neighborhood D of $(0,\infty)$ in $\mathbb{C}$ such that the solution $(v(t),\theta(t))$ to the Boussinesq problem (B) is analytic as the mappings $v: D\to\mathcal{D}(A)$, $\theta: D\to H^1(\Omega)$.*

(ii) *If $n=3$, there exist $T_0>0$ and an open neighborhood D_{T_0} of $(0,T_0)$ in $\mathbb{C}$ such that the solution of the Boussinesq problem (B) is analytic as the mappings $v: D_{T_0}\to\mathcal{D}(A)$, $\theta: D_{T_0}\to H^1(\Omega)$.*

REMARK 1.1. For $n = 3$, the interval $[0, T_0)$ in Theorem 1.1 is the maximal one on which the strong solutions exist.

THEOREM 1.2 (Backward uniqueness). *Let (v_1, θ_1), (v_2, θ_2) be two strong solutions of the Boussinesq problem (B).*

(i) *In the 2D case: Assume that the initial data $v_1(0)$, $v_2(0)$ are in $V(\Omega)$ and $\theta_1(0)$, $\theta_2(0)$ are in $H_0^1(\Omega)$. Suppose there exists $t_0 \geq 0$ such that $(v_1(t_0), \theta_1(t_0)) = (v_2(t_0), \theta_2(t_0))$. Then $(v_1(t), \theta_1(t)) = (v_2(t), \theta_2(t))$ for all $t \geq 0$.*

(ii) *In the 3D case: Assume that the initial data $v_1(0)$, $v_2(0)$ are in $V(\Omega)$ and $\theta_1(0)$, $\theta_2(0)$ are in $H_0^1(\Omega)$. Set $T_0 = \min(T_1, T_2)$, where $[0, T_i)$ is the existence interval of (v_i, θ_i), $i = 1, 2$. If, for some $t_0 \in [0, T_0)$, $(v_1(t_0), \theta_1(t_0)) = (v_2(t_0), \theta_2(t_0))$ then $(v_1(t), \theta_1(t)) = (v_2(t), \theta_2(t))$ for all $t \in [0, T_0)$.*

2. Preliminaries. In this section, we list some notations, and present multilinear product estimates used in the proofs of the main theorems.

In order to extend system (B_1) to complex time t, we need to complexify the spaces H, V, $\mathcal{D}(A)$ and the corresponding operators. The complexification of H is the Hilbert space

$$H_{\mathbb{C}} = \{u_1 + iu_2 \,|\, u_1 \in H,\, u_2 \in H\}, \qquad i = \sqrt{-1},$$

with the scalar product

$$(u, v)_{\mathbb{C}} = (u_1 + iu_2, v_1 + iv_2)_{\mathbb{C}} = (u_1, v_1) + (u_2, v_2) + i[(u_2, v_1) - (u_1, v_2)].$$

Other spaces and corresponding operators will be defined in the analogous way. Moreover, for simplicity of exposition, we omit the subscript $\mathbb{C}$.

Following the usual practice, we denote the norm of $H_0^1(\Omega)$ by $\|\cdot\|$, the norm of $L^2(\Omega)$ by $|\cdot|$; $(\cdot\,,\cdot)$ is the scalar product in complex $L^2(\Omega)$ and $((\cdot\,,\cdot))$ is the scalar product in complex $H_0^1(\Omega)$.

Finally, we recall two classical lemmas, which can be obtained from the Schwarz inequality and the Gagliardo–Nirenberg inequality (also see [2, pp. 49 and 55]).

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2, $n = 2, 3$. There exists a constant C depending on Ω such that, for all $u \in H_0^1(\Omega)$ and $v \in H^2(\Omega)$,*

$$|b(u, v, w)| \leq C\|u\|\,\|v\|^{1/2}\|v\|_{H^2}^{1/2}|w|,$$

where

$$b(u, v, w) = \int_{\Omega} u_j \frac{\partial v_i}{\partial x_j} w_i \, dx.$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2, $n = 2, 3$. There exists a constant C depending on Ω such that, for all $u \in H_0^1(\Omega)$ and $v \in H^2(\Omega)$,*

$$|B(u,v)| \leq C\|u\|\,\|v\|^{1/2}\|v\|_{H^2}^{1/2}.$$

Remark 2.1. By Lemma 2.1, for any $w \in L^2(\Omega)$,

$$|(u \cdot \nabla v, w)| = |b(u,v,w)| \leq C\|u\|\,\|v\|^{1/2}\|v\|_{H^2}^{1/2}|w|,$$

hence,

$$|u \cdot \nabla v| \leq C\|u\|\,\|v\|^{1/2}\|v\|_{H_2}^{1/2}.$$

3. Proof of Theorems 1.1 and 1.2. First, we prove Theorem 1.1. Then Theorem 1.2 is obtained as its direct consequence.

In the following proof, it suffices to consider problem (B_1) by the classical existence theory (see [1], [5], [7], [9]).

We use the Galerkin method to construct the approximation system of (B_1) as follows:

$$(B_2)\begin{cases} \dfrac{\partial v_m}{\partial t} + \nu A v_m + B(v_m, v_m) = \mathbb{P}\theta_m e_n, \\ \dfrac{\partial \theta_m}{\partial t} + (v_m \cdot \nabla)\theta_m = \kappa \Delta \theta_m, \\ v_m(x,0) = v_{0m}(x), \quad \theta_m(x,0) = \theta_{0m}(x), \\ v_m(\cdot,t)|_{\partial\Omega} = 0, \quad \theta_m(\cdot,t)|_{\partial\Omega} = 0, \end{cases}$$

where

$$v_m = \sum_{k=1}^{m} c_k^m(t)\omega^k(x), \qquad \theta_m = \sum_{k=1}^{m} d_k^m(t)\bar{\omega}^k(x),$$

$\omega^k(x)$ is an orthonormal basis in $\mathcal{D}(A)$, and $\bar{\omega}^k$ is an orthonormal basis in H_0^1.

The solution (v_m, θ_m) of the Galerkin approximation system (B_2) is obviously analytic in time, because (B_2) is a finite-dimensional system with a polynomial nonlinearity.

Proof of Theorem 1.1. The proof is divided into three steps. First, we obtain estimates of $\|v_m(se^{i\varphi})\|$ and $\|\theta_m(se^{i\varphi})\|$; then we give uniform estimates of higher order derivatives of v_m, θ_m; finally we take the limit as $m \to \infty$ to achieve our goals.

Step 1. Let $\varphi \in (-\pi/4, \pi/4)$ and take the time variable of the form $t = se^{i\varphi}$ for $s > 0$. Since the Stokes operator A is selfadjoint (see [2, p. 32, Th. 4.3]), we have the identity

$$\frac{d}{ds}\|v_m(se^{i\varphi})\|^2 = \frac{d}{ds}(v_m(se^{i\varphi}), Av_m(se^{i\varphi})),$$

which implies

$$\frac{1}{2}\frac{d}{ds}\|v_m(se^{i\varphi})\|^2 = \frac{1}{2}\left(e^{i\varphi}\frac{dv_m}{dt}, Av_m\right) + \frac{1}{2}\left(v_m, e^{i\varphi}A\frac{dv_m}{dt}\right)$$
$$= \operatorname{Re} e^{i\varphi}\left(\frac{dv_m}{dt}, Av_m\right).$$

Multiplying the first equation of (B_2) by $Av_m e^{i\varphi}$, integrating, and taking the real part, we deduce by the above identity that

$$\frac{1}{2}\frac{d}{ds}\|v_m(se^{i\varphi})\|^2 + \nu\cos\varphi|Av_m(se^{i\varphi})|^2$$
$$= \operatorname{Re} e^{i\varphi}\left(\frac{\partial v_m}{\partial t}, Av_m\right) + \nu\cos\varphi|Av_m(se^{i\varphi})|^2$$
$$= -\operatorname{Re} e^{i\varphi}(B(v_m, v_m), Av_m) + \operatorname{Re} e^{i\varphi}(\theta_m e_n, Av_m).$$

By Lemma 2.1, we obtain

$$|\operatorname{Re} e^{i\varphi}(B(v_m, v_m), Av_m)| \le C\|v_m\|^{3/2}\|v_m\|_{H^2}^{1/2}|Av_m|.$$

Thus, using the inequality $\|v_m\|_{H^2} \le C|Av_m|$ (see [2, p. 36, Prop. 4.7] this inequality is still valid in the complex case), and the Young inequality, we have

$$|\operatorname{Re} e^{i\varphi}(B(v_m, v_m), Av_m)| \le C\|v_m\|^{3/2}|Av_m|^{3/2}$$
$$\le \frac{\nu\cos\varphi}{4}|Av_m|^2 + \frac{C}{\nu^3\cos^3\varphi}\|v_m\|^6.$$

Similarly, we get

$$|\operatorname{Re} e^{i\varphi}(\theta_m e_n, Av_m)| \le \frac{\nu\cos\varphi}{4}|Av_m|^2 + \frac{|\theta_m|^2}{\nu\cos\varphi}.$$

Hence,

$$\frac{d}{ds}\|v_m\|^2 + \nu\cos\varphi\,|Av_m|^2 \le \frac{2|\theta_m|^2}{\nu\cos\varphi} + \frac{C}{\nu^3\cos^3\varphi}\|v_m\|^6.$$

Multiplying scalarly the second equation of (B_2) by $(-\Delta)\theta_m$ and by $e^{i\varphi}$, and taking the real part, we get

$$\frac{1}{2}\frac{d}{ds}\|\theta_m\|^2 + \kappa\cos\varphi\,|\Delta\theta_m|^2 = \operatorname{Re} e^{i\varphi}(v_m\cdot\nabla\theta_m, \Delta\theta_m).$$

Using Lemma 2.1, the inequality $\|\theta_m\|_{H^2} \le C|\Delta\theta_m|$, and the Young inequality we have

$$|\operatorname{Re} e^{i\varphi}(v_m\cdot\nabla\theta_m, \Delta\theta_m)| \le \|v_m\|\,\|\theta_m\|^{1/2}\|\theta_m\|_{H^2}^{1/2}|\Delta\theta_m|$$
$$\le C\left(\frac{\|v_m\|\,\|\theta_m\|^{1/2}}{(\kappa\cos\varphi)^{3/4}}\right)^4 + \delta\left(\left(\frac{\kappa}{2}\cos\varphi\right)^{1/4}\|\theta_m\|_{H^2}^{1/2}\right)^4 + \left(\left(\frac{\kappa}{2}\cos\varphi\right)^{1/2}|\Delta\theta_m|\right)^2$$

$$
\begin{aligned}
&= C\frac{1}{\kappa^3\cos^3\varphi}\|v_m\|^4\|\theta_m\|^2 + \left(\frac{\kappa}{2}\cos\varphi\right)\delta\|\theta_m\|_{H^2}^2 + \frac{\kappa}{2}\cos\varphi\,|\Delta\theta_m|^2 \\
&= C\left(\frac{1}{\kappa^2\cos^2\varphi}\|v_m\|^4\right)\left(\frac{1}{\kappa\cos\varphi}\|\theta_m\|^2\right) \\
&\quad + \left(\frac{\kappa}{2}\cos\varphi\right)\delta\|\theta_m\|_{H^2}^2 + \left(\frac{\kappa}{2}\cos\varphi\right)|\Delta\theta_m|^2 \\
&\le C\frac{\|v_m\|^6}{\kappa^3\cos^3\varphi} + C\frac{\|\theta_m\|^6}{\kappa^3\cos^3\varphi} + \left(\frac{\kappa}{2}\cos\varphi\right)\delta\|\theta_m\|_{H^2}^2 + \left(\frac{\kappa}{2}\cos\varphi\right)|\Delta\theta_m|^2.
\end{aligned}
$$

For δ so small that $\delta\|\theta_m\|_{H^2}^2 \le \frac{1}{2}|\Delta\theta_m|^2$, combining the estimates above we obtain

$$
\begin{aligned}
&\frac{d}{ds}(\|v_m\|^2 + \|\theta_m\|^2) + \nu\cos\varphi\,|Av_m|^2 + 2\kappa\cos\varphi\,|\Delta\theta_m|^2 \\
&\qquad\le \frac{2|\theta_m|^2}{\nu\cos\varphi} + C\frac{\|v_m\|^6}{\nu^3\cos^3\varphi} + 2C\frac{\|v_m\|^6}{\kappa^3\cos^3\varphi} + 2C\frac{\|\theta_m\|^6}{\kappa^3\cos^3\varphi} \\
&\qquad\quad + \frac{\kappa}{2}\cos\varphi\,|\Delta\theta_m|^2 + \kappa\cos\varphi\,|\Delta\theta_m|^2.
\end{aligned}
$$

Hence, we deduce that

$$
\frac{d}{ds}(\|v_m\|^2 + \|\theta_m\|^2) \le \frac{2|\theta_m|^2}{\nu\cos\varphi} + C\frac{\|v_m\|^6}{\nu^3\cos^3\varphi} + C\frac{\|v_m\|^6}{\kappa^3\cos^3\varphi} + C\frac{\|\theta_m\|^6}{\kappa^3\cos^3\varphi}.
$$

As $\varphi \in (-\pi/4, \pi/4)$, we have $\sqrt{2}/2 \le \cos\varphi \le 1$, $1 \le 1/\cos\varphi \le \sqrt{2}$, so that

$$
\frac{d}{ds}(\|v_m\|^2 + \|\theta_m\|^2) \le C_{\kappa,\nu}(\|\theta_m\|^2 + \|v_m\|^6 + \|\theta_m\|^6), \tag{3.1}
$$

where $C_{\kappa,\nu}$ depends only on κ, ν. Denoting

$$
X_m(s) = \|v_m(se^{i\varphi})\|^2 + \|\theta_m(se^{i\varphi})\|^2, \qquad X(0) = \|v_0\|^2 + \|\theta_0\|^2,
$$

we get the following estimate from (3.1):

$$
\frac{d}{ds}X_m(s) \le C_{\kappa,\nu}(1 + X_m(s))^3,
$$

so

$$
\frac{d}{ds}(1 + X_m(s)) \le C_{\kappa,\nu}(1 + X_m(s))^3.
$$

Integrating the above inequality in time over $[0, s]$, we obtain

$$
-\frac{1}{2}(1 + X_m(s))^{-2}|_0^s \le C_{\kappa,\nu}s.
$$

Thus

$$
\frac{1 - 2C_{\kappa,\nu}s(1 + X_m(0))^2}{(1 + X_m(0))^2} \le \frac{1}{(1 + X_m(s))^2}.
$$

Choose s such that $1 - 2C_{\kappa,\nu}s(1 + X(0))^2 > 1/4$, i.e.

$$s < \frac{3}{8C_{\kappa,\nu}(1 + \|v_0\|^2 + \|\theta_0\|^2)^2}.$$

Then

$$\begin{aligned}1 + X_m(s) &\le \frac{1 + X_m(0)}{\sqrt{1 - 2s(1 + X_m(0))^2}} \le \frac{1 + X_m(0)}{\sqrt{1 - 2s(1 + X(0))^2}}\\ &\le 2(1 + X_m(0)) \le 2(1 + X(0)),\end{aligned}$$

that is,

$$\|v_m(se^{i\varphi})\|^2 + \|\theta_m(se^{i\varphi})\|^2 \le 2(\|v_0\|^2 + \|\theta_0\|^2) + 1 \quad \text{for all } t \in \mathbf{D}, \tag{3.2}$$

where

$$\mathbf{D} = \left\{ t = se^{i\varphi} \;\middle|\; \varphi \in (-\pi/4, \pi/4),\ 0 < s < \frac{3}{8C_{\kappa,\nu}(1 + \|v_0\|^2 + \|\theta_0\|^2)^2} \right\}.$$

Step 2. In order to obtain a priori bounds for $|Av_m(t)|$ for $t \in \mathbf{D}$ we use the first Cauchy formula to obtain a priori bounds for $\left\|\frac{dv_m}{dt}\right\|$. Indeed, for $t \in \mathbf{D}$ and $k \in \mathbb{N}$, $k \ge 1$,

$$\frac{d^k v_m}{dt^k}(t) = \frac{k!}{2\pi i} \int\limits_{|z-t|=r/2} \frac{v_m(z)}{(z-t)^{k+1}}\, dz, \tag{3.3}$$

where $r = r(t, \partial\mathbf{D})$ is the distance of the time t to the boundary $\partial\mathbf{D}$ of $\mathbf{D}$. So we have

$$\left\| \frac{d^k v_m}{dt^k}(t) \right\| \le \frac{2^k k!}{r^k} \sup_{t\in\mathbf{D}} \|v_m(t)\|.$$

By (3.2) for any compact set $K \subset \mathbf{D}$, we get

$$\sup_{t\in K} \left\| \frac{d^k v_m}{dt^k}(t) \right\| \le \frac{2^k k!}{[r(K, \partial\mathbf{D})]^k} [2(\|v_0\|^2 + \|\theta_0\|^2) + 1]^{1/2}, \tag{3.4}$$

where $r(K, \partial\mathbf{D}) = d(K, \partial\mathbf{D})$ is the distance of K to the boundary $\partial\mathbf{D}$, $k = 0, 1, 2, \dots$. In particular, taking $k = 1$ and using the first equation of (B_2), we deduce that

$$\sup_{t\in K} |Av_m(t)| \le C(K), \tag{3.5}$$

where $C(K)$ is a positive constant depending on ν, $\|v_0\|$, $\|\theta_0\|$, $r(K, \partial\mathbf{D})$, but not on m, namely

$$\begin{aligned}C(K) = \frac{2}{\nu}\left(1 + \frac{2}{r(K, \partial\mathbf{D})}\right)[2(\|v_0\|^2 + \|\theta_0\|^2) + 1]^{1/2}\\ + \frac{C}{\nu^2}[2(\|v_0\|^2 + \|\theta_0\|^2) + 1]^{3/2}.\end{aligned}$$

The proof of (3.5) is straightforward. In fact, taking the L^2 norm in the first equation of (B_2), using Lemma 2.2, the inequality $\|v_m\|_{H^2} \leq C|Av_m|$, and the Young inequality, we deduce that

$$\begin{aligned}\nu|Av_m(t)| &\leq |\theta_m(t)| + \left|\frac{dv_m}{dt}(t)\right| + |B(v_m, v_m)| \\ &\leq |\theta_m(t)| + \left|\frac{dv_m}{dt}(t)\right| + C\|v_m(t)\|^{3/2}|Av_m(t)|^{1/2} \\ &\leq |\theta_m(t)| + \left|\frac{dv_m}{dt}(t)\right| + \frac{\nu}{2}|Av_m(t)| + \frac{C}{2\nu}\|v_m(t)\|^3.\end{aligned}$$

Hence

$$|Av_m| \leq \frac{2}{\nu}|\theta_m| + \frac{C}{\nu^2}\|v_m\|^3 + \frac{2}{\nu}\left|\frac{dv_m}{dt}\right|.$$

Thus we deduce that

$$|Av_m(t)| \leq C(K).$$

Now we can use (3.5) instead of (3.2) in the estimate of the Cauchy integral (3.3) and obtain, for every compact subset K of $\mathbf{D}$, $k \in \mathbb{N}$,

$$\left|A\frac{d^k v_m}{dt^k}(t)\right| \leq \frac{2^k k!}{[r(K, \partial\mathbf{D})]^k} \sup_{t\in K'} |Av_m(t)|,$$

where K' is the set

$$\big\{t \in \mathbf{D} \,|\, d(t, \partial\mathbf{D}) \geq \tfrac{1}{2} d(K, \partial\mathbf{D})\big\}. \tag{3.6}$$

Thus

$$\sup_{t\in K}\left|A\frac{d^k v_m}{dt^k}(t)\right| \leq \frac{2^k k! C(K')}{[r(K, \partial\mathbf{D})]^k}. \tag{3.7}$$

Similarly, we obtain estimates for $\left\|\frac{d^k\theta_m}{dt^k}\right\|$ and $|\Delta\theta_m|$, for $k \in \mathbb{N}$, $k \geq 1$:

$$\sup_{t\in K}\left\|\frac{d^k\theta_m}{dt^k}(t)\right\| \leq \frac{2^k k!}{[r(K, \partial\mathbf{D})]^k}[2(\|v_0\|^2 + \|\theta_0\|^2) + 1]^{1/2}, \tag{3.8}$$

$$\sup_{t\in K}|\Delta\theta_m(t)| \leq C'(K), \tag{3.9}$$

where the positive constant $C'(K)$ depends on κ, $\|v_0\|$, $\|\theta_0\|$, K, but not on m, and its expression is as follows:

$$\begin{aligned}C'(K) = &\frac{C}{\kappa^2}[2(\|v_0\|^2 + \|\theta_0\|)^2 + 1]^{3/2} \\ &+ \frac{4}{\kappa r(K, \partial\mathbf{D})}[2(\|v_0\|^2 + \|\theta_0\|)^2 + 1]^{1/2}.\end{aligned}$$

Actually, taking the L^2 norm in the second equation of (B_2), using Remark 2.1, the inequality $\|\theta_m\|_{H^2} \leq C|\Delta\theta_m|$ and the Young inequality, we obtain

$$\begin{aligned}\kappa|\Delta\theta_m| &\leq |(v_m \cdot \nabla)\theta_m| + \left|\frac{d\theta_m}{dt}\right| \\ &\leq \|v_m\| \cdot \|\theta_m\|^{1/2}\|\theta_m\|_{H^2}^{1/2} + \left|\frac{d\theta_m}{dt}\right| \\ &\leq C\|v_m\| \cdot \|\theta_m\|^{1/2}|\Delta\theta_m|^{1/2} + \left|\frac{d\theta_m}{dt}\right| \\ &\leq \frac{C}{2\kappa}\|v_m\|^2\|\theta_m\| + \frac{\kappa}{2}|\Delta\theta_m| + \left|\frac{d\theta_m}{dt}\right|,\end{aligned}$$

so that

$$|\Delta\theta_m| \leq \frac{C}{\kappa^2}\|v_m\|^2\|\theta_m\| + \frac{2}{\kappa}\left|\frac{d\theta_m}{dt}\right|.$$

Thus we deduce that

$$|\Delta\theta_m| \leq C'(K).$$

Using again the Cauchy formula and (3.9), we also obtain, for every $t \in K$ and $k \in \mathbb{N}$,

$$\left|\Delta\frac{d^k\theta_m}{dt^k}(t)\right| \leq \frac{2^k k!}{[r(K,\partial\mathbf{D})]^k} \sup_{t\in K'} |\Delta\theta_m(t)|,$$

$$\sup_{t\in K}\left|\Delta\frac{d^k\theta_m}{dt^k}(t)\right| \leq \frac{2^k k! C'(K')}{[r(K,\partial\mathbf{D})]^k},$$

where K' is defined in (3.6).

STEP 3. Now we can pass to the limit as $m \to \infty$.

This limit process is similar to that in Temam's book (see [8, pp. 62 and 63]). For completeness of exposition, we give the details here.

Since the set $\{v_m \in V_{\mathbb{C}} \mid \|v_m\| \leq R\}$ is compact in $H_{\mathbb{C}}$ for any $R \in (0,\infty)$, by the classical Rellich compactness theorem, we can extract a subsequence $\{v_{m_j}\} \subset \{v_m\}$ which converges in $H_{\mathbb{C}}$, uniformly on every compact subset of $\mathbf{D}$, to $v^*(t) \in H_{\mathbb{C}}$ which is analytic in $\mathbf{D}$ and by the lower semicontinuity of the norm, satisfies

$$\sup_{t\in\mathbf{D}} \|v^*(t)\|^2 \leq 2(\|v_0\|^2 + \|\theta_0\|^2) + 1.$$

Since the restriction of v_m to the real axis coincides with the solution to the Galerkin approximation in $\mathbb{R}_+$ of the Boussinesq equations, we deduce that the restriction of $v^*(t)$ to $(0, \infty)$ in 2D and to some interval $(0,T')$ of the real axis in 3D coincides with the unique (strong) solution v of the Boussinesq equations. Hence v^* is nothing but the analytic continuation of v to $\mathbf{D}$. Without loss of generality, we denote the limit by $v(t)$ instead of v^*. Further, we conclude that the whole sequence $v_m(\cdot)$ converges to $v(\cdot)$ uniformly on compact subsets of $\mathbf{D}$ in the norm of $H_{\mathbb{C}}$.

Because the embedding of $\mathcal{D}(A)$ in V is compact, it also follows from (3.5) and the compactness theorem that, on every compact subset of $\mathbf{D}$,

$$v_m \to v \quad \text{in } V.$$

Moreover, we have

$$\sup_{t\in K} |Av(t)| \le C(K),$$

with the same constant $C(K)$ as in (3.5). Finally, the estimates (3.4), (3.7) imply that $\frac{d^k v_m}{dt^k}$ converges to $\frac{d^k v}{dt^k}$ in V uniformly on every compact subset K of $\mathbf{D}$, and that

$$\sup_{t\in K} \left\| \frac{d^k v}{dt^k}(t) \right\| \le \frac{2^k k!}{[r(K,\partial\mathbf{D})]^k} [2(\|v_0\|^2 + \|\theta_0\|^2) + 1]^{1/2},$$

$$\sup_{t\in K} \left| A\frac{d^k v}{dt^k}(t) \right| \le \frac{2^k k! C(K')}{[r(K,\partial\mathbf{D})]^k},$$

where K' is defined in (3.6).

With the same arguments, we deduce that θ_m converges to θ uniformly on every compact subset K of $\mathbf{D}$, and

$$\sup_{t\in K} \left\| \frac{d^k \theta(t)}{dt^k} \right\| \le \frac{2^k k!}{[r(K,\partial\mathbf{D})]^k} [2(\|v_0\|^2) + \|\theta_0\|^2) + 1]^{1/2},$$

$$\sup_{t\in K} \left| \Delta\frac{d^k \theta(t)}{dt^k} \right| \le \frac{2^k k! C'(K')}{[r(K,\partial\mathbf{D})]^k}.$$

Finally, we observe that the reasoning conducted at $t = 0$ can be shifted to any other point $t_0 \in (0,\infty)$ such that $v(t_0) \in V$, $\theta(t_0) \in H^1$. We infer that v is a $\mathcal{D}(A)$-valued analytic function and θ is an H^1-valued analytic function in the region

$$\{t_0 + \mathbf{D}(\|v(t_0)\|, \|\theta(t_0)\|)\}$$

of $\mathbb{C}$, for all $t_0 \in (0,\infty)$ such that $v(t_0) \in V$, $\theta(t_0) \in H^1$.

In the 2D case, we know that the strong solutions globally exist and have a uniform bound, i.e. there exists C such that for any $t \in (0,\infty)$ we have $\|v(t)\|, \|\theta(t)\| \le C$. Moreover, $\mathbf{D}(\|v(t_0)\|, \|\theta(t_0)\|)$ depends only on the norms in H^1 of $v(t_0)$ and $\theta(t_0)$, and decreases as $\|v(t_0)\|$ or $\|\theta(t_0)\|$ increases. Therefore, v and θ are analytic in the region

$$\bigcup_{t_0\in(0,\infty)} \{t_0 + \mathbf{D}(C,C)\}.$$

Hence, the proof of Theorem 1.1 in the 2D case is complete.

In the 3D case, we only know the local well-posedness of strong solutions, i.e. there exist $T_0 \in (0, \infty)$ and constant $C > 0$ such that for any $t \in (0, T_0)$ we have $\|v(t)\|, \|\theta(t)\| \leq C$. Thus, the solutions are analytic in the region

$$\bigcup_{t_0 \in (0,T_0)} \{t_0 + \mathbf{D}(C, C)\}.$$

Hence, the proof in the 3D case is also complete. ∎

Proof of Theorem 1.2. According to Theorem 1.1, we can assume that the function couples $(v_1(t), \theta_1(t))$, $(v_2(t), \theta_2(t))$ are both analytic in $(0, T_0)$ for the 3D case and in $(0, \infty)$ for the 2D case, respectively. If $(v_1(t_0), \theta_1(t_0)) = (v_2(t_0), \theta_2(t_0))$ then from the uniqueness of strong solutions to the Boussinesq equations it follows that $(v_1(t), \theta_1(t)) = (v_2(t), \theta_2(t))$ for all $t \geq t_0$. Using the analyticity of solutions, we obtain

$$(v_1(t), \theta_1(t)) = (v_2(t), \theta_2(t)) \quad \text{for all } \ t > 0.$$

But $(v_1(t), \theta_1(t))$, $(v_2(t), \theta_2(t))$ tend strongly in (H, L^2) to $(v_1(0), \theta_1(0))$, $(v_2(0), \theta_2(0))$ as $t \to 0$, respectively. Thus $(v_1(0), \theta_1(0)) = (v_2(0), \theta_2(0))$ must hold, too. This completes the proof of Theorem 1.2. ∎

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Qianqian Hou, Xiaojing Xu
School of Mathematical Sciences
Beijing Normal University
and
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875, P.R. China
E-mail: langsa@mail.bnu.edu.cn
xjxu@bnu.edu.cn

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