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POTENTIAL THEORY OF HYPERBOLIC BROWNIAN MOTION IN TUBE DOMAINS

BY

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Abstract. Let $X = \{X(t); t \geq 0\}$ be the hyperbolic Brownian motion on the real hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$. We study the Green function and the Poisson kernel of tube domains of the form $D \times (0, \infty) \subset \mathbb{H}^n$, where D is any Lipschitz domain in $\mathbb{R}^{n-1}$. We show how to obtain formulas for these functions using analogous objects for the standard Brownian motion in $\mathbb{R}^{2n}$. We give formulas and uniform estimates for the set $D_a = \{x \in \mathbb{H}^n : x_1 \in (0, a)\}$. The constants in the estimates depend only on the dimension of the space.

1. Introduction. Potential theory on hyperbolic spaces is governed by the Laplace–Beltrami operator. It is the unique (up to a multiplicative constant) differential operator of order two which is invariant under isometries of the space. One of the main objects in the theory are the Green function and the Poisson kernel of subdomains. Although a purely analytical approach to this subject is possible, we rely on a probabilistic method, which is particularly convenient when dealing with subdomains. Our basic object of study is the hyperbolic Brownian motion (HBM), which is the canonical diffusion on the hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ with half the Laplace–Beltrami operator as generator. In recent years there is a growing interest in the hyperbolic Brownian motion. We refer the reader to [BCF], [B–Y], [G], [M], where such fundamental objects as the heat kernel and the global Green function were investigated. The hyperbolic Brownian motion is strongly related to the geometric Brownian motion and the Bessel processes. For details see [Y2] and [B–Y]. This process is also interesting from the point of view of physics (see [GS]) and risk theory in financial mathematics (see [D], [Y3]).

Recently, many papers have appeared concerning harmonic measures of subdomains (equivalently: Poisson kernels for HBM). We point out three of them. In [BGS] the authors provided some formulas along with the asymp-

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totics of the Poisson kernel of the set $\{x \in \mathbb{H}^n : x_n > a\}$, whose boundary is a horocycle. The Poisson kernel of the ball was considered in the real hyperbolic space (see [BM]) and also in the complex one (see [Z]). Furthermore, [MS] dealt with HBM with drift exiting the set $\{x \in \mathbb{H}^n : x_1 > 0\}$. The paper [CFZ] shows that the Green functions and the Poisson kernels for bounded sets (in hyperbolic metric) are comparable with analogous objects in Euclidean space. For unbounded sets this result does not hold. Although explicit formulas are really intricate and often expressed by special functions, they seem to be crucial for obtaining estimates.

In the present paper we denote by $X = \{X(t);\, t \geq 0\}$ the HBM on the half-space model $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ of the real hyperbolic space. We investigate the potential theory for tube domains S of the form

$$S = \{x \in \mathbb{H}^n : (x_1, \ldots, x_{n-1}) \in D\}, \tag{1}$$

where D is any Lipschitz domain in $\mathbb{R}^{n-1}$. The set $A \subset \mathbb{R}^d$, $d = 1, 2, \ldots$, is a *Lipschitz domain* if for every $x \in \partial A$ there exist a radius $r > 0$ and a bijection $f_x : B_r(x) \to B_1(0)$ such that f_x and f_x^{-1} are Lipschitz functions and $f_x(\partial A \cap B_x(r)) = \{y \in B_1(0) : y_n = 0\}$, $f_x(A \cap B_x(r)) = \{y \in B_1(0) : y_n > 0\}$. Here, $B_r(x)$ is the open ball of radius r centered at x. Note that S is unbounded in $\mathbb{H}^n$. Its boundary ∂S, as a subset of $\mathbb{R}^n$, consists of two parts of totally different nature:

$$\partial_1 S = \partial D \times (0, \infty), \qquad \partial_2 S = D \times \{0\}\,.$$

The first part coincides with the boundary of S as a subset of $\mathbb{H}^n$. The other part does not belong to $\mathbb{H}^n$ but it can be reached by HBM in infinite time. We define

$$\tilde{S} = \{x \in \mathbb{R}^{2n} : (x_1, \ldots, x_{n-1}) \in D\}, \tag{2}$$

which is an analogue of the set S in $2n$-dimensional Euclidean space. We exhibit in Theorem 1.1 the relationship between hyperbolic and Euclidean potential theory on S and $\tilde{S}$, respectively.

We will consider several diffusions, so we introduce some universal notations and definitions for related objects. Let $\{\Psi(t);\, t \geq 0\}$ be a continuous process in $\mathbb{R}^m$ starting from any $x \in \mathbb{R}^m$. For any Lipschitz domain $A \subset \mathbb{R}^m$ we define the *first exit time* from A for the process $\Psi(t)$ as $\tau_A^\Psi = \inf\{t > 0 : \Psi(t) \notin A\}$. Analogously, we define the *Green function* $G_A^\Psi(x,y)$ and the *Poisson kernel* $P_A^\Psi(x,y)$ of A as follows:

$$G_A^\Psi(x,y) = \int_0^\infty \mathbb{P}^x(\Psi(t) \in dy,\, t < \tau_A^\Psi)\, dt/dy, \qquad x, y \in A,$$

$$P_A^\Psi(x,y) = \mathbb{P}^x(\Psi(\tau_A^\Psi) \in dy)/dy, \qquad x \in A,\ y \in \partial A,$$

where the measure $\mathbb{P}^x$ corresponds to the process starting from $x \in \mathbb{R}^m$. For $n < m$ and $x \in \mathbb{R}^m$ we denote $x = (x^-, x^+)$, where $x^- = (x_1, \ldots, x_{n-1})$ and $x^+ = (x_n, \ldots, x_m)$. Let $W = \{W(t);\, t \geq 0\}$ be the standard $2n$-dimensional Brownian motion. Since $\tau^W_{\tilde{S}}$ is independent of the vector $(W_n(t), \ldots, W_{2n}(t))$, the Green function $G^W_{\tilde{S}}(x,y)$ and the Poisson kernel $P^W_{\tilde{S}}(x,y)$ depend only on x^-, y^- and the squared Euclidean distance $|x^+ - y^+|^2$ between x^+ and y^+. We will write

$$G^W_{\tilde{S}}(x,y) = G^W_{\tilde{S}}(x^-, y^-, |x^+ - y^+|^2), \qquad x, y \in \tilde{S},$$
$$P^W_{\tilde{S}}(x,y) = P^W_{\tilde{S}}(x^-, y^-, |x^+ - y^+|^2), \qquad x \in \tilde{S},\, y \in \partial\tilde{S}.$$

THEOREM 1.1. *Let S be of the form* (1). *The Green function of the set S for HBM is given by*

$$G^X_S(x,y) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{x_n^{n-1}}{y_n} \int_{-1}^{1} (1-u^2)^{n/2-1} G^W_{\tilde{S}}(x^-, y^-, x_n^2 + y_n^2 + 2x_n y_n u)\, du,$$

where $x, y \in S$ and $\tilde{S}$ is defined by (2). *The Poisson kernel of the set S for HBM is given by*

$$P^X_S(x,y) = 2\pi^{n/2} x_n^{n-1}$$
$$\times \begin{cases} \dfrac{y_n}{\Gamma(n/2)} \displaystyle\int_{-1}^{1} (1-u^2)^{n/2-1} P^W_{\tilde{S}}(x^-, y^-, x_n^2 + y_n^2 + 2x_n y_n u)\, du, & y \in \partial_1 S, \\ \dfrac{\sqrt{\pi}}{\Gamma((n-1)/2)}\, G^W_{\tilde{S}}(x^-, y^-, x_n^2), & y \in \partial_2 S, \end{cases}$$

where $x \in S$.

The proof of Theorem 1.1 given in Section 2 is based on considering a Brown–Bessel process. This general method was introduced by Molchanov and Ostrowski [MO]. An example of a set of the form (1) is $D_a = \{x \in \mathbb{H}^n : 0 < x_1 < a\}$, $a > 0$. Reflections with respect to the hyperplanes $\{x \in \mathbb{H}^n : x_1 = 0\}$ and $\{x \in \mathbb{H}^n : x_1 = a\}$ are isometries of $\mathbb{H}^n$, therefore it is natural to call D_a a *strip*. We apply Theorem 1.1 and provide formulas for the Green function $G^X_{D_a}(x,y)$ and the Poisson kernel $P^X_{D_a}(x,y)$ of the set D_a in Theorem 1.2. We also give their estimates in Theorem 1.3.

THEOREM 1.2. *For $x, y \in D_a$ we have*

$$G^X_{D_a}(x,y) = \frac{(-1/2)^{n-1}}{\pi^{n/2}\Gamma(n/2) y_n^n} \int_{-1}^{1} (1-s^2)^{n/2-1}$$
$$\times \frac{\partial^{n-1}}{\partial s^{n-1}} \ln\left[1 + \frac{2\sin(\pi x_1/a)\sin(\pi y_1/a)}{\cosh(\pi\sqrt{2x_n y_n(\cosh\tilde{\rho} + s)}/a) - \cos(\pi(x_1 - y_1)/a)}\right] ds.$$

For $x \in D_a$, $y \in \partial_1 D_a$, the Poisson kernel of D_a is given by

$$P^X_{D_a}(x,y) = \frac{(-1/2)^{n-1}\pi^{1-n/2}}{a\Gamma(n/2)y_n^{n-2}} \int_{-1}^{1} (1-u^2)^{n/2-1} \times \frac{\partial^{n-1}}{\partial u^{n-1}} \frac{\sin(\pi(x_1-y_1)/a)}{\cosh(\pi\sqrt{2x_n y_n(\cosh\tilde{\rho}+u)}/a) - \cos(\pi(x_1-y_1)/a)}\, du,$$

where $\tilde{x} = (0, x_2, x_3, \ldots, x_n)$ and $\tilde{\rho} = d_{\mathbb{H}^n}(\tilde{x}, \tilde{y})$ is the hyperbolic distance between $\tilde{x}$ and $\tilde{y}$. For $x \in D_a$, $y \in \partial_2 D_a$ we have

$$P^X_{D_a}(x,y) = \frac{(-1)^{n-1}x_n^{n-1}}{\Gamma((n-1)/2)\pi^{(n-1)/2}} \times \frac{\partial^{n-1}}{\partial \xi^{n-1}} \ln\left[1 + \frac{2\sin(\pi x_1/a)\sin(\pi y_1/a)}{\cosh(\pi\sqrt{\xi}/a) - \cos(\pi(x_1-y_1)/a)}\right]_{\xi=|\tilde{x}-\tilde{y}|^2}.$$

The integral in the first two formulas can be computed for even n by integrating by parts $n-2$ times. However, this leads to a sum of oscillating components, and the above-given integral form is much more useful for finding the estimates given below. We write $f \overset{c}{\asymp} g$ whenever there exists $c > 1$ such that $c^{-1}f(x) < g(x) < cf(x)$ for all arguments x.

THEOREM 1.3. *There exists $c = c(n)$ such that for $x \in D_a$, we have*

$$G^X_{D_a}(x,y) \overset{c}{\asymp} \frac{e^{-\pi|x-y|/a}}{a^{n+1}x_n y_n}\left(\frac{x_n}{y_n}\right)^{n/2} \frac{|x-y|^2 \wedge (\delta_a(x_1)\delta_a(y_1))}{(\cosh\rho + |x-y|/a)^{n/2}}\, \frac{a^{n+1}+|x-y|^{n+1}}{|x-y|^n},$$

where $y \in D_a$, $\rho = d_{\mathbb{H}^n}(x,y)$ and $\delta_a(s) = \min\{s, a-s\}$. Moreover,

$$P^X_{D_a}(x,y) \overset{c}{\asymp} \begin{cases} \dfrac{\delta_a(x_1)e^{-\pi|x-y|/a}}{a^{n+1}|x-y|^n}\left(\dfrac{x_n}{y_n}\right)^{n/2} \dfrac{a^{n+1}+|x-y|^{n+1}}{(\cosh\rho + |x-y|/a)^{n/2}}, & y \in \partial_1 D_a, \\ \dfrac{x_n^{n-1}}{a^{n+1}} \exp(-\pi|x-y|/a)(|x-y|^2 \wedge [\delta_a(x_1)\delta_a(y_1)]) \times \dfrac{a^{n+1}+|x-y|^{n+1}}{|x-y|^{2n}}, & y \in \partial_2 D_a. \end{cases}$$

Such precise estimates for unbounded subsets of the hyperbolic space were known only for the set $\{x \in \mathbb{H}^n : x_n > a\}$ [BMR] and the hyperbolic half-space $D_\infty = \{x \in \mathbb{H}^n : x_1 > 0\}$ [MS]. We refer the reader to [MS] for simplified formulas when $a = \infty$. For arguments in some domains, $P^X_{D_a}(x,y)$ on $\partial_1 D_a$ and the Poisson kernel of $(0,1) \times \mathbb{R}^{n-1}$ for standard Brownian motion in $\mathbb{R}^n$ are comparable, for example when $|x-y| < a$ and when x_n, y_n are bounded and bounded away from zero. However, they are not comparable in general.

The organization of the paper is as follows. In the Preliminaries we provide some facts about Bessel processes, since they play an important role in the proof of Theorem 1.1. Next we introduce the half-space model of the real hyperbolic space and describe the structure of the hyperbolic Brownian motion. In Section 3 we give the proof of Theorem 1.1 together with some comments. In Section 4 we prove Theorems 1.2 and 1.3. In the Appendix we collect some technical lemmas and compute the Poisson kernel and the Green function of the strip in $\mathbb{R}^2$.

2. Preliminaries

2.1. Bessel process. We denote by $R^{(\nu)} = \{R^{(\nu)}(t);\, t \geq 0\}$ the Bessel process with index $\nu \in \mathbb{R}$, starting from $R^{(\nu)}(0) = x > 0$. As we will see in Section 2.2, studying HBM requires using Bessel processes with negative index. For $\nu \leq -1$ the point 0 is killing. In the case $-1 < \nu < 0$, that is, when the point 0 is non-singular, we impose the killing condition at 0. Then, the transition density function, with respect to the Lebesgue measure, is given by (see [BS, p. 134])

$$g_t^{(\nu)}(x,w) = \frac{w}{t}\left(\frac{w}{x}\right)^{\nu} \exp\left(-\frac{x^2+w^2}{2t}\right) I_{|\nu|}\left(\frac{xw}{t}\right), \qquad x,w>0, \tag{3}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind.

Let us denote by $B = \{B(t);\, t \geq 0\}$ the one-dimensional Brownian motion starting from 0 and by $B^{(\nu)} = \{B(t)+\nu t;\, t \geq 0\}$ the Brownian motion with constant drift $\nu \in \mathbb{R}$. The Bessel process is related to the geometric Brownian motion $\{x \exp(B^{(\nu)}(t));\, t \geq 0\}$, $x > 0$, by the *Lamperti relation*,

$$\{x \exp(B^{(\nu)}(t));\, t \geq 0\} \stackrel{d}{=} \{R^{(\nu)}(A_x^{(\nu)}(t));\, t \geq 0\},$$

where the integral functional $A_x^{(\nu)}(t)$ is defined by

$$A_x^{(\nu)}(t) = x^2 \int_0^t \exp(2B_s + 2\nu s)\, ds. \tag{4}$$

The density function $f_{x,t}^{(\nu)}$ of $\big(A_x^{(\nu)}(t), x\exp(B_n(t)+\nu t)\big)$ was computed in [Y3]. We have

$$f_{x,t}^{(\nu)}(u,v) = \left(\frac{v}{x}\right)^{\nu} e^{-\nu^2 t/2} \frac{1}{uv} \exp\left(-\frac{x^2+v^2}{2u}\right) \theta_{xv/u}(t), \qquad x,u,v,t>0, \tag{5}$$

where

$$\theta_r(t) = \frac{r}{(2\pi^3 t)^{1/2}} \int_0^\infty e^{(\pi^2-b^2)/(2t)} e^{-r\cosh b} \sinh(b) \sin\left(\frac{\pi b}{t}\right) db, \qquad r,t>0. \tag{6}$$

Moreover, the Laplace transform of the function θ_r is given by (see [Y1])

$$\int_0^\infty e^{-\lambda t}\theta_r(t)\,dt = I_{\sqrt{2\lambda}}(r). \tag{7}$$

Whenever ν is strictly negative, the limit of $A_x^{(\nu)}(t)$ as $t \to \infty$ exists a.s. The density function $h_x^{(\nu)}(u)$ of $A_x^{(\nu)}(\infty)$ is given by (see [D])

$$h_x^{(\nu)}(u) = \frac{x^{-2\nu}}{\Gamma(-\nu)2^{-\nu}}\,\frac{e^{-x^2/(2u)}}{u^{1-\nu}}\,\mathbb{1}_{(0,\infty)}(u). \tag{8}$$

2.2. Hyperbolic space $\mathbb{H}^n$ and hyperbolic Brownian motion. For $n = 1, 2, \ldots$ we consider the half-space model of the n-dimensional real hyperbolic space

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}.$$

The Riemannian volume element is given by

$$dV_n = \frac{1}{x_n^n} dx_1 \ldots dx_n,$$

where $dx_1 \ldots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$. The hyperbolic distance $d_{\mathbb{H}^n}(x, y)$ between $x, y \in \mathbb{H}^n$ is described by the formula

$$\cosh d_{\mathbb{H}^n}(x, y) = 1 + \frac{|x-y|^2}{2x_n y_n},$$

where $|x - y|$ is the Euclidean distance between x and y. The Laplace–Beltrami operator is given by

$$\Delta = x_n^2 \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} - \frac{n-2}{2} x_n \frac{\partial}{\partial x_n}.$$

We define the hyperbolic Brownian motion $X(t)$ as the canonical diffusion on the whole of $\mathbb{H}^n$ with generator $\frac{1}{2}\Delta$. The structure of the process is described by the following representation. If we denote by $B(t) = (B_1(t), \ldots, B_n(t))$ the standard n-dimensional Brownian motion starting from $(x_1, \ldots, x_{n-1}, 0)$, then

$$X(t) \stackrel{d}{=} \Big(B_1(A_{x_n}^{(-(n-1)/2)}(t)), \ldots, B_{n-1}(A_{x_n}^{(-(n-1)/2)}(t)),\; x_n \exp\Big(B_n(t) - \frac{n-1}{2}t\Big)\Big). \tag{9}$$

Here, the functional $A_{x_n}^{(-(n-1)/2)}(t)$ defined by (4) is associated with $B_n(t)$. In addition, using the Lamperti relation, we get

$$\{X(t);\, t \ge 0\} \stackrel{(d)}{=} \{Y(A_{x_n}^{(-(n-1)/2)}(t));\, t \ge 0\}, \tag{10}$$

where

$$Y(t) = (B_1(t), \ldots, B_{n-1}(t), R^{(-(n-1)/2)}(t)). \tag{11}$$

The process $R^{(-(n-1)/2)}(t)$ is the Bessel process starting from $x_n > 0$ and independent of $(B_1(t), \ldots, B_{n-1}(t))$.

3. Proof of Theorem 1.1. Let us note that the representation (11) of HBM simplifies many arguments. In particular, we have

LEMMA 3.1. *For any Lipschitz domain $U \subset \mathbb{H}^n$,*

$$X(\tau_U^X) \stackrel{d}{=} Y(\tau_U^Y).$$

Proof. From (10) the process $Z(t) = Y(A_{x_n}^{(-(n-1)/2)}(t))$ is a hyperbolic Brownian motion. Since the functional $A_{x_n}^{(-(n-1)/2)}(t)$ is continuous and increasing a.s., we obtain $\tau_U^Y = A_{x_n}^{(-(n-1)/2)}(\tau_U^Z)$ a.s. Hence

$$X(\tau_U^X) \stackrel{d}{=} Z(\tau_U^Z) = Y(A_{x_n}^{(-(n-1)/2)}(\tau_U^Z)) \stackrel{\text{a.s.}}{=} Y(\tau_U^Y). \quad \blacksquare$$

From now on, we will consider sets of the form (1). The absolute continuity relationship for the laws of the Bessel processes with different indices (see [MY, (2.2), p. 314]) implies

$$\mathbb{P}^x(R^{(-(n-1)/2)}(t) \in dy) = \left(\frac{x}{y}\right)^{n-1} \mathbb{P}^x(R^{((n-1)/2)}(t) \in dy), \qquad x, y > 0.$$

Moreover, the process $R^{((n-1)/2)}(t)$ has an interpretation as the Euclidean norm of $(n+1)$-dimensional Brownian motion. Since the first $n-1$ coordinates of the process $Y(t)$ are identical with $(n-1)$-dimensional Brownian motion, we can deduce the relationship between $Y(t)$ and $2n$-dimensional Brownian motion.

Proof of Theorem 1.1. Let us denote by $p_t^D(\cdot,\cdot)$ the density function of $(n-1)$-dimensional Brownian motion killed on exiting the set D. Since (9) holds and $A_{x_n}^{(-(n-1)/2)}(t)$ is continuous and increasing a.s., the density function of HBM killed on exiting S is given by

$$\int_0^\infty p_u^D(x^-, y^-) f_{x_n,t}^{(-(n-1)/2)}(u, y_n)\, du.$$

Hence, using the Fubini–Tonelli theorem, we get

$$G_S^X(x,y) = \int_0^\infty \int_0^\infty p_u^D(x^-, y^-) f_{x_n,t}^{(-(n-1)/2)}(u, y_n)\, du\, dt$$

$$\overset{(5)}{=} \int_0^\infty p_u^D(x^-,y^-)\left(\frac{x_n}{y_n}\right)^{(n-1)/2}\frac{1}{uy_n}\exp\left(-\frac{x_n^2+y_n^2}{2u}\right) \times \int_0^\infty e^{-((n-1)/2)^2 t/2}\theta_{x_n y_n/u}(t)\,dt\,du$$

$$\overset{(7)}{=} \int_0^\infty p_u^D(x^-,y^-)\left(\frac{x_n}{y_n}\right)^{(n-1)/2}\frac{\exp\left(-\frac{x_n^2+y_n^2}{2u}\right)}{uy_n} I_{(n-1)/2}\left(\frac{x_n y_n}{u}\right)du.$$

We will use the following integral formula (see [GR, 8.431]):

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_{-1}^1 (1-s^2)^{\nu-1/2}e^{-zs}\,ds, \tag{12}$$

where $z>0$ and $\nu>-1/2$. It follows that

$$\begin{aligned}
&G_S^X(x,y)\\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{x_n^{n-1}}{y_n}\int_{-1}^1 (1-s^2)^{n/2-1}\int_0^\infty p_u^D(x^-,y^-)\frac{\exp\left(-\frac{x_n^2+y_n^2+2x_ny_ns}{2u}\right)}{(2\pi u)^{(n+1)/2}}\,du\,ds\\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{x_n^{n-1}}{y_n}\int_{-1}^1 (1-s^2)^{n/2-1}G^W_{D\times\mathbb{R}^{n+1}}(x^-,y^-,x_n^2+y_n^2+2x_ny_ns)\,ds.
\end{aligned}$$

To prove formulas for the Poisson kernel we use Lemma 3.1. It implies that $P_S^X(x,y)=P_S^Y(x,y)$ for $x\in S$ and $y\in\partial S$. We define two exit times

$$\begin{aligned}
\tau_1 &= \inf\{t>0 : (W_1(t),\ldots,W_{n-1}(t))\notin D\} = \tau^W_{D\times\mathbb{R}^{n+1}},\\
\tau_2 &= \inf\{t>0 : R^{(-(n-1)/2)}(t)=0\}.
\end{aligned}$$

We have

$$\tau^Y_{D\times(0,\infty)} = \tau_1\wedge\tau_2,$$

and the continuity of sample paths of the hyperbolic Brownian motion implies $Y(\tau_i)\in\partial_i S$ for $i=1,2$. Let us denote the density function of the vector $((W_1(\tau^W_{D\times\mathbb{R}^{n+1}}),\ldots,W_{n-1}(\tau^W_{D\times\mathbb{R}^{n+1}})),\tau^W_{D\times\mathbb{R}^{n+1}})$ by $k_{x^-}(y^-,t)$. The exit time $\tau^W_{D\times\mathbb{R}^{n+1}}$ depends only on the first $n-1$ coordinates of $W(t)$. Since $(W_1(t),\ldots,W_{n-1}(t))$ is independent of $(W_{n+1}(t),\ldots,W_{2n-1}(t),W_{2n}(t))$, we obtain

$$P^W_{D\times\mathbb{R}^{n+1}}(x^-,y^-,|x^+-y^+|^2) = \int_0^\infty \frac{\exp\left(-\frac{1}{2t}|x^+-y^+|^2\right)}{(2\pi t)^{(n+1)/2}}k_{x^-}(y^-,t)\,dt. \tag{13}$$

Using the independence of the processes $(B_1(t),\ldots,B_{n-1}(t))$ and $R^{(-\frac{n-1}{2})}(t)$, we get

$$P_S^Y(x,y) = \int_0^\infty g_t^{(-(n-1)/2)}(x_n, y_n) k_{x^-}(y^-, t)\, dt$$

$$\stackrel{(3)}{=} \int_0^\infty \frac{y_n}{t}\left(\frac{x_n}{y_n}\right)^{(n-1)/2} \exp\left(-\frac{x_n^2+y_n^2}{2t}\right) I_{(n-1)/2}\left(\frac{x_n y_n}{t}\right) k_{x^-}(y^-, t)\, dt.$$

By (12) we get

$$P_S^Y(x,y) = \frac{2\pi^{n/2} x_n^{n-1} y_n}{\Gamma(n/2)} \int_{-1}^1 (1-u^2)^{n/2-1} \times \int_0^\infty \frac{\exp\left(-\frac{1}{2t}(x_n^2+y_n^2+2x_n y_n u)\right)}{(2\pi t)^{(n+1)/2}} k_{x^-}(y^-, t)\, dt\, du$$

$$\stackrel{(13)}{=} \frac{2\pi^{n/2} x_n^{n-1} y_n}{\Gamma(n/2)} \int_{-1}^1 (1-u^2)^{n/2-1} P^W_{D\times\mathbb{R}^{n+1}}(x^-, y^-, x_n^2+y_n^2+2x_n y_n u)\, du.$$

The Poisson kernel on $\partial_2 D_a$ is given by

$$P_S^Y(x,y) = \mathbb{P}^x\left(Y(\tau_2)\in dy;\ \tau_2<\tau_1\right).$$

According to the Lamperti relation we have

$$X_n(t) = x_n \exp(B_n(t) - (n-1)t) \stackrel{(d)}{=} R^{(-(n-1)/2)}(A_{x_n}(t)).$$

The left-hand side of the above equation tends to 0 as $t\to\infty$ a.s. It follows that

$$\tau_2 \stackrel{(d)}{=} A_{x_n}(\infty).$$

As before, τ_2 is independent of $(B_1(t),\ldots,B_{n-1}(t))$, hence by (8) we get

$$P_S^Y(x,y) = \int_0^\infty p_t^D(x^-, y^-) \frac{x_n^{n-1}}{\Gamma((n-1)/2) 2^{(n-1)/2}} \frac{e^{-x_n^2/(2t)}}{t^{(n+1)/2}}\, dt$$

$$= \frac{2x_n^{n-1}\pi^{(n+1)/2}}{\Gamma((n-1)/2)} G^W_{\tilde S}(x^-, y^-, x_n^2). \ \blacksquare$$

Remark. Theorem 1.1 can be proven using the interpretation of the process $R^{((n-1)/2)}(t)$ as the Euclidean norm of the $(n+1)$-dimensional Brownian motion. This would lead to integration over a sphere, which explains the appearance of the factor $(1-u^2)^{n/2-1}$ in the formulas. However, that proof would be longer and more complicated than the one given above.

4. Poisson kernel of the strip. Let us recall that

$$D_a = \{x\in\mathbb{H}^n;\ 0<x_1<a\}, \qquad a>0.$$

The boundary ∂D_a consists of two parts: $\partial_1 D_a = \{x \in \mathbb{H}^n : x_1 \in \{0,a\}\}$ and $\partial_2 D_a = \{x \in \mathbb{R}^n : x_n = 0,\, 0 \le x_1 \le a\}$. Let $\tau^X_{D_a}$ be the first exit time from D_a for the hyperbolic Brownian motion

$$\tau^X_{D_a} = \inf\{s > 0 : X(s) \notin D_a\}.$$

Whenever $\tau^X_{D_a} < \infty$, the variable $X(\tau^X_{D_a})$ is supported on the set $\partial_1 D_a$. For $\tau^X_{D_a} = \infty$, the random variable $X(\tau^X_{D_a})$ takes values in the set $\partial_2 D_a$. It is well defined because $\lim_{t\to\infty} X(t)$ exists a.s.

According to Theorem 1.1 we start by considering the Euclidean case. Let us recall that $W(t) = (W_1(t), \ldots, W_{2n}(t))$ is the standard BM. The set in $\mathbb{R}^{2n}$ which corresponds to D_a is $\tilde{D}_a = (0,a) \times \mathbb{R}^{2n-1}$. We define $\tau^{W_1}_{(0,a)}$ as the first exit time of the process $W_1(t)$ from the set $\tilde{D}_a$,

$$\tau^{W_1}_{(0,a)} = \inf\{t > 0 : W_1(t) \notin (0,a)\} = \inf\{t > 0 : W(t) \notin \tilde{D}_a\}.$$

We denote

$$\eta(w,t) = \sum_{k=-\infty}^{\infty} \frac{w + 2ka}{\sqrt{2\pi}\, t^{3/2}} \exp\left(-\frac{(w+2ka)^2}{2t}\right), \qquad w \in \mathbb{R}.$$

Then we have [BS, 3.0.6(a), (b), p. 212]

$$\begin{aligned}(14)\qquad \mathbb{P}^{x_1}(W_1(\tau^{W_1}_{(0,a)}) = y_1,\, \tau^{W_1}_{(0,a)} \in dt)&\\ = \eta(|x_1 - y_1|, t), \qquad x_1 \in (0,a),\ y_1 \in \{0,a\}.&\end{aligned}$$

Since the random variables $\tau^{W_1}_{(0,a)}$ and $(W_2(t), W_3(t), \ldots, W_{2n}(t))$ are independent, we obtain the following formula for the Poisson kernel of the set $\tilde{D}_a$:

$$P^W_{\tilde{D}_a}(x,y) = \int_0^\infty \frac{1}{(2\pi t)^{n-1/2}} e^{-|\tilde{x}-\tilde{y}|^2/(2t)} \eta(|x_1 - y_1|, t)\, dt, \tag{15}$$

where $\tilde{x} = (0, x_2, x_3, \ldots, x_n)$ and $\tilde{y} = (0, y_2, y_3, \ldots, y_n)$. For $w \in \mathbb{R}$ and $\xi > 0$ we define

$$\Phi^a_n(w,\xi) := \frac{1}{(2\pi)^{(n-1)/2}} \int_0^\infty t^{(1-n)/2} e^{-\xi/(2t)} \eta(w,t)\, dt. \tag{16}$$

Note that

$$P^W_{\tilde{D}_a}(x,y) = \Phi^a_{2n}(|x_1 - y_1|, |\tilde{x} - \tilde{y}|^2). \tag{17}$$

In the next lemma we present a differential formula for the function $\Phi^a_n(w,\xi)$.

Lemma 4.1. *Let $a > 0$ and $n = 1, 2, \ldots$. For $w \in \mathbb{R}$ and $\xi > 0$ we have*

$$\Phi^a_{2n}(w,\xi) = \frac{(-1)^{n-1}}{2a\pi^{n-1}} \frac{\partial^{n-1}}{\partial \xi^{n-1}} \frac{\sin(\pi w/a)}{\cosh(\pi\sqrt{\xi}/a) - \cos(\pi w/a)}. \tag{18}$$

Proof. For $w \in (0,a)$ we have

$$\eta(w,t) = \mathbb{P}^w(\tau_a \in dt,\, W_1(\tau_a) = 0) \le \mathbb{P}^w(\inf\{s > 0 : W_1(s) > 0\} \in dt) = \frac{w}{\sqrt{2\pi}\, t^{3/2}} e^{-w^2/(2t)}.$$

By continuity of both functions, the above inequality also holds for $w = 0$. This implies the uniform integrability of (16) for fixed $w \in [0,a)$ and ξ from every compact subset of $(0,\infty)$. It allows us to change the order of differentiation and integration in the expression

$$\frac{\partial}{\partial\xi}\Phi_n^a(w,\xi) = \frac{1}{(2\pi)^{(n-1)/2}}\frac{\partial}{\partial\xi}\int_0^\infty t^{-(n-1)/2} e^{-\xi/(2t)}\eta(w,t)\,dt = \frac{1}{(2\pi)^{(n-1)/2}}\int_0^\infty t^{-(n-1)/2}\frac{\partial}{\partial\xi} e^{-\xi/(2t)}\eta(w,t)\,dt = -\pi\Phi_{n+2}^a(w,\xi).$$

Consequently, for $n = 1,2,\ldots,$

$$\Phi_{2n}^a(w,\xi) = \frac{(-1)^{n-1}}{\pi^{n-1}}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\Phi_2^a(w,\xi). \tag{19}$$

From (17) and Theorem 5.1 (see Appendix) we get

$$\Phi_2^a(w,\xi) = \frac{1}{2a}\frac{\sin(\pi w/a)}{\cosh(\pi\sqrt{\xi}/a) - \cos(\pi w/a)}.$$

The result for all $w \in \mathbb{R}$ comes from the fact that both sides of (18) are odd and $2a$-periodic with respect to w. ∎

Proof of Theorem 1.2. Fix $x \in D_a$. For $y \in \partial_1 D_a$ we combine (18), (17) and Theorem 1.1 to get

$$\begin{aligned} P_{D_a}^X(x,y) &= \frac{2\pi^{n/2}}{\Gamma(n/2)} x_n^{n-1} y_n \int_{-1}^1 (1-u^2)^{(n-2)/2} \\ &\quad\times \Phi_{2n}^a\Big(|x_1 - y_1|, \sum_{k=2}^{n-1}|x_k - y_k|^2 + x_n^2 + y_n^2 + 2x_n y_n u\Big)\,du \\ &= \frac{(-1/2)^{n-1}\pi^{1-n/2}}{a\Gamma(n/2)y_n^{n-2}}\int_{-1}^1 (1-u^2)^{(n-2)/2} \\ &\quad\times \frac{\partial^{n-1}}{\partial u^{n-1}}\frac{\sin(\pi(x_1-y_1)/a)}{\cosh(\pi\sqrt{2x_n y_n(\cosh\tilde\rho + u)}/a) - \cos(\pi(x_1-y_1)/a)}\,du. \end{aligned} \tag{20}$$

The Green function of $(0,a)\times\mathbb{R}^{2n-1}$ for the $2n$-dimensional Brownian motion $W(t)$ is given by

$$G^W_{(0,a)\times\mathbb{R}^{2n-1}}(x,y) = \int_0^\infty \gamma(t;x_1,y_1)\frac{1}{(2\pi t)^{(2n-1)/2}}\exp\Big(-\frac{|\tilde x - \tilde y|^2}{2t}\Big)\,dt,$$

where $\gamma(t;x,y)$ is the transition density of the one-dimensional standard Brownian motion killed on exiting $(0,a)$, given by [BS, p. 122]:

$$\gamma(t;v,w)=\frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\Big[\exp\Big(-\frac{(v-w+2ka)^2}{2t}\Big)-\exp\Big(-\frac{(v+w+2ka)^2}{2t}\Big)\Big],\qquad v,w\in(0,a).$$

Let us define

$$\phi_{2n}^a(v,w,\xi)=\int_0^\infty \gamma(t;v,w)\frac{1}{(2\pi t)^{(2n-1)/2}}\exp\Big(-\frac{\xi}{2t}\Big)\,dt.$$

Since $\gamma(t;v,w)$ is less than $\frac{1}{\sqrt{2\pi t}}e^{-(v-w)^2/(2t)}$ (the density of standard Brownian motion), we can differentiate under the integral sign to obtain

$$\phi_{2n}^a(v,w,\xi)=\frac{1}{(-\pi)^{n-1}}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\phi_2^a(v,w,\xi).\tag{21}$$

Using the second part of Theorem 5.1, we obtain

$$\phi_2^a(x_1,y_1,\xi)=\frac{1}{2\pi}\ln\Big[1+\frac{2\sin(\pi x_1/a)\sin(\pi y_1/a)}{\cosh(\pi\sqrt{\xi}/a)-\cos(\pi(x_1-y_1)/a)}\Big].\tag{22}$$

Then

$$G^W_{(0,a)\times\mathbb{R}^{2n-1}}(x,y)=\phi_{2n}^a(x_1,y_1,|\tilde{x}-\tilde{y}|^2)=\frac{(-1)^{n-1}}{2\pi^n}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\ln\Big[1+\frac{2\sin(\pi x_1/a)\sin(\pi y_1/a)}{\cosh(\pi\sqrt{\xi}/a)-\cos(\pi(x_1-y_1)/a)}\Big]_{\xi=|\tilde{x}-\tilde{y}|^2}.$$

By Theorem 1.1 we get

$$P^X_{D_a}(x,y)=\frac{(-1)^{n-1}x_n^{n-1}}{\Gamma((n-1)/2)\pi^{(n-1)/2}}\times\frac{\partial^{n-1}}{\partial\xi^{n-1}}\ln\Big[1+\frac{2\sin(\pi x_1/a)\sin(\pi y_1/a)}{\cosh(\pi\sqrt{\xi}/a)-\cos(\pi(x_1-y_1)/a)}\Big]_{\xi=|\tilde{x}-\tilde{y}|^2},$$

where $y\in\partial_2 D_a$. Moreover

$$G^X_{D_a}(x,y)=\frac{2\pi^{n/2}x^{n-1}}{\Gamma(n/2)y_n}\int_{-1}^{1}(1-s^2)^{n/2-1}\phi_{2n}^a(x_1,y_1,2x_ny_n(\cosh\tilde{\rho}+s))\,ds$$

$$=\frac{(-1/2)^{n-1}}{\pi^{n/2}\Gamma(n/2)y_n^n}\int_{-1}^{1}(1-s^2)^{n/2-1}\times\frac{\partial^{n-1}}{\partial s^{n-1}}\ln\Big[1+\frac{2\sin(\pi x_1/a)\sin(\pi y_1/a)}{\cosh(\pi\sqrt{2x_ny_n(\cosh\tilde{\rho}+s)}/a)-\cos(\pi(x_1-y_1)/a)}\Big]\,ds.\ \blacksquare$$

To estimate the Poisson kernel of D_a we need the estimates of $\Phi^a_{2n}(w,\xi)$ in Lemma 4.3. In the proof we consider the function

$$\Lambda_n(\xi,w) := \frac{2(-\pi)^n}{\sin w}\Phi^{\pi}_{2(n+1)}(w,\xi) = \frac{\partial^n}{\partial\xi^n}\,\frac{1}{\cosh\sqrt{\xi}-\cos w}, \tag{23}$$

where $w\in\mathbb{R}$, $\xi>0$ and $n=0,1,2,\ldots$. A more convenient form of this function is a sum of elementary functions, which is given in Lemma 4.2.

Lemma 4.2. *For $w\in\mathbb{R}$, $\xi>0$ and $n=1,2,\ldots$,*

$$\Lambda_n(\xi,w) = \sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^n_{\alpha^k}\,\frac{\prod_{i=1}^k \frac{\partial^{1+\alpha^k_i}}{\partial\xi^{1+\alpha^k_i}}\cosh\sqrt{\xi}}{(\cosh\sqrt{\xi}-\cos w)^{k+1}}, \tag{24}$$

where α^k denotes a k-dimensional multi-index. Moreover, the constants $c^n_{\alpha^k}$ satisfy

(1) $c^n_{\alpha^k}\ge 0$.
(2) $c^n_{\alpha^n}=n!$ *for* $\alpha^n=(0,\ldots,0)$.
(3) $\sum_{k=1}^n\sum_{|\alpha^k|=n-k}(-1)^k c^n_{\alpha^k}=(-1)^n$.

Proof. We use induction on n. Let us denote

$$h^{(n)}(\xi)=\frac{\partial^n}{\partial\xi^n}\cosh\sqrt{\xi},\qquad \xi>0.$$

For $n=1$ we have

$$\Lambda_1(\xi,w) = -\frac{h'(\xi)}{(\cosh\sqrt{\xi}-\cos w)^2}.$$

Assume that (24) is true for a fixed $n\in\mathbb{N}$. Differentiating the right-hand side, we get

$$\begin{aligned}\sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^n_{\alpha^k}\Bigg[&\frac{\sum_{m=1}^k\prod_{i=1}^k h^{(1+(\alpha^k+e^k_m)_i)}(\xi)}{(\cosh\sqrt{\xi}-\cos w)^{k+1}}\\ &-(k+1)\frac{h^{(1)}(\xi)\prod_{i=1}^k h^{(1+\alpha^k_i)}(\xi)}{(\cosh\sqrt{\xi}-\cos w)^{k+2}}\Bigg]\\ =\sum_{k=1}^{n}\sum_{|\alpha^k|=n+1-k}&(-1)^k\sum_{m=1}^{k}c^n_{\alpha^k-e^k_m}\frac{\prod_{i=1}^k h^{(1+\alpha^k_i)}(\xi)}{(\cosh\sqrt{\xi}-\cos w)^{k+1}}\\ +\sum_{k=2}^{n+1}\sum_{\substack{|\alpha^k|=n+1-k\\ \alpha^k_k=0}}&(-1)^k k\, c^n_{\alpha^k\restriction\mathbb{N}^{k-1}}\frac{\prod_{i=1}^k h^{(1+\alpha^k_i)}(\xi)}{(\cosh\sqrt{\xi}-\cos w)^{k+1}},\end{aligned} \tag{25}$$

where e^k_m is the k-dimensional multi-index with the mth coordinate 1 and the others 0. We also make the convention that $c^n_{\alpha^k-e^k_m}=0$ if $\alpha^k_m=0$. We can see that all components of the above sums are of the form (24) and have proper signs. The coefficient $c^{n+1}_{n,\alpha^{n+1}}$ which appears in the last sum for $k = n+1$ is equal to $(n+1)!$. Moreover, (25) yields the formula for the sum of the coefficients:

$$\begin{aligned}\sum_{k=1}^{n+1}\sum_{|\alpha^k|=n+1-k}(-1)^k c^{n+1}_{\alpha^k} &= \sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^{n}_{\alpha^k}\Big(\sum_{m=1}^{k}1-(k+1)\Big)\\ &= \sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^{n}_{\alpha^k}(k-(k+1))\\ &= -\sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^{n}_{\alpha^k} = (-1)^{n+1}.\end{aligned}$$

This yields (24) for $n+1$, completing the proof. ∎

LEMMA 4.3. *For $\xi > 0$ and $w \in (0, a)$ we have*

$$\Phi^a_{2n}(w,\xi) \overset{c}{\asymp} \frac{w(a-w)}{a^{n+2}}\exp\left(-\frac{\pi}{a}\sqrt{\xi}\right)\frac{a^{n+1}+(\xi+w^2)^{(n+1)/2}}{(\xi+w^2)^n}.$$

Proof. Since $\Phi^a_{2n}(w,\xi)$ has the following scaling property (see Lemma 5.2):

$$\Phi^a_n(w,\xi) = \frac{1}{a^{n-1}}\Phi^1_n\left(\frac{w}{a},\frac{\xi}{a^2}\right),$$

it is enough to prove the lemma for $a = \pi$. Using (23) we can write

$$\Phi^{\pi}_{2(n+1)}(w,\xi) \overset{c}{\asymp} w(\pi-w)\Lambda_n(\xi,w).$$

Recall that

$$h^{(n)}(\xi) = \frac{\partial^n}{\partial\xi^n}\cosh\sqrt{\xi} = \sum_{k=n}^{\infty}\frac{\xi^{k-n}}{(2k)!}\frac{k!}{(k-n)!}.$$

It is easy to see that

$$h^{(n)}(0^+) := \lim_{\xi\to0^+} h^{(n)}(\xi) = \frac{n!}{(2n)!} \tag{26}$$

and

$$\begin{aligned}\lim_{\xi+w^2\to0}\frac{\xi+w^2}{\cosh\sqrt{\xi}-\cos w} &= \lim_{\xi+w^2\to0}\frac{\xi+w^2}{1+\frac12\xi+o(\xi)-1+\frac12 w^2+o(w^2)}\\ &= \lim_{\xi+w^2\to0}\frac{2}{1+\frac{o(\xi)+o(w^2)}{\xi+w^2}} = 2.\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{\xi+w^2\to 0}(\xi+w^2)^{n+1}\Lambda_n(\xi,w)\\
&\quad\overset{(24)}{=}\lim_{\xi+w^2\to 0}\sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^n_{\alpha^k}\prod_{i=1}^{k}h^{(1+\alpha^k_i)}(\xi)(\xi+w^2)^{n-k}\\
&\qquad\qquad\times\left(\frac{\xi+w^2}{\cosh\sqrt{\xi}-\cos w}\right)^{k+1}\\
&\quad=2(-1)^n n!.
\end{aligned}$$

The last equality follows from (26) and Lemma 4.2(2). Consequently,

$$\lim_{\xi+w^2\to 0} e^{\sqrt{\xi}}\frac{(\xi+w^2)^{n+1}}{1+(\xi+w^2)^{n/2+1}}(-1)^n\Lambda_n(\xi,w)=2n!.$$

We will show now that

$$\lim_{\xi\to\infty} h^{(n)}(\xi)\xi^{n/2}e^{-\sqrt{\xi}}=2^{-n-1}. \tag{27}$$

We have

$$\begin{aligned}
2^{n+1}h^{(n)}(\xi)\xi^{n/2}e^{-\sqrt{\xi}}&=2e^{-\sqrt{\xi}}\sum_{k=n}^{\infty}\frac{\xi^{k-n/2}}{(2k-n)!}\,\frac{2^n k!(2k-n)!}{(k-n)!(2k)!}\\
&=2e^{-\sqrt{\xi}}\sum_{k=n}^{\infty}\frac{\xi^{k-n/2}}{(2k-n)!}\left(\prod_{j=0}^{n-1}\frac{2(k-j)}{(2k-j)}\right).
\end{aligned}$$

The series $\sum_{k=n}^{\infty}\xi^{k-n/2}/(2k-n)!$ is equal to $\sinh\sqrt{\xi}$ for odd n and to $\cosh\sqrt{\xi}$ for even n up to the first $\lceil n/2\rceil$ components, so it behaves like $e^{\sqrt{\xi}}/2$ for large ξ. Moreover

$$\lim_{k\to\infty}\prod_{j=0}^{n-1}\frac{2(k-j)}{(2k-j)}=1.$$

In order to obtain (27) we use Lemma 5.4. Thus

$$\begin{aligned}
&\lim_{\xi+w^2\to\infty}(\xi+w^2)^{n/2}e^{\sqrt{\xi}}\Lambda_n(\xi,w)\\
&\overset{(24)}{=}\lim_{\xi+w^2\to\infty}\left(\frac{\xi+w^2}{\xi}\right)^{n/2}\sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^n_{\alpha^k}\\
&\qquad\qquad\times\prod_{i=1}^{k}\frac{h^{(1+\alpha^k_i)}(\xi)\xi^{(1+\alpha^k_i)/2}}{e^{\sqrt{\xi}}}\left(\frac{e^{\sqrt{\xi}}}{\cosh\sqrt{\xi}-\cos w}\right)^{k+1}\\
&\overset{(27)}{=}2^{-n+1}\sum_{k=1}^{n}\sum_{|\alpha^k|=n-k}(-1)^k c^n_{\alpha^k}=(-1)^n 2^{-n+1}.
\end{aligned}$$

The factor $\left(\frac{\xi+w^2}{\xi}\right)^{n/2}$ vanishes because $w \in (0,\pi)$, so $\lim_{\xi+w^2\to\infty} \frac{\xi+w^2}{\xi} = 1$. It follows that

$$\lim_{\xi+w^2\to\infty} e^{\sqrt{\xi}} \frac{(\xi+w^2)^{n+1}}{1+(\xi+w^2)^{n/2+1}} (-1)^n \Lambda_n(\xi,w) = 2^{-n+1}.$$

What we have already proved is that there exists a constant $c > 0$ such that

$$\frac{1}{c} < \Lambda_n(\xi,w)\left(\frac{1+(\xi+w^2)^{n/2+1}}{(\xi+w^2)^{n+1}} e^{-\sqrt{\xi}}\right)^{-1} < c \tag{28}$$

for $\xi+w^2 < \varepsilon$ and for $\xi+w^2 > M$, for some $\varepsilon, M > 0$. The function $\Lambda_n(\xi,w)$ is continuous on the set $A = ([0,\infty)\times[0,\pi]) - \{0,0\}$, by (24). Moreover, it is positive on $(0,\infty)\times(0,\pi)$, by (17). It is also positive on $(0,\infty)\times\{0,\pi\}$, since the Poisson kernel of the strip $(0,\pi)\times\mathbb{R}^{2n-1}$ for Brownian motion (given by (17)) behaves locally as the distance from the boundary, so it is positive on the whole set A. The set $\{(\xi,w)\in A : \varepsilon \le \xi+w^2 \le M\}$ is compact, so there exists a constant $c > 0$ such that (28) holds for every $(\xi,w)\in A$. ∎

Proof of Theorem 1.3. By Theorem 1.1 and scaling properties for standard Brownian motion we get

$$G^X_{D_a}(x,y) = \left(\frac{\pi}{a}\right)^{n-1} G^X_{D_\pi}\left(\frac{\pi}{a}x, \frac{\pi}{a}y\right),$$
$$P^X_{D_a}(x,y) = \left(\frac{\pi}{a}\right)^{n-1} P^X_{D_\pi}\left(\frac{\pi}{a}x, \frac{\pi}{a}y\right),$$

so that it is sufficient to prove the result for $a=\pi$. For $y \in \partial_1 D_\pi$ we apply Lemma 4.3 to (20) to get

$$P^X_{D_\pi}(x,y) \stackrel{c}{\asymp} x_1(\pi-x_1)x_n^n \int_{-1}^{1} (1-u^2)^{(n-2)/2} e^{-\sqrt{2x_ny_n(\cosh\rho+u)}} \times \frac{1+(2x_ny_n(\cosh\rho+u))^{(n+1)/2}}{(2x_ny_n(\cosh\rho+u))^n}\,du.$$

Now we use Lemma 5.3 with $p = 2x_ny_n$, $q = \cosh\rho-1$, $r=0$, $\alpha=(n-2)/2$, $\beta=(n+1)/2$ and $\gamma=n$ to obtain the required formula. To get asymptotic behavior for $y \in \partial_2 D$, we rewrite (22) for $a=\pi$ as

$$\phi_2^\pi(x_1,y_1,\xi) = \int_{x_1-y_1}^{x_1+y_1} \Phi_2^\pi(w,\xi)\,dw.$$

Using (21) we get

$$\begin{aligned}\phi_{2n}^{\pi}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) &= \frac{1}{(-\pi)^{n-1}}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\int_{x_1-y_1}^{x_1+y_1}\Phi_2^{\pi}(w,\xi)\,dw\Big|_{\xi=|\tilde{x}-\tilde{y}|^2}\\ &= \frac{1}{(-\pi)^{n-1}}\int_{x_1-y_1}^{x_1+y_1}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\Phi_2^{\pi}(w,\xi)\,dw\Big|_{\xi=|\tilde{x}-\tilde{y}|^2}\\ &\overset{(19)}{=} \int_{x_1-y_1}^{x_1+y_1}\Phi_{2n}^{\pi}(w,|\tilde{x}-\tilde{y}|^2)\,dw.\end{aligned}$$

The properties

$$\Phi_n^{\pi}(-w,\xi) = -\Phi_n^{\pi}(w,\xi), \qquad \Phi_n^{\pi}(\pi+w,\xi) = -\Phi_n^{\pi}(\pi-w,\xi)$$

give

$$\phi_{2n}^{\pi}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) = \int_{|x_1-y_1|}^{\pi-|x_1+y_1-\pi|}\Phi_{2n}^{\pi}(w,|\tilde{x}-\tilde{y}|^2)\,dw.$$

Note that for $x_1, y_1 \in (0,\pi)$ we have

$$0 \le |x_1-y_1| < \pi - |x_1+y_1-\pi| \le \pi.$$

By Lemma 4.3 we see that

$$\begin{aligned}&\phi_{2n}^{\pi}(x_1,y_1,|\tilde{x}-\tilde{y}|^2)\\ &\quad\overset{c}{\asymp} e^{-|\tilde{x}-\tilde{y}|}\int_{|x_1-y_1|}^{\pi-|x_1+y_1-\pi|} w(\pi-w)\frac{\pi^{n+1}+(|\tilde{x}-\tilde{y}|^2+w^2)^{(n+1)/2}}{(|\tilde{x}-\tilde{y}|^2+w^2)^n}\,dw.\end{aligned}\tag{29}$$

We will show that

$$\begin{aligned}&\phi_{2n}^{\pi}(x_1,y_1,|\tilde{x}-\tilde{y}|^2)\\ &\quad\overset{c}{\asymp} e^{-|x-y|}\frac{\delta_\pi(x_1)\delta_\pi(y_1)}{|x-y|^2+\delta_\pi(x_1)\delta_\pi(y_1)}\,\frac{\pi^{n+1}+(|x-y|^2)^{(n+1)/2}}{(|x-y|^2)^{n-1}},\end{aligned}\tag{30}$$

where $\delta_\pi(s) = \min\{s, \pi-s\}$. Since $x_1, y_1 \in (0,\pi)$, we have

$$e^{-|\tilde{x}-\tilde{y}|} \overset{c}{\asymp} e^{-|x-y|}.$$

For $|x_1-y_1| \le \pi/2$ we apply Lemma 5.5 to obtain

$$\begin{aligned}\phi_{2n}^{\pi}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) &\overset{c}{\asymp} e^{-|x-y|}\int_{|x_1-y_1|}^{\pi-|x_1+y_1-\pi|} w\frac{\pi^{n+1}+(|\tilde{x}-\tilde{y}|^2+w^2)^{(n+1)/2}}{(|\tilde{x}-\tilde{y}|^2+w^2)^n}\,dw\\ &= \frac{e^{-|x-y|}}{2}\int_{|x_1-y_1|^2}^{(\pi-|x_1+y_1-\pi|)^2}\frac{\pi^{n+1}+(|\tilde{x}-\tilde{y}|^2+w)^{(n+1)/2}}{(|\tilde{x}-\tilde{y}|^2+w)^n}\,dw.\end{aligned}$$

Then, for $|\tilde{x}-\tilde{y}|>\pi$, we have

$$\phi^{\pi}_{2n}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) \stackrel{c}{\asymp} e^{-|x-y|}\int_{|x_1-y_1|^2}^{(\pi-|x_1+y_1-\pi|)^2}\frac{1}{|\tilde{x}-\tilde{y}|^{n-1}}\,dw$$
$$= e^{-|x-y|}\frac{(\pi-|x_1+y_1-\pi|)^2-(x_1-y_1)^2}{(|\tilde{x}-\tilde{y}|^2)^{(n-1)/2}}.$$

Putting $\alpha=x_1$ and $\beta=\pi-y_1$ in Lemma 5.7, we get

$$(\pi-|x_1+y_1-\pi|)^2-(x_1-y_1)^2 \stackrel{c}{\asymp} \frac{x_1y_1(\pi-x_1)(\pi-y_1)}{\pi^2-(x_1-y_1)^2}.$$

Together with the condition $|x_1-y_1|<\pi/2$ this gives

$$(\pi-|x_1+y_1-\pi|)^2-(x_1-y_1)^2 \stackrel{c}{\asymp} \delta_\pi(x_1)\delta_\pi(y_1). \tag{31}$$

Consequently,

$$\phi^{\pi}_{2n}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) \stackrel{c}{\asymp} e^{-|x-y|}\frac{x_1y_1(\pi-x_1)(\pi-y_1)}{|\tilde{x}-\tilde{y}|^{n-1}},$$

which satisfies (30) under the current assumptions. For $|\tilde{x}-\tilde{y}|\le\pi$ we have

$$\phi^{\pi}_{2n}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) \stackrel{c}{\asymp} e^{-|x-y|}\int_{|x_1-y_1|^2}^{(\pi-|x_1+y_1-\pi|)^2}\frac{1}{(|\tilde{x}-\tilde{y}|^2+w)^n}\,dw$$
$$= e^{-|x-y|}\left(\frac{1}{(|x-y|^2)^{n-1}}-\frac{1}{(|\tilde{x}-\tilde{y}|^2+(\pi-|x_1+y_1-\pi|)^2)^{n-1}}\right).$$

Now, we use Lemma 5.6 and (31) to get

$$\phi^{\pi}_{2n}(x_1,y_1,|\tilde{x}-\tilde{y}|^2) \stackrel{c}{\asymp} \frac{(\pi-|x_1+y_1-\pi|)^2-(x_1-y_1)^2}{|\tilde{x}-\tilde{y}|^2+(\pi-|x_1+y_1-\pi|)^2}\,\frac{1}{(|x-y|^2)^{n-1}}$$
$$\stackrel{c}{\asymp} \frac{\delta_\pi(x_1)\delta_\pi(y_1)}{|\tilde{x}-\tilde{y}|^2+(\pi-|x_1+y_1-\pi|)^2}\,\frac{1}{(|x-y|^2)^{n-1}}.$$

Again, it is comparable with (30). Since $\frac{fg}{f+g}\stackrel{c}{\asymp} f\wedge g$ for positive functions f and g, we put $f=|x-y|^2$, $g=\delta_a(x_1)\delta_a(y_1)$ and get the desired formula.

Let us now consider the case when $|x_1-y_1|>\pi/2$. This implies that $\pi/2<|x_1-y_1|<\pi-|x_1+y_1-\pi|<\pi$, so we can estimate the variable w

in (29) by π or by $|x_1 - y_1|$. Thus,

$$
\begin{aligned}
(32)\qquad & \int_{|x_1-y_1|}^{\pi-|x_1+y_1-\pi|} w(\pi-w)\frac{\pi^{n+1}+(|\tilde{x}-\tilde{y}|^2+w^2)^{(n+1)/2}}{(|\tilde{x}-\tilde{y}|^2+w^2)^n}\,dw \\
&\overset{c}{\asymp} \int_{|x_1-y_1|}^{\pi-|x_1+y_1-\pi|} (\pi-w)\frac{\pi^{n+1}+|x-y|^{n+1}}{|x-y|^{2n}}\,dw \\
&= \frac{\pi^{n+1}+|x-y|^{n+1}}{|x-y|^{2n}}\big((|x_1+y_1|-\pi)^2-(x+y-\pi)^2\big) \\
&\overset{c}{\asymp} \frac{x_1y_1(\pi-x_1)(\pi-y_1)}{(x_1+y_1)(2\pi-x_1-y_1)}\,\frac{\pi^{n+1}+|x-y|^{n+1}}{|x-y|^{2n}},
\end{aligned}
$$

where the last equivalence follows from Lemma 5.7. By the assumptions $|x_1 - y_1| > \pi/2$ and $|\tilde{x} - \tilde{y}| \le \pi$ we get

$$
\begin{aligned}
\pi/2 &< x_1 + y_1 < 3\pi/2, \\
\pi/2 &< 2\pi - x_1 - y_1 < 3\pi/2, \\
|x-y| &\overset{c}{\asymp} |x-y| + \delta_\pi(x_1)\delta_\pi(y_1).
\end{aligned}
$$

Applying these formulas to (32) we obtain (30). Since $G^W_{(0,\pi)\times\mathbb{R}^{2n-1}}(x,y) = \phi^\pi_{2n}(x_1,y_1,|\tilde{x}-\tilde{y}|^2)$, taking into account Theorem 1.1 completes the estimation of the Poisson kernel $P^X_{D_a}(x,y)$. The estimates of the Green function $G^W_{(0,\pi)\times\mathbb{R}^{2n-1}}(x,y)$ imply also

$$
\begin{aligned}
G^X_{D_\pi}(x,y) \overset{c}{\asymp} & \frac{\delta_\pi(x_1)\delta_\pi(y_1)x_n^{n-1}}{y_n}\int_{-1}^{1}(1-u^2)^{(n-2)/2} \\
&\times \frac{e^{-\sqrt{2x_ny_n(\cosh\rho+u)}}}{x_ny_n(\cosh\rho+u)+\delta_a(x_1)\delta_a(y_1)}\,\frac{1+(2x_ny_n(\cosh\rho+u))^{(n+1)/2}}{(2x_ny_n(\cosh\rho+u))^{n-1}}\,du.
\end{aligned}
$$

We apply Lemma 5.3 with $p = 2x_ny_n$, $q = \cosh\rho - 1$, $r = \delta_a(x_1)\delta_a(y_1)$, $\alpha = (n-2)/2$, $\beta = (n+1)/2$ and $\gamma = n-1$ to obtain

$$
\begin{aligned}
G^X_{D_\pi}(x,y) \overset{c}{\asymp} & \frac{e^{-|x-y|}}{x_ny_n}\left(\frac{x_n}{y_n}\right)^{n/2} \\
&\times \frac{\delta_a(x_1)\delta_a(y_1)}{\big(|x-y|^2+\delta_a(x_1)\delta_a(y_1)\big)|x-y|^{n-2}}\,\frac{1+|x-y|^{n+1}}{(|x-y|+\cosh\rho)^{n/2}}. \qquad \blacksquare
\end{aligned}
$$

5. Appendix

THEOREM 5.1. *Let $W(t) = (W_1(t), W_2(t))$ be a two-dimensional Brownian motion. Then*

$$P^W_{(-\pi/4,\pi/4)\times\mathbb{R}}(x, y) = \frac{1}{\pi}\frac{\sin 2|x_1 - y_1|}{\cosh 2(x_2 - y_2) - \cos 2(x_1 - y_1)},$$

where $x \in (-\pi/4, \pi/4)\times\mathbb{R}$, $y \in \{-\pi/4, \pi/4\}\times\mathbb{R}$. For $x, y \in (-\pi/4, \pi/4)\times\mathbb{R}$ we get

$$G^W_{(-\pi/4,\pi/4)\times\mathbb{R}}(x, y) = \frac{1}{2\pi}\ln\left[1 + \frac{2\cos 2x_1 \cos 2y_1}{\cosh 2(x_2 - y_2) - \cos 2(x_1 - y_1)}\right].$$

Proof. For $x \in \mathbb{R}$ and $y \in (-\pi/4, \pi/4)$,

$$\tan(x + iy) = \frac{\sin 2x}{\cosh 2y + \cos 2x} + i\frac{\sinh 2y}{\cosh 2y + \cos 2x} = u(x, y) + iv(x, y).$$

This function is a continuous bijection from the set $[-\pi/4, \pi/4]\times\mathbb{R}$ onto the set $\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \le 1\} - \{(0, 1), (0, -1)\}$. For fixed $y_0 \in \mathbb{R}$ we have

$$\{\tan(\pi/4 + it) : t < y_0\} = \left\{\frac{1}{\cosh 2t} + i\frac{\sinh 2t}{\cosh 2t} : t < y_0\right\}$$

$$= \{u + iv : u > 0, u^2 + v^2 = 1, v/u < \sinh 2y_0\} \tag{33}$$

$$= \{e^{i\varphi} : \varphi \in (-\pi/2, \arctan(\sinh 2y_0))\}. \tag{34}$$

Brownian motion is space-homogeneous, so it is sufficient to consider $x = (x_1, 0)$. Since the function $\tan z$ is holomorphic on $\{z \in \mathbb{C} : |\Re(z)| < \pi/2\}$, the process $Y(t) = (Y_1(t), Y_2(t)) = (u(W_1(t), W_2(t)), v(W_1(t), W_2(t)))$ is another Brownian motion with a continuous time-change. Hence, the density function of $Y(\tau^W_{(-\pi/4,\pi/4)\times\mathbb{R}})$ is the Poisson kernel of the ball $B(0, 1)$, which is well known and given by

$$P^Y_{B(0,1)}(u, w) = \frac{1 - |u|^2}{2\pi|u - w|^2},$$

where $u \in B(0, 1)$ and $w \in \partial B(0, 1)$. Denoting $\tau := \tau^W_{(-\pi/4,\pi/4)\times\mathbb{R}}$ we get

$$\begin{aligned}
\mathbb{P}^x(W_1(\tau) = \pi/4, W_2(\tau) < t)
&\overset{(33)}{=} \mathbb{P}^{(\tan x_1,0)}\big(Y_1(\tau) > 0,\ Y_1(\tau)/Y_2(\tau) < \sinh 2t\big) \\
&\overset{(34)}{=} \frac{1}{2\pi}\int_{-\pi/2}^{\arctan(\sinh 2t)} \frac{1 - (\tan x_1)^2}{(\tan x_1 - \cos\varphi)^2 + \sin^2\varphi}\,d\varphi \\
&= \frac{1}{2\pi}\int_{-\pi/2}^{\arctan(\sinh 2t)} \frac{\cos 2x_1}{1 - \cos\varphi \sin 2x_1}\,d\varphi.
\end{aligned}$$

Changing variables according to $\tan\varphi = \sinh 2u$, we obtain $\frac{d\varphi}{\cos^2\varphi} = 2\cosh 2u\, du$. Moreover,

$$1/\cos^2\varphi = \tan^2\varphi + 1 = \sinh^2 2u + 1 = \cosh^2 2u,$$
$$\cos\varphi = 1/\cosh 2u,$$
$$d\varphi = \frac{2\cosh 2u\, du}{\sinh^2 2u + 1}\, du = \frac{2du}{\cosh 2u}.$$

Consequently, for $y_1 = \pi/4$, we have

$$\mathbb{P}\big(B_1(\tau^B) = \pi/4, B_2(\tau^B) < t\big) = \frac{1}{\pi}\int_{-\infty}^{t} \frac{\cos 2x_1}{\cosh 2u - \sin 2x_1} du = \frac{1}{\pi}\int_{-\infty}^{t} \frac{\sin 2(y_1 - x_1)}{\cosh 2u - \cos 2(x_1 - y_1)} du.$$

The symmetry of Brownian motion gives us, for $y_1 = -\pi/4$,

$$\begin{aligned}\mathbb{P}^{(x_1,0)}&\big(B_1(\tau^B) = -\pi/4,\ B_2(\tau^B) \in dt\big)\\ &= \mathbb{P}^{(-x_1,0)}\big(-B_1(\tau^B) = \pi/4,\ B_2(\tau_B) \in dt\big)\\ &\overset{(35)}{=} \frac{1}{\pi}\,\frac{\cos(-2x_1)}{\cosh 2t - \sin(-2x_1)}\\ &= \frac{1}{\pi}\,\frac{\sin 2(x_1 - y_1)}{\cosh 2t - \cos 2(x_1 - y_1)}.\end{aligned}$$

The Green function $G^W_{(-\pi/4,\pi/4)\times\mathbb{R}}(x,y)$ is harmonic in the interior of the set $(-\pi/4,\pi/4)\times\mathbb{R}$, vanishes at the boundary and its derivative with respect to the normal vector is equal at the boundary to the Poisson kernel (see [C]). It is easy to check that the formula given in the theorem satisfies all these conditions. ∎

LEMMA 5.2. *The function* $\Phi^a_n(w,\xi)$ *defined by* (16) *has the following scaling property:*

$$\Phi^a_n(w,\xi) = \frac{1}{a^{n-1}}\Phi^1_n\bigg(\frac{w}{a},\frac{\xi}{a^2}\bigg).$$

Proof. From (16) and (14) we obtain

$$\begin{aligned}\Phi^a_n(w,\xi) &= \int_0^\infty \frac{1}{(2\pi t)^{(n-1)/2}} e^{-\xi/(2t)} \sum_{k=-\infty}^{\infty} \frac{w + 2ka}{\sqrt{2\pi}\, t^{3/2}} \exp\bigg(-\frac{(w+2ka)^2}{2t}\bigg)\, dt\\ &= a\int_0^\infty \frac{1}{(2\pi t)^{(n-1)/2}} e^{-\frac{\xi/a^2}{2t/a^2}} \sum_{k=-\infty}^{\infty} \frac{w/a + 2k}{\sqrt{2\pi}\, t^{3/2}} \exp\bigg(-\frac{(w/a+2k)^2}{2t/a^2}\bigg)\, dt.\end{aligned}$$

Changing variables $t/a^2 = u$ ends the proof. ∎

LEMMA 5.3. *There exists $c = c(\alpha,\beta,\gamma)$ such that for $p, q > 0$, $r \ge 0$ and $\alpha \ge 0$, $\beta > (\alpha+1)/2$, $\gamma > \alpha+1$ we have*

$$\int_{-1}^{1} (1-u^2)^\alpha \frac{e^{-\sqrt{p(q+1+u)}}}{p(q+1+u)+r} \frac{1+(p(q+1+u))^\beta}{(p(q+1+u))^\gamma}\,du \overset{c}{\asymp} \frac{e^{-\sqrt{pq}}p^{-\gamma-1}q^{\alpha-\gamma}}{(1+r/(pq))} \frac{(1+pq)^\beta}{(\sqrt{pq}+q+1)^{\alpha+1}}.$$

If $r = 0$, then it is sufficient that $\gamma > \alpha$.

Proof. Let us denote the integral on the left-hand side of the above equivalence by $\mathcal{I}(p,q)$. The function $(1-u^2)^\alpha$ is symmetric and non-negative, and the other two factors under the integral are decreasing. It follows that the integral over $(-1,1)$ is smaller than twice the integral over $(-1,0)$. Thus, using the estimate $1+u \le 1-u^2 \le 2(1+u)$ for $u \in (-1,0)$, we get

$$\mathcal{I}(p,q) \overset{c}{\asymp} \int_{-1}^{0} (1+u)^\alpha \frac{e^{-\sqrt{p(q+1+u)}}}{p(q+1+u)+r} \frac{1+(p(q+1+u))^\beta}{(p(q+1+u))^\gamma}\,du.$$

Substituting $u+1 = q[(t+1)^2-1] = qt(t+2)$ with $q+1+u = q(t+1)^2$ and $du = 2q(t+1)dt$ we get

$$\mathcal{I}(p,q) \overset{c}{\asymp} \frac{e^{-\sqrt{pq}}q^{\alpha-\gamma}}{p^{\gamma+1}} \int_{0}^{w(q)} \frac{[t(t+2)]^\alpha e^{-\sqrt{pq}\,t}}{(t+1)^2+r/(pq)} \frac{1+(\sqrt{pq}(t+1))^{2\beta}}{(t+1)^{2\gamma-1}}\,dt, \tag{35}$$

where $w(q) = \sqrt{1+1/q}-1$. If $w(q) < 1$, then

$$\begin{aligned}\mathcal{I}(p,q) &\overset{c}{\asymp} e^{-\sqrt{pq}}p^{-\gamma-1}q^{\alpha-\gamma}\frac{(1+pq)^\beta}{1+r/(pq)} \int_{0}^{w(q)} t^\alpha e^{-\sqrt{pq}\,t}\,dt \\ &\overset{c}{\asymp} \frac{e^{-\sqrt{pq}}p^{-\gamma-1}q^{\alpha-\gamma}}{1+r/(pq)} \frac{(1+pq)^\beta}{(\sqrt{pq}+1/w(q))^{\alpha+1}}.\end{aligned} \tag{36}$$

The second estimate comes from [BMR, Lemma 12]. For $w(q) \ge 1$ we divide the integral in (35) into $\int_0^1 + \int_1^{w(q)} =: I_1 + I_2$. Then

$$I_1 \overset{c}{\asymp} \frac{(1+pq)^\beta}{1+r/(pq)} \int_0^1 t^\alpha e^{-\sqrt{pq}\,t}\,dt \overset{c}{\asymp} \frac{(1+pq)^\beta}{(1+r/(pq))(1+\sqrt{pq})^{\alpha+1}},$$

$$I_2 \overset{c}{\asymp} \int_{0}^{w(q)-1} \frac{e^{-\sqrt{pq}(t+1)}}{(t+1)^2+r/(pq)} \frac{1+(\sqrt{pq}(t+1))^{2\beta}}{(t+1)^{2\gamma-2\alpha-1}}\,dt$$

$$\le \frac{\sup_{s>0} e^{-s}(1+s^{2\beta})}{1+r/(pq)} \int_1^\infty \frac{1}{(t+1)^{2\gamma-2\alpha-1+2\operatorname{sgn} r}}\, dt$$
$$\overset{c}{\asymp} \frac{1}{1+r/(pq)}.$$

Since $\beta > (\alpha+1)/2$, we get $I_1+I_2 \overset{c}{\asymp} I_1$. Combining this with (36) we obtain

$$\mathcal{I}(p,q) \overset{c}{\asymp} \frac{e^{-\sqrt{pq}} p^{-\gamma-1} q^{\alpha-\gamma}}{1+r/(pq)} \, \frac{(1+pq)^\beta}{(\sqrt{pq}+1/w(q)+1)^{\alpha+1}}.$$

Moreover,

$$1/w(q)+1 = q+\sqrt{q^2+q}+1 \overset{c}{\asymp} 1+q,$$

which ends the proof. ■

Lemma 5.4. *Let*

$$f(x) = \sum_{k=0}^\infty a_k x^k, \qquad g(x) = \sum_{k=0}^\infty b_k x^k$$

for $x>0$ *and* $a_k \in \mathbb{R}$, $b_k>0$, $k=0,1,2,\ldots$. *If* $\lim_{k\to\infty} a_k/b_k = 1$, *then* $\lim_{x\to\infty} f(x)/g(x) = 1$.

Proof. Fix $\varepsilon>0$. We will show that $|f(x)/g(x)-1| < \varepsilon$ for large x. We have

$$\left|\frac{f(x)}{g(x)} - 1\right| = \left|\frac{\sum_{k=0}^\infty (a_k-b_k)x^k}{g(x)}\right|.$$

By the assumption $\lim_{k\to\infty} a_k/b_k = 1$, there exists $N\in\mathbb{N}$ such that $|a_k/b_k-1| < \varepsilon/2$ for all $k\ge N$. Then

$$\left|\frac{\sum_{k=0}^\infty (a_k-b_k)x^k}{g(x)}\right| \le \left|\frac{\sum_{k=0}^{N-1} (a_k-b_k)x^k}{g(x)}\right| + \left|\frac{\sum_{k=N}^\infty b_k(a_k/b_k-1)x^k}{g(x)}\right|$$
$$\le \left|\frac{\sum_{k=0}^{N-1} (a_k-b_k)x^k}{g(x)}\right| + \varepsilon/2.$$

Since $b_k>0$, the first term in the last expression is less than $\varepsilon/2$ for large x, which ends the proof. ■

Lemma 5.5. *Let* f *be a positive and decreasing function on* $(0,\pi)$. *For* $u\in(0,\pi/2)$ *and* $u<v<\pi$,

$$\int_u^v w(\pi-w) f(w)\, dw \overset{c}{\asymp} \pi \int_u^v w f(w)\, dw.$$

Proof. If $v < 3\pi/4$, then $\pi/4 < \pi - w < \pi$ for $w \in (u, v)$, and the formula is clearly true. For $v \geq 3\pi/4$ we have

$$\int_u^{3\pi/4} wf(w)\,dw > \int_{\pi/2}^{3\pi/4} wf(w)\,dw > \frac{\pi}{2}\int_{\pi/2}^{3\pi/4} f(w)\,dw$$
$$> \frac{1}{2}\int_{3\pi/4}^{v} \pi f(w)\,dw > \frac{1}{2}\int_{3\pi/4}^{v} wf(w)\,dw,$$

hence

$$\int_u^{3\pi/4} wf(w)\,dw > \frac{1}{2}\int_{3\pi/4}^{v} wf(w)\,dw. \tag{37}$$

It follows that

$$\begin{aligned}\int_u^v w(\pi - w)f(w)\,dw &\geq \int_u^{3\pi/4} w(\pi-w)f(w)\,dw > \frac{1}{4}\pi\int_u^{3\pi/4} wf(w)\,dw\\ &= \frac{1}{8}\pi\int_u^{3\pi/4} wf(w)\,dw + \frac{1}{8}\pi\int_u^{3\pi/4} wf(w)\,dw\\ &\overset{(37)}{>} \frac{1}{8}\pi\int_u^{3\pi/4} wf(w)\,dw + \frac{1}{16}\pi\int_{3\pi/4}^{v} wf(w)\,dw\\ &> \pi\frac{1}{16}\int_u^v wf(w)\,dw.\end{aligned}$$

The opposite inequality is obvious since $\pi - w \leq \pi$. ∎

We now recall [MS, Lemma 4].

LEMMA 5.6. *We have*

$$\frac{1}{x^\alpha} - \frac{1}{y^\alpha} \overset{c}{\asymp} \frac{y-x}{yx^\alpha} \qquad \text{for all } y > x > 0.$$

LEMMA 5.7. *We have*

$$(|\alpha - \beta| - \pi)^2 - (\alpha + \beta - \pi)^2 \overset{c}{\asymp} \frac{\alpha\beta(\pi-\alpha)(\pi-\beta)}{(\alpha+\beta)(2\pi-\alpha-\beta)}$$

for all $\alpha, \beta \in (0, \pi)$.

Proof. Rewriting the left-hand side we get

$$\begin{aligned}(|\alpha - \beta| - \pi)^2 - (\alpha + \beta - \pi)^2 &= 2\pi(\alpha + \beta - |\alpha - \beta|) - 4\alpha\beta\\ &= 4(\pi(\alpha \wedge \beta) - \alpha\beta)\\ &= 4(\alpha \wedge \beta)((\pi - \alpha) \wedge (\pi - \beta)).\end{aligned}$$

In the last step of the proof we use the fact that $\alpha \wedge \beta \overset{c}{\asymp} \frac{\alpha\beta}{\alpha+\beta}$. ∎

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