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ON ISOMETRIES OF THE SYMMETRIC SPACE $P_1(3,\mathbb{R})$

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Abstract. We classify the isometries in the non-identity component of the whole isometry group of the symmetric space of positive 3×3 matrices of determinant 1: we determine the translation lengths, minimal spaces and fixed points at infinity.

1. Introduction. Let M be a simply connected complete Riemannian manifold of non-positive sectional curvature. For any given point $x \in M$, there is a well-defined map $S_x\colon M \to M$ sending $\gamma(t)$ to $\gamma(-t)$ for each geodesic γ with $\gamma(0) = x$. If the symmetries S_x are Riemannian isometries, the manifold M is called *symmetric*. If, in addition, M is not isometric to a Riemannian direct product with a Euclidean factor, M is said to be of *non-compact type*. A fundamental example is the space $P_1(n,\mathbb{R})$ of positive definite $n \times n$ matrices of determinant 1 equipped with the Riemannian metric $\langle X, Y\rangle_P = \mathrm{Tr}(XP^{-1}YP^{-1})$. (The tangent space at $P \in P_1(n,\mathbb{R})$ can be readily identified with the space of symmetric matrices X with $\mathrm{Tr}(XP^{-1}) = \mathrm{Tr}(\sqrt{P^{-1}}X\sqrt{P^{-1}}) = 0$.) In fact if M is any symmetric manifold of non-compact type, there exists a diffeomorphism onto a totally geodesic submanifold of some $P_1(n,\mathbb{R})$. The pull-back metric on M obtained by means of the embedding coincides with the original metric on M up to a constant multiple on each irreducible de Rham factor. See Eberlein [3] for a more detailed account of symmetric manifolds.

An important aspect in the study of a Riemannian manifold is the investigation of its isometries as well as the group of all isometries. In this paper we address the problem of classification of the Riemannian isometries of $P_1(n,\mathbb{R})$. Our vantage point, however, is that of CAT(0) geometry, as it affords greater flexibility and lucidity by neglecting the differentiable structure wherever possible. If M is a simply connected Riemannian manifold of non-positive sectional curvature, then it is a CAT(0) space when equipped with its Riemannian distance as metric. Our basic reference for CAT(0) geometry is Bridson and Haefliger [1].

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The *displacement function* of an isometry α of M is the assignment $d_\alpha(x) = d(x, \alpha(x))$. If d_α attains its minimum on M, the isometry α is called *semi-simple*, in which case it makes sense to define the *minimal space* of α, $\mathrm{Min}(\alpha)$, as the set of all points where the minimum is attained. When the minimal space is actually the set of fixed points $\mathrm{Fix}(\alpha)$, we call the isometry α *elliptic*. If α is semi-simple but not elliptic, it is called *hyperbolic*. If d_α does not attain a minimum, α is called *parabolic*. In any case we can define the *translation length* of α as $|\alpha| = \inf_{x\in X} d_\alpha(x)$.

The concept of a fixed point can be extended to include also non-elliptic isometries by considering the boundary at infinity of M. The points at infinity correspond to equivalence classes of geodesic rays in M where two rays are *equivalent* if they remain within a fixed distance for all times. The boundary at infinity is denoted by ∂M. Since an isometry acts on the set of geodesics, it induces a bijection α on ∂M. By introducing a metric on ∂M we can talk about its geometry and consequently about the geometry of $\mathrm{Fix}_\infty(\alpha)$, the fixed point set of α on ∂M.

The classification of all isometries of $P_1(n, \mathbb{R})$ is by no means trivial. Recently, Fujiwara, Nagano, and Shioya [5] classified the isometries and their fixed point sets for the connected component of the identity in the full group of isometries of $P_1(3, \mathbb{R})$. To some extent that achievement was an application of their more general investigation of parabolic isometries of CAT(0) spaces. Here we classify the isometries of $P_1(3, \mathbb{R})$ in the connected component of the inversion $\sigma(P) = P^{-1}$. In particular, we note that there are parabolic isometries in that component and we determine their fixed point set at infinity, thereby solving a problem posed by Fujiwara (see [4, Problem 4.1]). To do that, we prove a couple of general results of independent interest concerning the isometries of CAT(0) spaces.

To every matrix $g \in \mathrm{SL}(n, \mathbb{R})$ we can associate the Riemannian isometry $g\colon P_1(n, \mathbb{R}) \to P_1(n, \mathbb{R})$ sending each P to gPg^T. The resulting representation $\mathrm{SL}(n, \mathbb{R}) \to \mathrm{Iso}(P_1(n, \mathbb{R}))$ induces an isomorphism of $\mathrm{PSL}(n, \mathbb{R})$ and the identity component of $\mathrm{Iso}(P_1(n, \mathbb{R}))$. (See [1, Chapter II, §10] as well as [3, §2.13] for details.) By that isomorphism, we view $\mathrm{PSL}(n, \mathbb{R})$ as a subgroup of $\mathrm{Iso}(P_1(n, \mathbb{R}))$. Similarly, we associate to every element $g \in \mathrm{SL}(n, \mathbb{R})$ the Riemannian isometry $\widetilde{g}\colon P_1(n, \mathbb{R}) \to P_1(n, \mathbb{R})$ sending each P to $gP^{-1}g^T$. This results in a diffeomorphism of $\mathrm{PSL}(n, \mathbb{R})$ and the component of inversion in $\mathrm{Iso}(P_1(n, \mathbb{R}))$ which we denote by $\mathrm{PSL}(n, \mathbb{R})\sigma$. Note that for odd n we can identify $\mathrm{SL}(n, \mathbb{R}) = \mathrm{PSL}(n, \mathbb{R})$.

The isometry group $\mathrm{Iso}(P_1(n, \mathbb{R}))$ has exactly two components for each $n > 2$. To see this, recall that the outer automorphism group $\mathrm{Out}(\mathrm{PSL}(n, \mathbb{R}))$ has order 2 and consider the natural morphism

$$\mathrm{Iso}(P_1(n, \mathbb{R})) \to \mathrm{Out}(\mathrm{PSL}(n, \mathbb{R})).$$

The kernel can be expressed as the product

$$\mathrm{PSL}(n,\mathbb{R}) \cdot \mathcal{Z}_{\mathrm{Iso}(P_1(n,\mathbb{R}))}(\mathrm{PSL}(n,\mathbb{R})).$$

However, if α is an element of the centralizer $\mathcal{Z}_{\mathrm{Iso}(P_1(n,\mathbb{R}))}(\mathrm{PSL}(n,\mathbb{R}))$, then the displacement function d_α is constant on the orbits of the action of $\mathrm{PSL}(n,\mathbb{R})$ on $P_1(n,\mathbb{R})$. Since the action in question is transitive (see [1, Lemma II.10.52]), d_α is in fact constant. If that constant were positive, then $P_1(n,\mathbb{R})$ would be isometric to a non-trivial product by [1, Theorem II.6.8(4)]. However, $P_1(n,\mathbb{R})$ is irreducible (see [1, Proposition II.10.53]), and hence d_α is constantly zero, which means that α is the identity. Thus the index $[\mathrm{Iso}(P_1(n,\mathbb{R})) : \mathrm{PSL}(n,\mathbb{R})]$ equals two.

We are in a position to start determining the geometric objects $\mathrm{Min}(g)$ (the minimal space) and $\mathrm{Fix}_\infty(g)$ (the fixed point set at infinity) associated to any isometry g of $P_1(n,\mathbb{R})$.

1.1. Organization and outline. The paper is organized as follows. We begin with some preliminaries in Section 2. In Section 3 we explore the interplay between an isometry and its powers in the context of CAT(0) spaces. This enables us to reduce the classification of the non-identity component of $\mathrm{Iso}(P_1(3,\mathbb{R}))$ to the classification of the identity component. We specialize to the symmetric space $P_1(n,\mathbb{R})$ in Section 4. We apply the results of Section 3 and perform some calculations to determine all the possibilities for minimal spaces of isometries in $\mathrm{PSL}(3,\mathbb{R})\sigma$. Our effort culminates in the main result, Theorem 4.5, in which we fully classify the isometries in the component $\mathrm{PSL}(3,\mathbb{R})\sigma$ of $\mathrm{Iso}(P_1(3,\mathbb{R}))$ and their fixed points at infinity viewed as subspaces of the boundary at infinity of $P_1(3,\mathbb{R})$ equipped with the Tits metric. In (4a) in the proof of Theorem 4.5, we also express the fixed point set of a certain isometry from the identity component of $\mathrm{Iso}(P_1(3,\mathbb{R}))$, which is missed in [5, §6.3]. In the final section we develop a method for decomposing any isometry from the identity component of $\mathrm{Iso}(P_1(n,\mathbb{R}))$ into the product of three commuting isometries: hyperbolic, elliptic, and non-ballistic (i.e., of zero translation length) parabolic. This decomposition enables us to calculate the translation length of any isometry.

We refer the reader to [1] for general theory of isometries of CAT(0) spaces, and to [2] for structure theory of isometry groups.

2. Preliminaries. Here we recall the concept of boundary at infinity of a CAT(0) space X with metric d, and then concentrate on the case where $X = P_1(n,\mathbb{R})$. This section is a brief summary of [1, Chapter II, §8–10]. A *geodesic ray* in X is an isometric embedding $r\colon [0,\infty) \to X$ or, by abuse of language, the image of such an embedding. Similarly, by *geodesic line* we mean either an isometric embedding $\mathbb{R} \to X$ or its image. We call two rays r and s *equivalent* if the function $t \mapsto d(r(t),s(t))$ is bounded. The convexity of CAT(0) metric d allows us to define the *boundary at infinity* for any CAT(0) space X as the set of equivalence classes of geodesic rays. We denote the set of equivalence classes of rays in X by ∂X. From now on, let X be a proper CAT(0) space. Then for every point $\xi \in \partial X$ and every point $x \in X$ there exists a geodesic ray $r\colon [0,\infty) \to X$ with $r(0) = x$ such that r belongs to the equivalence class ξ. (See [1, Chapter II, §8].)

Given an isometry α of a CAT(0) space X, the action of α on X respects the equivalence relation on geodesic rays, described above. Hence the natural action of α on ∂X can be defined (see [1, Corollary II.8.9]).

Recall that the space $S_0(n,\mathbb{R})$ of symmetric $n \times n$ matrices with zero trace, viewed as the tangent space to $P_1(n,\mathbb{R})$ at the identity I, is equipped with the norm $\|X\|_2 = \sqrt{\mathrm{Tr}(X^2)}$. Any geodesic line (ray, resp.) through I in $P_1(n,\mathbb{R})$ is parametrized as $t \mapsto \exp(tX)$ for $t \in \mathbb{R}$ ($t \in [0,\infty)$, resp.) for some $X \in S_0(n,\mathbb{R})$ with $\|X\|_2 = 1$. This is a special case of [1, Corollary II.10.42].

Note that for $X \neq X'$, rays $t \mapsto \exp(tX)$ and $t \mapsto \exp(tX')$ are not equivalent. On the other hand, there is some $X \in S_0(n,\mathbb{R})$ for each ξ in $\partial P_1(n,\mathbb{R})$ such that $t \mapsto \exp(tX)$ belongs to the equivalence class ξ. Hence we can identify $\partial P_1(n,\mathbb{R})$ with the unit sphere in $S_0(n,\mathbb{R})$, denoted by $\mathcal{S}$. There we introduce a simplicial structure. A simplex of dimension m, for $m = 0, 1, \dots, n-2$, is determined by the following data: an ordered orthonormal basis $(e_1, \dots, e_n)$ (or, equivalently, a matrix $O \in O(n)$ whose columns are $e_1, \dots, e_n$) and a subset $\{i_1, \dots, i_{m+1}\}$ of $\{1, \dots, n-1\}$ of cardinality $m+1$. That simplex consists of all matrices $X \in \mathcal{S}$ such that $OXO^T = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, i.e. the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ on the diagonal, where

$$\begin{aligned}\lambda_1 = \dots = \lambda_{i_i} \geq \lambda_{i_1+1} = \dots = \lambda_{i_2} \geq \cdots \\ \geq \lambda_{i_m+1} = \dots = \lambda_{i_{m+1}} \geq \lambda_{i_{m+1}+1} = \dots = \lambda_n.\end{aligned}$$

We will use the notation $\partial_T P_1(n,\mathbb{R})$ when referring to the boundary at infinity equipped with this simplicial structure. Note that for two different orthogonal matrices we may get the same m-simplex.

We metrize each 1-simplex in $\partial_T P_1(n,\mathbb{R})$ so that it is isometric to the interval $[0,\pi/3]$. Furthermore, higher dimensional simplices are metrized as spherical simplices with all edge lengths equal to $\pi/3$. The whole simplicial complex is then equipped with the natural length (shortest path) metric. This is exactly the Tits metric on the boundary at infinity of $P_1(n,\mathbb{R})$ (see [1, Chapter II, §9] for a definition of the Tits metric on the boundary at infinity of a general CAT(0) space and [1, Chapter II, p. 337] for a discussion in the case of $P_1(n,\mathbb{R})$). For our purposes it suffices to view the Tits boundary $\partial_T P_1(n,\mathbb{R})$ just as a spherical simplicial complex.

Each simplex of dimension $n-2$ is called a *chamber*. We fix a basis $O \in \mathrm{SO}(n)$. The union of all simplices consisting of matrices X such that

OXO^T is diagonal is called an *apartment*. It is isometric (in the Tits metric) to the unit sphere $S^{n-2} \subseteq \mathbb{R}^{n-1}$. Furthermore, such an apartment corresponds to the boundary at infinity of an $(n-1)$-dimensional flat in $P_1(n,\mathbb{R})$ consisting of all matrices that are diagonal in the basis O. Recall that an *m-dimensional flat* in a CAT(0) space X is a subspace isometric to the Euclidean space $\mathbb{R}^m$.

3. Some properties of CAT(0)-space isometries. The following theorem and its consequence, Corollary 3.3, are the key tools for classifying isometries from the non-identity component of $\mathrm{Iso}(P_1(n,\mathbb{R}))$. Indeed, some power of any group element in a group of isometries with finitely many connected components lies in the identity component. In case $n = 3$, we have a complete characterization of the identity component of $\mathrm{Iso}(P_1(3,\mathbb{R}))$ by [5, §6.3].

Recall that for an isometry α of a CAT(0) space X, the *minimal space* of α, denoted $\mathrm{Min}(\alpha)$, is the set of points that are translated of the minimal distance $|\alpha| = \inf\{d(x,\alpha(x)) \mid x \in X\}$. In the case of an elliptic isometry α, its minimal space is also denoted by $\mathrm{Fix}(\alpha)$. We denote by $\mathrm{Fix}_\infty(\alpha)$ the set of fixed points of the induced α-action on ∂X.

THEOREM 3.1. *Let (X,d) be a proper* CAT(0) *space. An isometry α of X has the same type (elliptic, hyperbolic or parabolic) as its powers and the translation lengths relate as $|\alpha^n| = n|\alpha|$. Moreover,* $\mathrm{Fix}_\infty(\alpha) \subseteq \mathrm{Fix}_\infty(\alpha^n)$ *and in the semi-simple case* $\mathrm{Min}(\alpha) \subseteq \mathrm{Min}(\alpha^n)$.

Proof. Let $n \in \mathbb{N}$. Recall that

- if α is elliptic, then so is α^n, since $\mathrm{Fix}(\alpha) \subseteq \mathrm{Fix}(\alpha^n)$;
- if α^n is elliptic, then α itself is elliptic, because for $x \in \mathrm{Fix}(\alpha^n)$ the orbit of x under α is finite, hence its circumcenter is a fixed point for α (see [1, Proposition II.2.7]);
- if α is hyperbolic, so is α^n, since it acts as translation by $n|\alpha|$ on $\mathrm{Min}(\alpha)$;
- if α^n is hyperbolic, then α is hyperbolic (see [1, Theorem II.6.8]).

Consequently, α^n is parabolic if and only if α is parabolic, and the first statement of the theorem follows. For the second part let us first observe that the limit

$$\lim_{n\to\infty} \frac{1}{n} d(x,\alpha^n x)$$

exists and is independent of x (see [1, Exercise II.6.6(1)]). The existence follows from the fact that for a fixed x the function $f(n) = d(x,\alpha^n x)$ is subadditive. It is well known that for such functions the limit $\lim_{n\to\infty} f(n)/n$ exists. To show independence of x take another point y. The triangle inequality yields

$$\begin{aligned} d(x,\alpha^n x) - d(x,y) - d(\alpha^n x, \alpha^n y) &\le d(y,\alpha^n y) \\ &\le d(x,\alpha^n x) + d(x,y) + d(\alpha^n x,\alpha^n y). \end{aligned}$$

Note that $d(x,y) = d(\alpha^n x, \alpha^n y)$. Hence dividing by n and taking the limit we obtain:

$$\lim_{n\to\infty} \frac{1}{n} d(x,\alpha^n x) = \lim_{n\to\infty} \frac{1}{n} d(y,\alpha^n y).$$

The evaluation of this limit is also a part of the cited exercise, but only for the semi-simple case where the proof is easier because one can take $x \in \mathrm{Min}(\alpha)$. Here we give a proof for the general case. The triangle inequality implies

$$(\diamondsuit) \qquad d(x,\alpha^n x) \le d(x,\alpha x) + d(\alpha x, \alpha^2 x) + \cdots + d(\alpha^{n-1}x, \alpha^n x).$$

Choose an arbitrary $\varepsilon > 0$. Let x be such that $d(x,\alpha x) \le |\alpha| + \varepsilon$. It follows from $(\diamondsuit)$ that $\frac{1}{n} d(x,\alpha^n x) \le |\alpha| + \varepsilon$, hence $\lim_{n\to\infty} \frac{1}{n} d(x,\alpha^n x) \le |\alpha|$. For the reverse inequality, let x' and $x'' = \alpha x'$ be the midpoints of the geodesic segments $[x,\alpha x]$ and $[\alpha x, \alpha^2 x]$. It follows (by the convexity of the metric d on X) that $d(x,\alpha^2 x) \ge 2d(x',x'') = 2d(x',\alpha x')$. Applying this inductively we note that for each n there exists a point $\widetilde{x}_n \in X$ such that

$$d(x,\alpha^{2^n} x) \ge 2^n d(\widetilde{x}_n, \alpha \widetilde{x}_n).$$

Consequently, $2^{-n} d(x,\alpha^{2^n} x) \ge |\alpha|$ for each n. The asserted inequality follows by taking the limit.

To conclude the proof, note that

$$|\alpha^n| = \lim_{m\to\infty} \frac{1}{m} d(x,(\alpha^n)^m x) = n \lim_{m\to\infty} \frac{1}{nm} d(x,\alpha^{nm} x) = n|\alpha|.$$

If $\xi \in \mathrm{Fix}_\infty(\alpha)$, then obviously ξ is already contained in $\mathrm{Fix}_\infty(\alpha^n)$. In the hyperbolic case, every axis of α is also an axis for α^n, because α^n acts on it as translation by $n|\alpha|$. In the elliptic case, if $\alpha x = x$ then also $\alpha^n x = x$. ∎

Example 3.2. The inclusions in the statement of the theorem can be strict, as shown by the following examples.

For the semi-simple case let β be a rotation of order $n \ge 2$ on the Euclidean space $\mathbb{R}^3$, and let τ be a translation in the direction of the axis of β. Then $\alpha = \tau\beta$ is a semi-simple isometry (elliptic if τ is trivial and a hyperbolic glide-rotation if τ is non-trivial). On the one hand, the only fixed points at infinity of α are the ends of the axis of β. On the other hand, $\alpha^n = \tau^n$ is a translation, hence it fixes the whole of $\partial\mathbb{R}^3$.

For the parabolic case, let $X = \mathbb{R} \times \mathbb{H}^2$ and let α act as reflection across the origin on $\mathbb{R}$ and as an arbitrary parabolic isometry τ on $\mathbb{H}^2$ (for instance $\tau(x,y) = (x+1,y)$ in the upper halfplane model). Then $\mathrm{Fix}_\infty(\alpha)$ is a point, but $\mathrm{Fix}_\infty(\alpha^2) = S^0 * \mathrm{Fix}_\infty(\tau) \approx [0,\pi]$.

Theorem 3.1 yields the following corollary:

Corollary 3.3. *Let* $\widetilde{g} \in \mathrm{PSL}(n,\mathbb{R})\sigma$. *Then* $\widetilde{g}^2 = gg^{-T} \in \mathrm{PSL}(n,\mathbb{R})$, *hence the isometry* $\widetilde{g}$ *has the same type as the isometry* $gg^{-T} \in \mathrm{PSL}(n,\mathbb{R})$ *and* $|\widetilde{g}| = \frac{1}{2}|gg^{-T}|$.

Proof. A trivial computation shows that in $\mathrm{Iso}(P_1(n,\mathbb{R}))$, we have $(\widetilde{g})^2 = gg^{-T}$. Hence Theorem 3.1 applies. ∎

This means that the isometry $\widetilde{g} \in \mathrm{PSL}(n,\mathbb{R})\sigma$ is semi-simple if and only if the matrix gg^{-T} is diagonalizable over $\mathbb{C}$ (see [1, Proposition II.10.61]), and is elliptic if and only if gg^{-T} is conjugate (in $\mathrm{SL}(n,\mathbb{R})$) to an orthogonal matrix.

By using the classification of isometries in the identity component of $\mathrm{Iso}(P_1(n,\mathbb{R}))$ in [1, Chapter II, §10], Corollary 3.3 can be used to determine the type of any isometry $\widetilde{g} \in \mathrm{PSL}(n,\mathbb{R})\sigma$.

The next lemma shows a nice relation between the fixed point set of an elliptic isometry α of a complete CAT(0) space and the fixed point set of the induced action of α at infinity (see also [7, Lemma 10]). We are going to apply it in the proof of Theorem 4.5 below.

Lemma 3.4. *Let* α *be a semi-simple isometry of a complete* CAT(0) *space* X *and let* $F = \mathrm{Min}(\alpha)$. *Then* $\mathrm{Fix}_\infty(\alpha) = \partial F$.

Proof. Let us denote $F_\infty := \mathrm{Fix}_\infty(\alpha)$. Because of convexity of F the inclusion $\partial F \subseteq F_\infty$ is obvious. For the reverse inclusion, take an element ξ in F_∞. For an arbitrary $x \in F$ let $c\colon ([0,\infty), 0, \infty) \to (X, x, \xi)$ be the unique geodesic ray with initial point x in the class of geodesic rays representing ξ. As $\xi \in F_\infty$, the geodesic ray $\alpha \circ c$ is asymptotic to c, which means that $f(t) = d(\alpha(c(t)), c(t))$ is a bounded function of t. As the metric of a CAT(0) space is convex, f itself is convex and therefore decreasing. On the other hand, $f(t) \geq d(\alpha(c(0)), c(0)) = |\alpha|$, hence $t \mapsto f(t)$ is constant. This means that the image of c lies entirely in F, hence $\xi \in \partial F$. ∎

4. The non-identity component of $\mathrm{Iso}(P_1(3,\mathbb{R}))$**.** In this section, we dive into $\mathrm{PSL}(n,\mathbb{R})\sigma$ to explore the machinery needed for our main result, Theorem 4.5.

4.1. Jordan forms. Recall that the geometric properties of an isometry of a given CAT(0) space X behave nicely under conjugation. In particular, for given $\alpha, \beta \in \mathrm{Iso}(X)$, the isometries α and $\beta\alpha\beta^{-1}$ have the same type (elliptic, hyperbolic or parabolic) and their translation lengths are the same. Furthermore, $\operatorname{Min}(\beta\alpha\beta^{-1}) = \beta \operatorname{Min}(\alpha)$. The following result about conjugation in $\mathrm{PSL}(n, \mathbb{R})\sigma$ will be of use for us.

LEMMA 4.1. *Isometries* $\widetilde{g}, \widetilde{h} \in \mathrm{PSL}(n, \mathbb{R})\sigma$ *are conjugate if there exists* $A \in \mathrm{SL}(n, \mathbb{R})$ *such that* $g = AhA^T$.

Proof. Let $g = AhA^T$. For $P \in P_1(n, \mathbb{R})$, we have

$$\widetilde{g}.P = gP^{-1}g^T = (AhA^T)P^{-1}(Ah^TA^T) = A.(\widetilde{h}.(A^{-1}.P)). \blacksquare$$

To analyze the non-identity component of the isometry group $\mathrm{Iso}(P_1(3, \mathbb{R}))$, it is enough to classify all the isometries of the form $gg^{-T} \in \mathrm{SL}(3, \mathbb{R})$ by Corollary 3.3. Following the classification of isometries in $\mathrm{SL}(3, \mathbb{R})$ from [5, §6.3], we have to determine the real Jordan form of gg^{-T} for

Table 1

	Possible real Jordan form A for matrix in $\mathrm{SL}(3, \mathbb{R})$	Solutions g_x of $g = Ag^T$	Conjugacy relations among solutions
(1)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1/(4x^2) & 0 & 0 \\ 0 & x & 2x \\ 0 & -2x & 0 \end{bmatrix}$, $x \neq 0$	$g_x = A_{x,y} g_y A_{x,y}^T$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$
(2)	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} x & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $x \in \mathbb{R}$	$g_x = B_{x,y} g_y B_{x,y}^T$ for any x, y
(3)	$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $a^2 + b^2 = 1$, $b \neq 0$	$\begin{bmatrix} x & \frac{xb}{1+a} & 0 \\ \frac{-xb}{1+a} & x & 0 \\ 0 & 0 & \frac{1+a}{2x^2} \end{bmatrix}$, $x \neq 0$	$g_x = C_{x,y} g_y C_{x,y}^T$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$
(3′)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	g is any symmetric matrix	g and g' are conjugate iff either both are positive or both have two negative eigenvalues
(4)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & a \end{bmatrix}$, $a \notin \{0, 1\}$	$\begin{bmatrix} -a/x^2 & 0 & 0 \\ 0 & 0 & x/a \\ 0 & x & 0 \end{bmatrix}$, $x \neq 0$	$g_x = A_{x,y} g_y A_{x,y}^T$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$
(5)	all the rest	no solutions	

each $g \in \mathrm{SL}(3,\mathbb{R})$. Observe that conjugation of gg^{-T} by $A \in \mathrm{SL}(n,\mathbb{R})$ corresponds to conjugating the isometry $\widetilde{g}$ by A. Since conjugation in $\mathrm{SL}(3,\mathbb{R})$ does not change the isometry type and the translation length of gg^{-T}, we can restrict ourselves to solving the equation $g = Ag^T$ for all possible real Jordan matrices A that correspond to the isometries in $\mathrm{SL}(3,\mathbb{R})$. We can solve that equation as a homogeneous system of linear equations and then scale to land at $g \in \mathrm{SL}(3,\mathbb{R})$. By some lengthy but straightforward linear algebra, we get four families of solutions which we list below. The conjugacy relation between different solutions in each family is deduced by Lemma 4.1 and is given in the last column of Table 1. We employ the following notation:

$$A_{x,y} = \mathrm{diag}\big(y/x, \sqrt{x/y}, \sqrt{x/y}\big), \qquad B_{x,y} = \begin{bmatrix} 1 & 1 & (x-y)/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$C_{x,y} = \mathrm{diag}\big(\sqrt{x/y}, \sqrt{x/y}, y/x\big).$$

4.2. Minimal spaces. As Example 3.2 shows, there is no straightforward way to determine the minimal space of $\widetilde{g}$, given $\mathrm{Min}(gg^{-T}) = \mathrm{Min}(\widetilde{g}^2)$. Hence we have to calculate $\mathrm{Min}(\widetilde{g})$ by hand. To this end, we first retrieve some information about semi-simple isometries in $\mathrm{Iso}(P_1(n,\mathbb{R}))\sigma$ for general n. We use that information to determine all possible shapes of minimal spaces of semi-simple isometries in $\mathrm{SL}(3,\mathbb{R})\sigma$.

Assume first that $\widetilde{g}$ is hyperbolic. Without loss of generality we can take $I \in \mathrm{Min}(\widetilde{g})$ (otherwise we can conjugate $\widetilde{g}$ by $(\sqrt{R})^{-1}$ for $R \in \mathrm{Min}(\widetilde{g})$). Let $X \in S_0(n,\mathbb{R})$ with $\|X\|_2 = 1$ be such that $\widetilde{g}$ acts as a translation on $\exp(\mathbb{R}X)$. We can as well assume that X (or equivalently, $\exp(X)$) is diagonal since otherwise we can conjugate $\widetilde{g}$ by an orthogonal matrix O for which the O-conjugate of X is diagonal. For an arbitrary $t \in \mathbb{R}$ and $t_0 := |\widetilde{g}|$ this means

$$\widetilde{g}.\exp(tX) = g\exp(-tX)g^T = \exp((t+t_0)X).$$

Acting by $\exp(-t_0X/2) \in \mathrm{PSL}(n,\mathbb{R})$ on this equality gives

$$(\heartsuit) \qquad \exp(-t_0X/2)g\exp(-tX)g^T\exp(-t_0X/2) = \exp(tX),$$

which implies $\exp(-t_0X/2)g = O \in O(n)$ (it fixes I), hence $g = \exp(t_0X/2)O$ is a polar decomposition for g. Inserting $t - t_0$ in place of t in the equation $(\heartsuit)$ gives $g\exp(t_0X/2) = O' \in O(n)$, and hence $g = O'\exp(-t_0X/2)$ is another polar decomposition for g. A simple application of the above equalities yields $O = O'$:

$$O\exp(-t_0X/2)O^T = \exp(t_0X/2) = O'\exp(-t_0X/2)O^T,$$

where the last equality comes from both polar decompositions.

From this we can derive additional information in case $n = 3$.

LEMMA 4.2. *The minimal space* $\mathrm{Min}(\widetilde{g})$ *of a hyperbolic isometry* $\widetilde{g} \in \mathrm{SL}(3,\mathbb{R})\sigma$ *is isometric to* $\mathbb{R}$.

Proof. As explained above, we may assume that $I \in \mathrm{Min}(\widetilde{g})$ and $X = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in S_0(3,\mathbb{R})$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \neq 0$ is such that $\widetilde{g}$ acts as a translation on $\exp(\mathbb{R}X)$. Suppose also that $P = \exp(Y) \in \mathrm{Min}(\widetilde{g})$ where $Y \in S_0(3,\mathbb{R})$ is linearly independent of X. As $\exp(Y) \in \mathrm{Min}(\widetilde{g})$, there is a geodesic parallel to $c\colon t \mapsto \exp(tX)$ through $\exp(Y)$. We borrow the notation of [1, Proposition II.10.67]: let $F(b)$ denote the union of geodesics, parallel to b. With this notation it means that $\exp(Y) \in F(c)$, hence $\exp(Y)$ commutes with $\exp(X)$ (by the same proposition), and this implies $[X, Y] = 0$, i.e. X and Y are diagonalizable in some common basis.

As above, we show that $g = \exp(t_0X/2)O$ is a polar decomposition for g, where $t_0 = |\widetilde{g}|$. Regarding O, we have $O\exp(-tX)O^T = \exp(tX)$ or, equivalently, $OXO^T = -X$. Hence the spectrum of X must satisfy $\sigma(X) = -\sigma(X)$. Because X is non-zero, the only possibility is that $\sigma(X) = \{\lambda, 0, -\lambda\}$ for positive λ and that O is just a "permutation" of the basis, swapping $\mathrm{Lin}\{e_1\}$ and $\mathrm{Lin}\{e_3\}$, and leaving $\mathrm{Lin}\{e_2\}$ invariant. From $[X, Y] = 0$ we get also $[OYO^T, X] = 0$, hence X and OYO^T are diagonalizable in a common basis. But X has three different eigenvalues, hence it is diagonalizable in only one basis, which means that the three matrices X, Y and OYO^T are diagonalizable in that basis. Hence, Y is a diagonal matrix. The convexity of $\mathrm{Min}(\widetilde{g})$ implies that $\exp(tY) \in \mathrm{Min}(\widetilde{g})$ for all $0 \leq t \leq 1$, and we can calculate

$$\begin{aligned}
t_0 &= d(\exp(tY), g\exp(-tY)g^T) \\
&= d\big(I, \exp(-tY/2)\exp(t_0X/2)O\exp(-tY)O^T\exp(t_0X/2)\exp(-tY/2)\big) \\
&= d\big(I, \exp(-tY/2 + t_0X/2 - tOYO^T + t_0X/2 - tY/2)\big) \\
&= d(I, \exp(t_0X - tY - tOYO^T)) = \|t_0X - tY - tOYO^T\|_2.
\end{aligned}$$

Because Y is supposed to be linearly independent of X and the length of the vector $t_0X - tY - tOYO^T$ is independent of t, we have $OYO^T = -Y$. If we write $Y = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$, the last equality means $0 \neq \mu_1 = -\mu_3$ and $\mu_2 = 0$. Hence Y is linearly dependent on X, a contradiction. ∎

We proceed to the elliptic case where again we start with general n. If $\widetilde{g}$ is elliptic, then it fixes some $P \in P_1(n,\mathbb{R})$. This means that $P = gP^{-1}g^T$, which we rewrite as

$$I = \sqrt{P^{-1}}\,g\sqrt{P^{-1}}(\sqrt{P^{-1}}\,g\sqrt{P^{-1}})^T.$$

Therefore $\widetilde{g}$ is conjugate to $\widetilde{h}$, where $h = \sqrt{P^{-1}}\,g\sqrt{P^{-1}} = \sqrt{P^{-1}}g(\sqrt{P^{-1}})^T \in \mathrm{SO}(n)$. Conversely, if $g \in \mathrm{SO}(n)$, then $\widetilde{g}.I = I$, and obviously $\widetilde{g}$ is elliptic.

Suppose now that for $g \in \mathrm{SO}(n)$, the isometry $\widetilde{g}$ fixes some $P = \exp(X) \neq I$. As in the proof of Lemma 4.2, the property $gXg^T = -X$ implies that the spectrum of X is symmetric about 0 and that $\exp(\mathbb{R}X) \subseteq \mathrm{Fix}(\widetilde{g})$. If X has n different eigenvalues, then g acts as an involution on the set of n different eigenspaces of X. Hence, $\widetilde{g}$ has order either 2 or 4 (since g^2 may be minus the identity on each eigenspace for non-zero eigenvalue). Hence for $n = 3$ we have the following lemma:

Lemma 4.3. *The fixed point set of an elliptic isometry from* $\mathrm{SL}(3,\mathbb{R})\sigma$ *is either a single point or a hyperbolic plane.*

Proof. Let $\widetilde{g}$ be an elliptic isometry. Without loss of generality suppose $I \in \mathrm{Fix}(\widetilde{g})$, hence $g \in \mathrm{SO}(3)$. If there is a non-zero X with $\exp(X) \in \mathrm{Fix}(\widetilde{g})$, we have $gXg^T = -X$. In particular, the spectrum of X equals $\{\lambda, 0, -\lambda\}$ for some non-zero λ, and g swaps the eigenspaces corresponding to the non-zero eigenvalues and preserves the eigenspace of the eigenvalue 0. After another conjugation, we may assume that $X = \mathrm{diag}(\lambda, 0, -\lambda)$, and hence

$$g = \begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \mp 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

A computation shows that in each case, there is another linearly independent $Y \in S_0(3,\mathbb{R})$ such that $gYg^T = -Y$. Then for any linear combination $S = tX + sY$, we have $\exp(S) \in \mathrm{Fix}(\widetilde{g})$. Furthermore, Y and X do not commute, and hence $\mathrm{Fix}(\widetilde{g})$ is not a flat. But it still has constant curvature since it is homogeneous: $\exp(-(tX + sY)/2)$ conjugates $\widetilde{g}$ to itself by Lemma 4.1, hence it preserves $\mathrm{Fix}(\widetilde{g})$, but it also moves $\exp(tX + sY)$ to I. We conclude that $\mathrm{Fix}(\widetilde{g})$ is a scaled hyperbolic plane. ∎

4.3. Boundary at infinity. Recall from Section 2 that $\partial_T P_1(n,\mathbb{R})$ is a simplicial complex.

Lemma 4.4. *The inversion* σ *acts as a simplicial map on* $\partial_T P_1(n,\mathbb{R})$.

Proof. Let $\xi \in \partial_T P_1(n,\mathbb{R})$ be the class represented by a geodesic ray $[t \mapsto \exp(tX)]_{t>0}$ for $X \in S_0(n,\mathbb{R})$. Then $\sigma.\xi$ is represented by the geodesic ray $[t \mapsto \exp(-tX)]_{t>0}$. This means that σ maps the simplex determined by the ordered orthonormal basis $(e_1,\dots,e_n)$ and $\{i_1,\dots,i_{m+1}\} \subseteq \{1,\dots,n-1\}$ to the simplex determined by $(e_n,\dots,e_1)$ and $\{n-i_{m+1},\dots,n-i_1\}$. ∎

If we take an apartment $A \approx S^{n-2}$, which is the boundary of a flat containing I, then σ acts as a reflection across the center of S^{n-2}.

We know (see e.g. [1, Proposition II.10.75]) that for an isometry α in $\mathrm{PSL}(n,\mathbb{R})$, the set $\mathrm{Fix}_\infty(\alpha)$ is a simplicial subcomplex of $\partial_T P_1(n,\mathbb{R})$, but for α in $\mathrm{PSL}(n,\mathbb{R})\sigma$ that is generally not true: see Theorem 4.5 below for $\mathrm{SL}(3,\mathbb{R})\sigma$.

The tools developed above together with [5, Theorem 6.1] make the classification of isometries in $\mathrm{SL}(3,\mathbb{R})\sigma$ quite easy. In the next theorem, we use c_i, $i=1,\dots,6$, for the chambers consisting of equivalence classes of rays $t\mapsto \exp(tX)$ for diagonal matrices $X=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\in S_0(3,\mathbb{R})$. More accurately,

$$\begin{aligned}
c_1 &:= \{\text{equiv. cl. of } t\mapsto \exp(t\cdot \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)) \mid \lambda_1\geq\lambda_2\geq\lambda_3\},\\
c_2 &:= \{\text{equiv. cl. of } t\mapsto \exp(t\cdot \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)) \mid \lambda_2\geq\lambda_1\geq\lambda_3\},\\
c_3 &:= \{\text{equiv. cl. of } t\mapsto \exp(t\cdot \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)) \mid \lambda_2\geq\lambda_3\geq\lambda_1\},\\
c_4 &:= \{\text{equiv. cl. of } t\mapsto \exp(t\cdot \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)) \mid \lambda_3\geq\lambda_2\geq\lambda_1\},\\
c_5 &:= \{\text{equiv. cl. of } t\mapsto \exp(t\cdot \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)) \mid \lambda_3\geq\lambda_1\geq\lambda_2\},\\
c_6 &:= \{\text{equiv. cl. of } t\mapsto \exp(t\cdot \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)) \mid \lambda_1\geq\lambda_3\geq\lambda_2\}.
\end{aligned}$$

Furthermore, let v_i denote the common vertex of c_i and c_{i-1} (indices modulo 6) such that the simplex $[v_i,v_{i+1}]$, i.e. the simplex spanned on v_i and v_{i+1}, equals c_i. Let C_i denote the barycenter of the simplex c_i.

4.4. Classification

THEOREM 4.5. *Let $\widetilde{g}\in\mathrm{SL}(3,\mathbb{R})\sigma$ and let gg^{-T} have a (real) Jordan form as in the table above. Then*

(1) *$\widetilde{g}$ is parabolic, $\mathrm{Fix}_\infty(\widetilde{g})=C_2$ and $|\widetilde{g}|=0$;*
(2) *$\widetilde{g}$ is parabolic, $\mathrm{Fix}_\infty(\widetilde{g})=C_1$ and $|\widetilde{g}|=0$;*
(3) *$\widetilde{g}$ is elliptic, $\mathrm{Fix}(\widetilde{g})$ is a single point and $\mathrm{Fix}_\infty(\widetilde{g})=\emptyset$;*
(3′) *$\widetilde{g}$ is elliptic, and*

   (a) *if g is positive, then $\mathrm{Fix}(\widetilde{g})$ is a single point g and $\mathrm{Fix}_\infty(\widetilde{g})=\emptyset$;*
   (b) *if g is not positive, then $\mathrm{Fix}(\widetilde{g})$ is a hyperbolic plane and $\mathrm{Fix}_\infty(\widetilde{g})$ is its boundary;*

(4) *$\widetilde{g}$ is semi-simple, and*

   (a) *if $a=-1$, then $\widetilde{g}$ is elliptic, $\mathrm{Fix}(\widetilde{g})$ is a hyperbolic plane and $\mathrm{Fix}_\infty(\widetilde{g})$ is its boundary;*
   (b) *if $a\neq-1$, then $\widetilde{g}$ is hyperbolic, $|\widetilde{g}|=\sqrt{2}\,|\log|a|\,|$, $\mathrm{Fix}(\widetilde{g})$ is a single axis and $\mathrm{Fix}_\infty(\widetilde{g})$ consists of its ends.*

Proof. (1) By [5, Theorem 6.1], $|gg^{-T}|=0$ and $\mathrm{Fix}_\infty(gg^{-T})=c_1\cup c_2\cup c_3$, hence by Theorem 3.1 and Corollary 3.3, $|\widetilde{g}|=0$ and $\mathrm{Fix}_\infty(\widetilde{g})\subseteq c_1\cup c_2\cup c_3$.

For arbitrary $x \in \mathbb{R} \setminus \{0\}$ we calculate

$$\begin{aligned} &d\big(\widetilde{g}_x.\exp(\operatorname{diag}(-t,2t,-t)),\exp(\operatorname{diag}(t,t,-2t))\big) \\ &\qquad = d\big(\exp(\operatorname{diag}(-t/2,-t/2,t)).g_x.\exp(\operatorname{diag}(t,-2t,t)), I\big) \\ &\qquad = d\left(\begin{bmatrix} 1/(16x^4) & 0 & 0 \\ 0 & x^2(e^{-3t}+4) & -2x^2e^{-3t/2} \\ 0 & -2x^2e^{-3t/2} & 4x^2 \end{bmatrix}, I\right) \end{aligned}$$

which is bounded when $t \to \infty$. This means that $\widetilde{g}.v_3 = v_2$ (geodesic ray $\exp(\operatorname{diag}(-t,2t,-t))_{t>0}$ represents v_3 and $\exp(\operatorname{diag}(t,t,-2t))_{t>0}$ represents v_2 in $\partial_T P_1(3,\mathbb{R})$). Similarly we get $\widetilde{g}.v_2 = v_3$. Because $\operatorname{Fix}_\infty(\widetilde{g})$ is connected and non-empty (see [5, §1]), the only fixed point of $\widetilde{g}$ at infinity is the barycenter C_2 of $[v_2, v_3] = c_2$.

(2) As in (1), from $|gg^{-T}| = 0$ we get $|\widetilde{g}| = 0$, from $\operatorname{Fix}_\infty(gg^{-T}) = c_1$ we get $\operatorname{Fix}_\infty(\widetilde{g}) \subseteq c_1$, and for arbitrary $x \in \mathbb{R}$ we calculate that $\widetilde{g}_x.v_1 = v_2$ and $\widetilde{g}_x.v_2 = v_1$, hence the only fixed point of $\widetilde{g}$ at infinity is the barycenter C_1 of c_1.

(3) The matrix gg^{-T} is orthogonal and

$$\operatorname{Fix}(gg^{-T}) = \{\exp(\operatorname{diag}(t,t,-2t)) \mid t \in \mathbb{R}\}.$$

Since for any $x \neq 0$, $\operatorname{Fix}(\widetilde{g}_x) \subseteq \{\exp(\operatorname{diag}(t,t,-2t)) \mid t \in \mathbb{R}\}$ by Theorem 3.1, Lemma 4.3 shows that the fixed point set of $\widetilde{g}_x$ is a single point that can be calculated using the conjugacy relation from the table above and the fact that g_x is orthogonal exactly when $x = \pm\sqrt{(1+a)/2}$, in which case $\widetilde{g}_x$ fixes $\{I\}$. By Lemma 3.4, $\operatorname{Fix}_\infty(\widetilde{g}) = \emptyset$.

(3′a) Note that $\widetilde{g}$ is conjugate to $\widetilde{I}$ because $I = \sqrt{g^{-1}}g(\sqrt{g^{-1}})^T$. Inversion on $P_1(3,\mathbb{R})$ acts as a reflection around I on any line $t \to \exp(tX)$, hence $\operatorname{Fix}(\widetilde{I}) = \{I\}$ and $\operatorname{Fix}_\infty(\widetilde{I}) = \partial\{I\} = \emptyset$. Conjugating again to get $\widetilde{g}$ back yields $\operatorname{Fix}(\widetilde{g}) = \{g\}$.

(3′b) Since the matrix g is symmetric, not positive, and has determinant 1, it has exactly two negative eigenvalues, and thus by Lemma 4.1 the isometry $\widetilde{g}$ is conjugate to $\widetilde{g}'$, where $g' = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$. Observe that $\widetilde{g}'$ fixes two geodesic lines through I, namely

$$\exp(\operatorname{diag}(t,0,-t))_{t\in\mathbb{R}} \quad \text{and} \quad \exp\left(t\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)_{t\in\mathbb{R}}$$

(because $g'Xg'^T = -X$ for both possibilities above), and by Lemma 4.3 $\operatorname{Fix}(\widetilde{g}')$, and hence $\operatorname{Fix}(\widetilde{g})$, is isometric to a hyperbolic plane. By Lemma 3.4, $\operatorname{Fix}_\infty(\widetilde{g}') = \partial\operatorname{Fix}(\widetilde{g}')$.

(4a) The isometry $gg^{-T} = \mathrm{diag}(1,-1,-1)$ has a large (3-dimensional) fixed point set, parametrized as

$$P_{s,t,u} := \begin{bmatrix} e^{-2u} & 0 & 0 \\ 0 & e^u e^s \cosh t & e^u \sinh t \\ 0 & e^u \sinh t & e^u e^{-s} \cosh t \end{bmatrix}, \qquad s,t,u \in \mathbb{R}.$$

For arbitrary $x \neq 0$, the solution of the equation $g_x P_{s,t,u}^{-1} g_x^T = P_{s,t,u}$ is $u = \log|x|$. Hence the fixed point set for $\widetilde{g_x}$ is a hyperbolic plane by Lemma 4.3, and $\mathrm{Fix}_\infty(\widetilde{g})$ is its boundary.

(4b) The isometry gg^{-T}, and hence $\widetilde{g}$, is hyperbolic,

$$|\widetilde{g}| = \frac{1}{2}|gg^{-T}| = \frac{1}{2}2\sqrt{(\log|a|)^2 + \left(\log\tfrac{1}{|a|}\right)^2} = \sqrt{2}\,|\log|a|\,|.$$

Since for $x = \pm\sqrt{|a|}$ we have $\widetilde{g}_x.I = g_x g_x^T = \mathrm{diag}(1, 1/|a|, |a|)$ and

$$d(\mathrm{diag}(1,1/|a|,|a|), I) = \sqrt{2}|\log|a|\,| = |\widetilde{g}_x|,$$

we know that $I \in \mathrm{Min}(\widetilde{g}_x)$. Hence $\widetilde{g}_x$ acts as a translation on the geodesic line through I and $\widetilde{g}_x.I = \mathrm{diag}(1,1/|a|,|a|)$, i.e. on the geodesic $\exp(\mathrm{diag}(0,-t,t))$. The axis of $\widetilde{g}_x$ for $x \neq \pm\sqrt{|a|}$ can be expressed using the conjugacy relation among different solutions g_x from the table above. By Lemma 4.2, this single axis forms the whole minimal space. For the fixed point set of $\widetilde{g_x}$ at infinity we again use Lemma 3.4, which says that $\mathrm{Fix}_\infty(\widetilde{g_x}) = \partial\,\mathrm{Min}(\widetilde{g_x})$, hence the ends of the axis of $\widetilde{g_x}$ are the only fixed points at infinity of $\widetilde{g}_x$. ■

REMARK 4.6. The interested reader can verify that in each case where the fixed point set of an elliptic isometry $\widetilde{g} \in \mathrm{SL}(3,\mathbb{R})\sigma$ is isometric to a hyperbolic plane, the set $\mathrm{Fix}_\infty(\widetilde{g})$ consists of barycenters of certain chambers.

5. On translation lengths of isometries from $\mathrm{Iso}(P_1(n,\mathbb{R}))$**.** In this section we introduce a decomposition of an isometry of $P_1(n,\mathbb{R})$ from $\mathrm{PSL}(n,\mathbb{R})$ into three commuting isometries, one (if non-trivial) hyperbolic, one elliptic, and the third one (if non-trivial) parabolic with zero translation length. This result gives us a formula to calculate the translation length of any isometry of $P_1(n,\mathbb{R})$ for any $n \in \mathbb{N}$.

In every expression of the form $\sum_{\lambda\in\sigma(X)} \cdots$ below, eigenvalues λ from the spectrum $\sigma(X)$ are counted with multiplicities.

THEOREM 5.1. *Let $g \in \mathrm{PSL}(n,\mathbb{R})$ be an isometry of $P_1(n,\mathbb{R})$. Then g is conjugate to a product HUJ, where all the factors commute, H is a positive* *diagonal matrix, U is an orthogonal matrix (and hence both are semi-simple isometries), and J is an upper triangular matrix with 1s on the diagonal. Furthermore, g is semi-simple exactly when $J = I$ and the translation length of g equals to the translation length of H and can be expressed as*

$$|g| = 2\sqrt{\sum_{\lambda\in\sigma(g)} (\log|\lambda|)^2}.$$

Proof. Every matrix $g \in \mathrm{SL}(n,\mathbb{R})$ can be conjugate by another matrix in $\mathrm{SL}(n,\mathbb{R})$ to take on a *modified real Jordan form*, namely a matrix of block diagonal form

$$\mathrm{diag}(D, D_O, J_1, \dots, J_b, J_1^O, \dots, J_a^O),$$

where the blocks are as follows:

First, D is a pure diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_d)$. Next, D_O has 2×2 blocks on the diagonal, which are $\mu_i O_i$, $i = 1, \dots, c$, for some $\mu_i \in (0,\infty)$ and some $O_i \in O(2)$. Each J_i is a non-trivial modified Jordan block of dimension m_i for real eigenvalues ν_i, $i = 1, \dots, b$, which means that it has ν_i on the diagonal and also on the first upper superdiagonal (instead of 1s as in the classical Jordan form). Finally, J_i^O is a modified Jordan block of dimension $2k_i$ pertaining to complex eigenvalues, i.e. J_i^O is a block of the form

$$\begin{bmatrix} \kappa_i U_i & \kappa_i U_i & 0 & \dots & 0 & 0 \\ 0 & \kappa_i U_i & \kappa_i U_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \kappa_i U_i & \kappa_i U_i \\ 0 & 0 & 0 & \dots & 0 & \kappa_i U_i \end{bmatrix},$$

where $U_i \in O(2)$ and κ_i is the absolute value of the corresponding complex eigenvalue.

We will now express g as a product of commuting matrices H, U and J, and then use the formula

$$|g| = \lim_{r\to\infty} \frac{1}{r} d(g^r.I, I) = \lim_{r\to\infty} \frac{1}{r}\sqrt{\mathrm{Tr}(\log(g^r g^{rT})^2)}.$$

The factors H, U, J are as follows. First, the diagonal matrix

$$H = \mathrm{diag}\big(|\lambda_1|, \dots, |\lambda_d|, \mu_1, \mu_1, \dots, \mu_c, \mu_c, \underbrace{|\nu_1|, \dots, |\nu_1|}_{m_1 \text{ times}}, \dots, \underbrace{|\nu_b|, \dots, |\nu_b|}_{m_b \text{ times}}, \underbrace{\kappa_1, \dots, \kappa_1}_{2k_1 \text{ times}}, \dots, \underbrace{\kappa_a, \dots, \kappa_a}_{2k_a \text{ times}}\big).$$

Next, the orthogonal matrix

$$U = \operatorname{diag}\big(\operatorname{sgn}(\lambda_1), \dots, \operatorname{sgn}(\lambda_d), O_1, \dots, O_c, \underbrace{\operatorname{sgn}(\nu_1), \dots, \operatorname{sgn}(\nu_1)}_{m_1 \text{ times}}, \dots, \underbrace{\operatorname{sgn}(\nu_b), \dots, \operatorname{sgn}(\nu_b)}_{m_b \text{ times}}, \underbrace{U_1, \dots, U_1}_{k_1 \text{ times}}, \dots, \underbrace{U_a, \dots, U_a}_{k_a \text{ times}}\big),$$

and finally a Jordan form matrix J with only 1s on the diagonal,

$$J = \operatorname{diag}(\underbrace{1, \dots, 1}_{d+2c \text{ times}}, K_{m_1}, \dots, K_{m_b}, L_{k_1}, \dots, L_{k_a}),$$

where K_i is an $i \times i$ Jordan block with 1s on the diagonal and on the first upper superdiagonal, and L_i is a Jordan block with I_2s on the diagonal and on the first upper superdiagonal, hence a block of dimension $2i \times 2i$.

Note that g is diagonalizable over $\mathbb{C}$ (and hence a semi-simple isometry by [1, Proposition II.10.61]) exactly when there are no non-trivial (i.e. non-identity) blocks among K_i and no non-trivial blocks among L_i. Therefore g is a semi-simple isometry exactly when $J = I$.

Let us now compute the translation length of UJ. Because U and J commute, it follows that $(UJ)^r(UJ)^{rT} = J^rU^rU^{rT}J^{rT} = J^rJ^{rT}$, and we get

$$|UJ| = \lim_{r\to\infty} \frac{1}{r} d((UJ)^r(UJ)^{rT}, I) = \lim_{r\to\infty} \frac{1}{r} d(J^rJ^{rT}, I) = |J|.$$

Take the geodesic ray $\gamma(t) := \exp(t \operatorname{diag}(u_1, \dots, u_n))$, where $u_1 > \cdots > u_n$, and calculate

$$\begin{aligned} |J| &\le \lim_{t\to\infty} d(J.\gamma(t), \gamma(t)) = \lim_{t\to\infty} d(\gamma(-t/2).J.\gamma(t), I) \\ &= \lim_{t\to\infty} d((\gamma(-t/2)J\gamma(t/2)).I, I). \end{aligned}$$

Because J is an upper triangular matrix with 1s on the diagonal and the eigenvalues of the generator of the geodesic line γ are decreasing, the matrix $\gamma(-t/2)J\gamma(t/2)$ tends to the identity as t tends to infinity (see [1, Proposition 10.64]). Hence the above limit equals 0, and consequently $|J| = |UJ| = 0$.

Recall from the definition of H that it is a diagonal matrix with positive diagonal entries. Such a matrix acts as an elliptic isometry exactly when $H = I$, otherwise it acts as a translation on the geodesic line through I

and H. It moves I to H^2 and one can easily compute

$$|H| = d(I, H.I) = d(I, H^2) = \|\log(H^2)\|_2$$
$$= \sqrt{\sum_{\lambda\in\sigma(H)} (\log \lambda^2)^2} = 2\sqrt{\sum_{\lambda\in\sigma(H)} (\log \lambda)^2}.$$

Computing further, we get

$$\begin{aligned}
&2\sqrt{\sum_{\lambda\in\sigma(H)} (\log \lambda)^2} = |H| \\
&= \lim_{r\to\infty} \frac{1}{r} d(H^r H^{rT}, I) = \lim_{r\to\infty} \frac{1}{r} d(H^r H^{rT}, I) + \lim_{r\to\infty} \frac{1}{r} d((UJ)^r (UJ)^{rT}, I) \\
&= \lim_{r\to\infty} \frac{1}{r} d((UJ)^r H^r H^{rT} (UJ)^{rT}, (UJ)^r (UJ)^{rT}) + \lim_{r\to\infty} \frac{1}{r} d((UJ)^r (UJ)^{rT}, I) \\
&\geq \lim_{r\to\infty} \frac{1}{r} d((UJ)^r H^r H^{rT} (UJ)^{rT}, I) = \lim_{r\to\infty} \frac{1}{r} d((HUJ)^r (HUJ)^{rT}, I) = |g| \\
&\geq \lim_{r\to\infty} \frac{1}{r} d((UJ)^r H^r H^{rT} (UJ)^{rT}, (UJ)^r (UJ)^{rT}) - \lim_{r\to\infty} \frac{1}{r} d((UJ)^r (UJ)^{rT}, I) \\
&= \lim_{r\to\infty} \frac{1}{r} d(H^r H^{rT}, I) = |H|.
\end{aligned}$$

Because the absolute values of eigenvalues of g and their multiplicities are exactly the same as those of H, we infer that

$$|g| = 2\sqrt{\sum_{\lambda\in\sigma(g)} (\log |\lambda|)^2}. \blacksquare$$

Theorem 5.1 together with Corollary 3.3 yields the following corollary:

COROLLARY 5.2. *Given $\widetilde{g} \in \mathrm{PSL}(n,\mathbb{R})\sigma$, its translation length is*

$$|\widetilde{g}| = \sqrt{\sum_{\lambda\in\sigma(gg^{-T})} (\log |\lambda|)^2}.$$

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REFERENCES

[1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. 319, Springer, Berlin, 1999.

[2] P. E. Caprace and N. Monod, *Isometry groups of non-positively curved spaces: structure theory*, J. Topol. 2 (2009), 661–700.

[3] P. B. Eberlein, *Geometry of Non-Positively Curved Manifolds*, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1996.

[4] K. Fujiwara, CAT(0) *spaces for Riemannian geometers*, Geometry Symposium, 2004; http://www.math.tohoku.ac.jp/˜fujiwara/gs04.2.pdf.
[5] K. Fujiwara, K. Nagano and T. Shioya, *Fixed point sets of parabolic isometries of* CAT(0) *spaces*, Comment. Math. Helv. 81 (2006), 305–335.
[6] N. Monod, *Continuous Bounded Cohomology of Locally Compact Groups*, Lecture Notes in Math. 1758, Springer, Berlin, 2001.
[7] E. L. Swenson, *A cut point theorem for* CAT(0) *groups*, J. Differential Geom. 53 (1999), 327–358.

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