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# A REMARK ON THE TRANSPORT EQUATION WITH $b \in \mathrm{BV}$ AND $\mathrm{div}_x\, b \in \mathrm{BMO}$

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**Abstract.** We investigate the transport equation $\partial_t u(t,x)+b(t,x)\cdot D_x u(t,x)=0$. Our result improves the classical criteria of uniqueness of weak solutions in the case of irregular coefficients: $b \in \mathrm{BV}$, $\mathrm{div}_x\, b \in \mathrm{BMO}$. To obtain our result we use a procedure similar to DiPerna and Lions's one developed for Sobolev vector fields. We apply renormalization theory for BV vector fields and logarithmic type inequalities to obtain energy estimates.

**1. Introduction.** In this paper we investigate the problem of existence and uniqueness of solutions to the following transport equation:

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot D_x u(t,x) = 0, \\ u(0,x) = \bar{u}(x), \end{cases} \tag{1.1}$$

where $b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is given and $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is unknown. This equation is a fundamental example in partial differential equations theory. One can interpret it as describing the transport of a given quantity $u$ which is constant along streamlines. Indeed, for smooth data $\bar{u}$ and $b$ one can easily solve (1.1) using the method of characteristics to see that $u$ is *transported* along the trajectories of the corresponding ODE. The straightforward outcome of this method is the uniqueness result for smooth $\bar{u}$ and $b$ (it is also provided by the Cauchy–Lipschitz theorem). DiPerna and Lions [16] showed that this result is valid for $b \in W^{1,1} \cap L^\infty$ with $\mathrm{div}_x\, b \in L^\infty$ over space. What is even more important, they introduced an innovative idea of dealing with the Cauchy problem.

We can split DiPerna and Lions's scheme into two independent parts. The first is what we can now call renormalization theory. Its aim is to show that solutions have the renormalization property (for (1.1) this means that whenever $u \in L^\infty$ solves the problem with initial data $\bar{u}$, so does $\beta(u)$ for any $\beta \in C^1$ with initial data $\beta(\bar{u})$). We can also express the idea in terms of properties of the vector field $b$. We say a given vector field $b$ has the renormalization property iff all bounded solutions to (1.1) have the renormalization

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property. In general there are vector fields without the renormalization property, though Ambrosio [3] successfully dealt with the case when $b \in \mathrm{BV}$. This assumption is sharp in view of counterexamples presented in [1, 11]. On the other hand, $W^{1,1} \subset \mathrm{BV}$, so this naturally extends DiPerna and Lions's classical result. Hence, it is not unexpected that even though there are many recent papers on the transport equation like [15, 6, 5, 4] to name but a few, they all work in the BV setting (with an interesting exception: Valkonen's paper [22] where velocity is a special function of bounded deformation).

The second part of DiPerna and Lions's scheme is to show how (and under which assumptions) the renormalization property implies uniqueness; in this part of the scheme Ambrosio [3] assumed $\operatorname{div}_x b \in L^\infty$. Recently in [9, 6, 5, 15] authors working in the renormalization framework dealing with the Cauchy problem for the transport and continuity equations redirect the assumptions from the divergence—they assume that $b$ is nearly incompressible (that is the case iff any solution is bounded away from zero and infinity). However, their considerations require $b \in L^\infty$ over time against the usual $L^1$ regularity. Bouchut et al. [10] worked with one-sided Lipschitz coefficient but once more with the assumption of $L^\infty$ regularity over time. Mucha [19] proved a parallel result: he showed that $\operatorname{div}_x b \in \mathrm{BMO}$ is enough to derive uniqueness from renormalization. It is a natural extension since $L^\infty \subset \mathrm{BMO}$. On the other hand, the BMO space is on the boundary of known counterexamples [16].

The cost of obtaining better regularity (BMO instead of $L^\infty$) is the assumption on the boundedness of the support of $\operatorname{div}_x b$ (see Theorem 1.6).

Remark 1.1. We recall that a measure $\mu$ is absolutely continuous (a.c.) with respect to a measure $\nu$ and we write $\mu \ll \nu$ iff $\nu(A) = 0$ implies $\mu(A)$ for every Borel $A \subset \mathbb{R}^d$. It is well known that every $f \in L^1_{\mathrm{loc}}$ is the density of an absolutely continuous measure, that is, $\mu_f(A) := \int_A f\,dx$ is a.c. Conversely, every a.c. measure is represented by a density function in $L^1_{\mathrm{loc}}$. Hence, we will often identify these two objects. In particular, writing $f \ll \mathcal{L}^d$ we mean $\mu_f \ll \mathcal{L}^d$. For details we refer to [7] and [17].

We consider weak solutions to (1.1) in the following sense:

Definition 1.2. Let $\bar{u} \in L^\infty((0,T)\times\mathbb{R}^d)$, $b, \operatorname{div}_x b \in L^1_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d))$. We say that $u \in L^\infty((0,T)\times\mathbb{R}^d)$ is a *weak solution* to (1.1) if the following integral identity holds:

$$\int_0^T \int_{\mathbb{R}^d} u(t,x)\{\partial_t\varphi(t,x) + b(t,x)\cdot D_x\varphi(t,x) + \varphi(t,x)\operatorname{div}_x b(t,x)\}\,dt\,dx = -\int_{\mathbb{R}^d} \bar{u}(x)\varphi(0,x)\,dx \tag{1.2}$$

for each $\varphi \in C^\infty([0,T]; C_0^\infty(\mathbb{R}^d))$ such that $\varphi|_{t=T} = 0$.

Renormalization is the key concept in DiPerna–Lions's and Ambrosio's works:

DEFINITION 1.3. We say that a vector field $b \in L^1_{\mathrm{loc}}(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^d))$ has the *renormalization property* if for every solution $u$ to (1.1) in the sense of Definition 1.2 the following integral identity holds:

$$\int_0^T \int_{\mathbb{R}^d} \beta(u(t,x))\{\partial_t\varphi(t,x)+b(t,x)\cdot D_x\varphi(t,x)+\varphi(t,x)\operatorname{div}_x b(t,x)\}\,dt\,dx = -\int_{\mathbb{R}^d} \beta(\bar{u}(x))\varphi(0,x)\,dx \tag{1.3}$$

for each $\beta \in C^1(\mathbb{R})$ and all $\varphi \in C^\infty([0,T];C_0^\infty(\mathbb{R}^d))$ such that $\varphi|_{t=T} = 0$.

Both results mentioned above (Mucha's and Ambrosio's) extend DiPerna and Lions's classical result. In this paper we highlight the proofs of these extensions and show that they are independent. We show that we can weaken the assumptions in both theorems (that is, $\operatorname{div}_x b \in \mathrm{BMO}$ and $b \in \mathrm{BV}$) and still derive existence and uniqueness for (1.1). The following is the main result of this paper:

THEOREM 1.4. *Let $T > 0$, $b \in L^1(0,T;\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$, $\bar{u} \in L^\infty(\mathbb{R}^d)$, and suppose that*

$$\operatorname{div}_x b \in L^1(0,T;\mathrm{BMO}(\mathbb{R}^d)), \qquad \frac{b}{1+|x|} \in L^1(0,T;L^1(\mathbb{R}^d)),$$
$$\operatorname{supp}\operatorname{div}_x b(t,\cdot) \subset B_R(0) \quad \text{for a fixed } R > 0,$$

*where $B_R(0)$ is the ball centered at the origin with radius $R$. Then there exists a unique weak solution to* (1.1).

We follow a procedure similar to DiPerna–Lions's and we split the proof of Theorem 1.4 into two parts. The first one is the renormalization theorem for BV vector fields proved by Ambrosio [3]:

THEOREM 1.5 (Ambrosio). *Let $b \in L^1_{\mathrm{loc}}(0,T;\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ with $\operatorname{div}_x b \ll \mathcal{L}^d$ (see Remark* 1.1*). Then $b$ has the renormalization property in the sense of Definition* 1.3.

The second part is derived from what Mucha proved in [19]. The key role here is played by the logarithmic type estimate for BMO functions given in [20].

THEOREM 1.6 (Mucha). *Let $f \in \mathrm{BMO}(\mathbb{R}^d)$ with bounded support and let $g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then*

$$\Big|\int_{\mathbb{R}^d} fg\,dx\Big| \le C_0\|f\|_{\mathrm{BMO}(\mathbb{R}^d)}\|g\|_{L^1(\mathbb{R}^d)}\big[|\ln\|g\|_{L^1(\mathbb{R}^d)}|+\ln(e+\|g\|_{L^\infty(\mathbb{R}^n)})\big]. \tag{1.4}$$

Mucha views this result as belonging to logarithmic Sobolev inequalities [13, 14, 18]; however, we require no additional information on the derivatives, hence we can apply it to a BV vector field. The weakness of the result lies in the assumption on the boundedness of the support of $f \in \mathrm{BMO}$, which seems to be inevitable since the proof relies on the classical Zygmund result for $f \in \mathrm{BMO}$ (see Appendix (3.2)).

By a *measure* we always mean a Borel signed measure ($\mu$ is a Borel measure iff any Borel set $A \subset \mathbb{R}^d$ is $\mu$-measurable). We say that $f \in L^1(\mathbb{R}^d)$ is a *function with bounded variation*, $f \in \mathrm{BV}(\mathbb{R}^d)$, if its distributional derivative $Df$ is representable by a measure with finite total variation (we often simply say that the measure is finite). The *total variation* of a measure $Df$ is defined by

$$(1.5)\quad |Df|(C) := \sup\Big\{\sum_{i=1}^{\infty} |Df(C_i)| : C_i \text{ Borel, pairwise disjoint, } C_i \subset C\Big\}.$$

We say that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ has *locally bounded variation*, $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, if its distributional derivative $Du$ is representable by a measure with locally finite variation (finite on any compact set). We say that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ has *bounded mean oscillation*, $f \in \mathrm{BMO}(\mathbb{R}^d)$, if the seminorm

$$(1.6)\qquad \|f\|_{\mathrm{BMO}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d,\, r>0} \fint_{B_r(x)} \Big[f(y) - \fint_{B_r(x)} f(x)\,dx\Big]\,dy$$

is finite, where $\fint_{B_r(x)} f(y)\,dy = |B_r(x)|^{-1} \int f(y)\,dy$ and $B_r(x)$ is the ball of radius $r$ centered at $x$. In general the norm in $\mathrm{BMO}(\mathbb{R}^d)$ may be defined as the sum of (1.6) and the standard $L^1$ norm, but we are interested in the case when $\mathrm{div}_x\, b \in \mathrm{BMO}(\mathbb{R}^d)$ has compact support, which implies that

$$\|\mathrm{div}_x\, b\|_{L^1(\mathbb{R}^d)} \le |\mathrm{diam}\,\mathrm{supp}\,\mathrm{div}_x\, b|^d \|\mathrm{div}_x\, b\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

This paper is organized as follows: first under the assumption that the vector field $b$ has the renormalization property we prove uniqueness for (1.1) (see Theorem 1.4); next we sketch the proof that in our case the vector field $b$ has the renormalization property (see Theorem 1.5); and in the Appendix we prove existence of distributional solutions to (1.1) and recall some basic facts used in Ambrosio's and Mucha's results.

**2. Proof of Theorem 1.4.** In this section we give a reasonably detailed proof of Theorem 1.4. We follow Mucha's proof in [19]. The difference is that in our case $b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ instead of $b \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ over the spatial variable. We find that this does not cause any trouble and Mucha's proof remains valid. There is no need to use any special properties of $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ functions in this part of the proof.

*Proof of Theorem 1.4.* We claim there exists a unique weak solution to (1.1) in the sense of Definition 1.2. The proof of existence in our case is standard and we omit it.

Theorem 1.5 states that $b$ has the renormalization property (1.3). In Theorem 1.5 we set $\beta(u) = u^2$ and we take $\pi_r(x)\varphi(t)$ as a test function, where $\varphi \in C^\infty(\mathbb{R})$ is such that $\varphi|_{t=T} = 0$, and $\pi_r \in C_c^\infty(\mathbb{R}^d)$ is a non-negative cut-off function such that $\pi_r(x) = \pi_1(x/r)$, $\pi_1(x) = 1$ in $B_1(0)$ and additionally

$$|\nabla \pi_r| \le C/r.$$

Then we obtain

$$\int_0^T \int_{\mathbb{R}^d} u^2 \pi_r \partial_t \varphi \, dx\, dt + \int_0^T \int_{\mathbb{R}^d} u^2 \varphi b \cdot \nabla \pi_r \, dx\, dt + \int_0^T \int_{\mathbb{R}^d} u^2 \pi_r \varphi \operatorname{div}_x b \, dx\, dt = 0. \tag{2.1}$$

The second term in (2.1) can be estimated as follows:

$$\Big| \int_0^T \int_{\mathbb{R}^d} u^2 \varphi b \cdot \nabla \pi_r \, dx\, dt \Big| \le \|u^2\|_{L^\infty} \int_0^T \int_{\mathbb{R}^d} (1+|x|)|\nabla \pi_r| \frac{b}{1+|x|} \, dx\, \varphi\, dt. \tag{2.2}$$

By definition $(1+|x|)|\nabla \pi_r| \le C$, $\operatorname{supp} \nabla \pi_r \subset B_{2r}(0) \setminus B_r(0)$ and by assumption $b/(1+|x|) \in L^1$, hence the r.h.s. of (2.2) converges to 0 as $r \to \infty$.

Also by assumption the support of $\operatorname{div}_x b$ is bounded, so for every $r > r_0$ with sufficiently large $r_0$ we have

$$\operatorname{div}_x b \pi_r = \operatorname{div}_x b \pi_{r_0} = \operatorname{div}_x b. \tag{2.3}$$

From (2.1) and (2.2), fixing $r_0$ as demanded in (2.3) we obtain

$$-\int_0^T \int_{\mathbb{R}^d} u^2 \pi_{r_0} \partial_t \varphi \, dx\, dt = \int_0^T \int_{\mathbb{R}^d} u^2 \pi_{r_0} \varphi \ \operatorname{div}_x b \, dx\, dt. \tag{2.4}$$

Denoting $\alpha(t) := \|u^2(x,t)\pi_{r_0}\|_{L^1(\mathbb{R}^d)}$ we have the pointwise representation

$$\alpha(t) = \int_0^t \int_{\mathbb{R}^d} u^2 \pi_{r_0} \operatorname{div}_x b \, dx\, dt. \tag{2.5}$$

By Theorem 1.6, from (2.5) we obtain

$$\alpha(t) \le \int_0^t C_0 \gamma(s) \alpha(s) [|\ln \alpha(s)| + \ln(m^2 + e)] \, ds$$

where $m := \|u\|_{L^\infty}$ and $\gamma(t) := \|\operatorname{div}_x b(t)\|_{\mathrm{BMO}(\mathbb{R}^d)}$. For a fixed $\epsilon > 0$ let us define an auxiliary function $\alpha^*(t)$ (that bounds $\alpha(t)$ and is easier to work with) by

$$\alpha^*(t) = \epsilon + \int_0^t C_0\gamma(s)\alpha^*(s)[|\ln \alpha^*(s)| + \ln(m^2+e)]\,ds. \tag{2.6}$$

From (2.5) we deduce that there exists $T_1 \in (0,T)$ such that

$$\alpha(t) \le e^{-1} \quad \text{for } t \in [0,T_1]. \tag{2.7}$$

On the interval $[0,e^{-1}]$ the function $w \mapsto w|\ln w|$ is increasing and the definition (2.6) guarantees that $\alpha^*(t)$ is a continuous and positive function, hence we can deduce

$$0 \le \alpha(t) < \alpha^*(t) \quad \text{for } t \in [0,T_1]. \tag{2.8}$$

By an argument used in the proof of the Gronwall inequality, (2.6) leads to the implicit formula

$$\alpha^*(t) = \epsilon \exp\Big\{\int_0^t C_0\gamma(s)[|\ln \alpha^*(s)| + \ln(m^2+e)]\,ds\Big\}.$$

From this representation, there exists $T_2 \in (0,T_1]$ such that

$$\alpha^*(t) \le C\epsilon \quad \text{for every } t \in [0,T_2]. \tag{2.9}$$

Then (2.9) and (2.8) immediately imply

$$0 \le \alpha(t) \le \alpha^*(t) \le \epsilon \quad \text{for all } t \in [0,T_2],$$

and taking $\epsilon \to 0$ we obtain

$$u^2\pi_{r_0} = 0 \quad \text{for each } t \in [0,T_2].$$

Since $r_0$ is arbitrarily large, we deduce that $u \equiv 0$ in $[0,T_2]$, and hence we obtain uniqueness for (1.1) in the whole $[0,T)$.

As mentioned at the beginning of this section, the proof of Theorem 1.4 will be complete if we deduce that $b$ has the renormalization property. We only sketch this part of the proof because the assumption on divergence (that $\operatorname{div}_x b \ll \mathcal{L}^d$) is used only at the beginning and in the final step of the argument. We also point out that the assumptions of Theorem 1.4 imply those of Theorem 1.5. In particular $\operatorname{div}_x b \in \mathrm{BMO}(\mathbb{R}^d)$ implies $\operatorname{div}_x b \in L^1(\mathbb{R}^d)$, and hence the distribution $\operatorname{div}_x b$ may be represented as a locally finite measure which is absolutely continuous (see Remark 1.1).

Before proving Theorem 1.5 we introduce some notation and terminology. (For the basic terminology and facts about BV functions and measure theory we refer to [7] and [17].) The spatial distributional derivative of $b(t,\cdot)$ is a matrix-valued measure, and hence for every $t \in (0,T)$ it has a decomposition $Db(t,\cdot) = D^a b(t,\cdot) + D^s b(t,\cdot)$, where $D^a b(t,\cdot)$ is absolutely continuous and $D^s b(t,\cdot)$ is singular. Integrating the measures $|Db(t,\cdot)|$, $|D^a b(t,\cdot)|$ and

$|D^s b(t,\cdot)|$ over time, we obtain measures $|Db|$, $|D^a b|$ and $|D^s b|$ respectively, i.e.

$$\int_{I\times\mathbb{R}^d} \varphi(t,x)\,d|Db|(t,x) = \int_I\int_{\mathbb{R}^d} \varphi(t,x)\,d|Db(t,\cdot)|(x)\,dt, \tag{2.10}$$

$$\int_{I\times\mathbb{R}^d} \varphi(t,x)\,d|D^\sigma b|(t,x) = \int_I\int_{\mathbb{R}^d} \varphi(t,x)\,d|D^\sigma b(t,\cdot)|(x)\,dt. \tag{2.11}$$

Throughout this section, $I \subset \mathbb{R}$ denotes any subinterval and $K \subset \mathbb{R}^d$ any compact subset.

To prove Theorem 1.5 we need two technical lemmas. The first one allows us to estimate difference quotients of BV functions. We split a BV quotient into two parts, $B^{1,\epsilon}$ and $B^{2,\epsilon}$, to deal with the absolutely continuous part and the singular part (in the limit).

Lemma 2.1. *Let $B \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^m)$ and $z \in \mathbb{R}^d$. Then there exists a decomposition*

$$\frac{B(x+\epsilon z)-B(x)}{\epsilon} = B_z^{1,\epsilon}(x) + B_z^{2,\epsilon}(x)$$

*such that*

$$B_z^{1,\epsilon}(x) \underset{\epsilon\to 0}{\longrightarrow} \nabla B(x)\cdot z \quad \text{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^m) \tag{2.12}$$

*where $\nabla B \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^m\times\mathbb{R}^d)$ satisfies $D^a B = \nabla B\mathcal{L}^d$, and*

$$\limsup_{\epsilon\to 0} \int_K |B_z^{2,\epsilon}(x)|\,dx \le |D^s B\cdot z|(K) \tag{2.13}$$

*for each compact $K \subset \mathbb{R}^d$. Moreover* (2.12) *and* (2.13) *yield*

$$\sup_{\epsilon\in(0,\delta)} \int_K (|B_z^{1,\epsilon}(x)| + |B_z^{2,\epsilon}(x)|)\,dx \le |z|\,|D^s B|(K_\delta)$$

*where $K_\delta$ is the open $\delta$-neighbourhood of $K$.*

For a detailed proof of Lemma 2.1 we refer to [15, Proposition 4.3].

To optimize our estimates with respect to the chosen convolution kernel we use the following lemma

Lemma 2.2 (Alberti). *Set*

$$\mathcal{K} := \Big\{\eta \in C_c^\infty(B_1(0)) : \eta \ge 0 \text{ even and } \int_{B_1(0)} \eta = 1\Big\}. \tag{2.14}$$

*For any matrix $M \in \mathbb{R}^d\times\mathbb{R}^d$ we have*

$$\inf_{\eta\in\mathcal{K}} \Lambda(M,\eta) = |\mathrm{tr}\, M|$$

*where*

$$\Lambda(M,\eta) := \int\limits_{\mathbb{R}^d} |\nabla\eta(z)\cdot M\cdot z|\,dz. \tag{2.15}$$

For a detailed proof of Lemma 2.2 we refer to [12, Lemma 2.6.2].

*Proof of Theorem 1.5.* Let $\eta^\epsilon \in C_c^\infty(\mathbb{R}^d)$ be the family of mollifiers $\eta^\epsilon(x) := \epsilon^{-d}\eta(x/\epsilon)$ with an even convolution kernel. We often use the notation $f_\epsilon := f * \eta^\epsilon$. We begin with a standard procedure of mollifying (1.1). Direct computation (see for instance [3, (3.12)]) gives

$$\partial_t\beta(u_\epsilon) + b\cdot D_x\beta(u_\epsilon) = R^\epsilon\beta'(u_\epsilon) \tag{2.16}$$

where

$$R^\epsilon = b\cdot D_x(u*\eta^\epsilon) - (b\cdot D_x u)*\eta^\epsilon.$$

Again, a direct but tedious computation (where we use the fact that $\eta$ is even; see [3, (3.8)]) gives another formula for $R^\epsilon$:

$$\begin{aligned} R^\epsilon(x) = \int\limits_{\mathbb{R}^d} u(x-\epsilon z)\left(\frac{b(t,x-\epsilon z)-b(t,x)}{\epsilon}\cdot\nabla\eta(z)\right)dz \\ -(u\operatorname{div}_x b)*\eta^\epsilon. \end{aligned} \tag{2.17}$$

Using Lemma 1 (with respect to the spatial variable) we decompose (2.17) into

$$\begin{aligned} \int\limits_{\mathbb{R}^d} u(x-\epsilon z)b_z^{1,\epsilon}(t,x)\cdot\nabla\eta(z)\,dz - (u\operatorname{div}_x b)*\eta^\epsilon \\ + \int\limits_{\mathbb{R}^d} u(x-\epsilon z)b_z^{2,\epsilon}(t,x)\cdot\nabla\eta(z)\,dz. \end{aligned} \tag{2.18}$$

We apply (2.12) to the first term of (2.18). Since $\operatorname{div}_x b \ll \mathcal{L}^d$, the absolutely continuous part of the divergence coincides with the whole divergence. Thus the same phenomenon occurs as in DiPerna and Lions's classical result—the first two terms in (2.18) vanish. Therefore following the computations in [15] we have

$$\begin{aligned} \int\limits_I\int\limits_K |R^\epsilon(x)|\,dx\,dt &\le \|u\|_{L^\infty}\int\limits_I\int\limits_K\int\limits_{\mathbb{R}^d} |b_z^{2,\epsilon}(t,x)\cdot\nabla\eta(z)|\,dz\,dx\,dt \\ &\le \|u\|_{L^\infty}\int\limits_I\int\limits_{\operatorname{supp}(\eta)}\int\limits_K |b_z^{2,\epsilon}(t,x)\cdot\nabla\eta(z)|\,dx\,dz\,dt. \end{aligned}$$

Next, the Fatou lemma yields

$$\limsup_{\epsilon\to 0}\int\limits_I\int\limits_K |R^\epsilon(x)|\,dx\,dt \le \|u\|_{L^\infty}\int\limits_I\int\limits_{\mathbb{R}^d}\limsup_{\epsilon\to 0}\int\limits_K |b_z^{2,\epsilon}(t,x)\cdot\nabla\eta(z)|\,dx\,dz\,dt.$$

For any fixed $z$ we have $b_z^{2,\epsilon}(x)\cdot\nabla\eta(z) = [b\cdot\nabla\eta(z)]_z^{2,\epsilon}(x)$, so Lemma 2.1

implies

$$\limsup_{\epsilon\to 0}\int_I\int_K |R^\epsilon(x)|\,dx\,dt \tag{2.19}$$
$$\le \|u\|_{L^\infty}\int_I\int_{\mathbb{R}^d}\limsup_{\epsilon\to 0}\int_K |b_z^{2,\epsilon}(t,x)\cdot\nabla\eta(z)|\,dx\,dz\,dt$$
$$\le \|u\|_{L^\infty}\int_I\int_{\mathbb{R}^d}\limsup_{\epsilon\to 0}\int_K |[b\cdot\nabla\eta(z)]_z^{2,\epsilon}(t,x)|\,dx\,dz\,dt$$
$$\le \|u\|_{L^\infty}\int_I\int_{\mathbb{R}^d} |D^s(b\cdot\nabla\eta(z))(t,\cdot)\cdot z|(K)\,dz\,dt.$$

Additionally using the identity $D^s b(t,\cdot) = M(t,x)|D^s b(t,\cdot)|$ (where $M : I\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ is the Radon–Nikodym derivative of $D^s b(t,\cdot)$ with respect to $|D^s b(t,\cdot)|$) we have

$$|D^s(b\cdot\nabla\eta(z))(t,\cdot)\cdot z|(K) = \int_K |\nabla\eta(z)\cdot M(t,x)\cdot z|\,d|D^s b(t,\cdot)|(x). \tag{2.20}$$

From (2.20) applied to (2.19) we have

$$\limsup_{\epsilon\to 0}\int_I\int_K |R^\epsilon(x)|\,dx\,dt \tag{2.21}$$
$$\le \|u\|_{L^\infty}\int_{\mathbb{R}^d} |\nabla\eta(z)\cdot M(t,x)\cdot z|\,dz\,|D^s b|(I\times K).$$

The left-hand side of (2.16) converges distributionally to (1.1). Notice that $\{\beta'(u^\epsilon)\}$ is uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^d)$ (see (3.2) in Appendix with $C = \|\bar{u}\|_{L^\infty}$). Therefore from the estimate (2.21) we deduce that the sequence $\{R^\epsilon\beta'(u^\epsilon)\}$ has limit points in the sense of measures. Up to a subsequence the right-hand side of (2.16) converges to a locally finite measure (we call it a defect measure), i.e. taking $\epsilon\to 0$ in (2.16) we have

$$\partial_t\beta(u) + b\cdot D_x\beta(u) = \sigma. \tag{2.22}$$

Hence, we can rewrite (2.21) as

$$|\sigma| \le \|\beta'\|_{L^\infty}\|u\|_{L^\infty}\int_{\mathbb{R}^d} |\nabla\eta(z)\cdot M(t,x)\cdot z|\,dz\,|D^s b| \tag{2.23}$$

in the sense of measures on $I\times\mathbb{R}^d$. Using the definition (2.15) we obtain

$$|\sigma| \le \|\beta'\|_{L^\infty}\|u\|_{L^\infty}\Lambda(M,\eta)|D^s b| \tag{2.24}$$

in the sense of measures on $I\times\mathbb{R}^d$. Since the measure $\sigma$ does not depend on the convolution kernel $\eta$, to finish the estimates and show that $\sigma = 0$ we may optimize (2.24) with respect to the convolution kernels. Estimate (2.21) yields $\sigma \ll |D^s b|$. Hence we can use the Radon–Nikodym decomposition

$\sigma = f(t,x)|D^s b|$ (where $f(t,x)$ is a real, Borel function) and (2.24) yields

$$|f(t,x)| \le \|\beta'\|_{L^\infty}\|u\|_{L^\infty}\Lambda(M,\eta) \qquad |D^s b|\text{-a.e.} \tag{2.25}$$

Here one can follow Bouchut [8] and then use Alberti's rank-one theorem for derivatives of BV functions (see [2]). We apply though the relatively easier Lemma 2.2 (also due to Alberti). However, arbitrarily optimizing with respect to the convolution kernel $\eta$ we do not control the set on which the estimate (2.25) may fail to hold. Hence we choose a dense, countable subset $\mathcal{D}$ of $\mathcal{K}$ defined in (2.14). From (2.25) we get

$$|f(t,x)| \le \|\beta'\|_{L^\infty}\|u\|_{L^\infty} \inf_{\eta\in\mathcal{D}} \Lambda(M,\eta) \qquad |D^s b|\text{-a.e.} \tag{2.26}$$

Since the map $\theta \mapsto \Lambda(M,\theta)$ is continuous in the $W^{1,1}$ topology, the infima over $\mathcal{K}$ and $\mathcal{D}$ coincide. Applying Lemma 2.2 to (2.26) we have

$$|f(t,x)| \le \|\beta'\|_{L^\infty}\|u\|_{L^\infty}|\operatorname{tr} M| \qquad |D^s b|\text{-a.e.}$$

Hence

$$|\sigma| \ll |D^s b|.$$

We conclude that the measure $\sigma$ is absolutely continuous with respect to the singular part of the divergence of $b$. Together with the assumption $\operatorname{div}_x b \ll \mathcal{L}^d$ (see Remark 1.1) this gives us $\sigma \equiv 0$. Thus if $u$ is a solution to (1.1), from (2.22) we have

$$\partial_t \beta(u) + b \cdot D_x \beta(u) = 0.$$

Hence $b$ has the renormalization property.

## 3. Appendix

**3.1. Stability.** As a by-product of the renormalization procedure we obtain the structural stability property.

THEOREM 3.1. *Let $b$ and $\bar{u}$ satisfy the assumptions of Theorem* 1.4, *and suppose that*

$$b_\epsilon \to b \quad \textit{strongly in } L^1_{\mathrm{loc}}(0,T;\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)),$$
$$\bar{u}_\epsilon \to \bar{u} \quad \textit{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

*and the sequence $\|\bar{u}_\epsilon\|_\infty$ is uniformly bounded. Then*

$$u_\epsilon \to u \quad \textit{strongly in } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$$

*where $u_\epsilon$ are solutions to the corresponding transport equations and $u$ is a solution to* (1.1).

*Proof.* Consider the problem

$$\begin{cases} \partial_t u_\epsilon + b_\epsilon \cdot D_x u_\epsilon = 0, \\ u_\epsilon(0,x) = \bar{u}_\epsilon(x). \end{cases} \tag{3.1}$$

The method of characteristics implies the existence of smooth solutions to (3.1) for $t \in (0,T)$ together with the bound

$$\|u_\epsilon\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq \|\bar{u}_\epsilon\|_{L^\infty(\mathbb{R}^d)} \leq C. \tag{3.2}$$

Note that we do not need any uniform bound on $\operatorname{div}_x b_\epsilon$. The solutions to (1.1) are classical, in particular they satisfy the integral identity

$$\int_0^T \int_{\mathbb{R}^d} u_\epsilon(t,x)\big\{\partial_t\varphi(t,x) + b_\epsilon(t,x)\cdot D_x\varphi(t,x) + \varphi(t,x)\operatorname{div}_x b_\epsilon(t,x)\big\}\,dt\,dx = -\int_{\mathbb{R}^d} \bar{u}_\epsilon(x)\varphi(0,x)\,dx \tag{3.3}$$

for any $\varphi \in C^\infty([0,T];C_0^\infty(\mathbb{R}^d))$ such that $\varphi|_{t=T} = 0$. The estimate (3.2) implies that $\|u_\epsilon\|_{L^\infty([0,T]\times\mathbb{R}^d)}$ is equibounded, so we can choose a subsequence such that for a subsequence $\epsilon_k \to 0$ we have

$$u_{\epsilon_k} \rightharpoonup^* u \quad \text{in } L^\infty([0,T]\times\mathbb{R}^d). \tag{3.4}$$

Then taking the limit in (3.3), by the properties of the sequences $b_\epsilon$ and $\bar{u}_\epsilon$ we find that $u$ is a solution to (1.1). Hence by the uniqueness part of Theorem 1.4, the whole sequence $u_{\epsilon_k}$ converges to $u$. By the renormalization property (Theorem 1.5), $u_\epsilon^2$ is also a solution to the transport equation with initial data $\bar{u}_\epsilon^2$, and arguing as before shows that

$$u_\epsilon^2 \rightharpoonup^* u^2 \quad \text{in } L^\infty([0,T]\times\mathbb{R}^d) \tag{3.5}$$

where $u^2$ is again the unique solution to the transport equation with initial data $\bar{u}^2$. The convergence property (3.5) implies $\|u_\epsilon\|_{L^2} \to \|u\|_{L^2}$ on any compact subset of $[0,T]\times\mathbb{R}^d$, and since we have the weak convergence (3.4), we conclude

$$u_\epsilon \to u \quad \text{strongly in } L^2_{\text{loc}}([0,T]\times\mathbb{R}^d),$$

and hence

$$u_\epsilon \to u \quad \text{strongly in } L^1_{\text{loc}}([0,T]\times\mathbb{R}^d).$$

**3.2. BMO logarithmic estimate.** Following the reasonings in [19] we will now prove Theorem 1.4.

*Proof of Theorem 1.4.* Since we have assumed that $\operatorname{supp} f$ is bounded, we can restrict our considerations to a $d$-dimensional torus $\mathbb{T}^d = [0,m)^d$. Let us consider the (real) Hardy space $\mathcal{H}^1$ on $\mathbb{T}^d$ with the norm

$$\|g\|_{\mathcal{H}^1(\mathbb{T}^d)} = \|g\|_{L^1(\mathbb{T}^d)} + \sum_{k=1}^d \|R_k g\|_{L^1(\mathbb{T}^d)}, \tag{3.6}$$

where $R_k$ are the Riesz operators (see [21]). Since the dual space to $\mathcal{H}^1(\mathbb{T}^d)$ is $\mathrm{BMO}(\mathbb{T}^d)$, we get

$$\Big| \int_{\mathbb{T}^d} fg\,dx \Big| \le \|f\|_{\mathrm{BMO}(\mathbb{T}^d)} \|g\|_{\mathcal{H}^1(\mathbb{T}^d)}. \tag{3.7}$$

Hence to control the norm (3.6) an estimate of $\|R_k g\|_{L^1(\mathbb{T}^d)}$ is required. Zygmund's classical result (see [21], [23]) says that

$$\|R_k h\|_{L^1(\mathbb{T}^d)} \le C + C \int_{\mathbb{T}^d} |h| \ln^+ |h|\,dx \tag{3.8}$$

where $\ln^+ a = \max\{\ln a, 0\}$ and the constants $C$ depend on $m$, and thus on the diameter of $\operatorname{supp} f$. Observe that

$$\begin{aligned} \Big| \ln^+ \frac{g}{\|g\|_{L^1(\mathbb{T}^d)}} \Big| &\le \ln(1 + \|g\|_{L^\infty(\mathbb{T}^d)}) + \Big| \ln \frac{g}{1 + \|g\|_{L^\infty(\mathbb{T}^d)}} \Big| \\ &\le 2\ln(1 + \|g\|_{L^\infty(\mathbb{T}^d)}) + |\ln \|g\|_{L^1(\mathbb{T}^d)}|. \end{aligned} \tag{3.9}$$

Applying (3.9) to (3.8) (for $h := g/\|g\|_{L^1(\mathbb{T}^d)}$) and (3.7) with norm (3.6) we obtain the desired result. ■

*REFERENCES*


[1] M. Aizenman, *On vector fields as generators of flows: a counterexample to Nelson's conjecture*, Ann. of Math. (2) 107 (1978), 287–296.

[2] G. Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), 239–274.

[3] L. Ambrosio, *Transport equation and Cauchy problem for* BV *vector fields*, Invent. Math. 158 (2004), 227–260.

[4] L. Ambrosio, *Transport equation and Cauchy problem for non-smooth vector fields*, in: Calculus of Variations and Nonlinear Partial Differential Equations, Lecture Notes in Math. 1927, Springer, Berlin, 2008, 1–41.

[5] L. Ambrosio, G. Crippa, A. Figalli, and L. V. Spinolo, *Some new well-posedness results for continuity and transport equations, and applications to the chromatography system*, SIAM J. Math. Anal. 41 (2009), 1890–1920.

[6] L. Ambrosio, G. Crippa, A. Figalli, and L. V. Spinolo, *Existence and uniqueness results for the continuity equation and applications to the chromatography system*, in: Nonlinear Conservation Laws and Applications, IMA Vol. Math. Appl. 153, Springer, New York, 2011, 195–204.

[7] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Oxford Univ. Press, New York, 2000.

[8] F. Bouchut, *Renormalized solutions to the Vlasov equation with coefficients of bounded variation*, Arch. Ration. Mech. Anal. 157 (2001), 75–90.

[9] F. Bouchut and G. Crippa, *Uniqueness, renormalization, and smooth approximations for linear transport equations*, SIAM J. Math. Anal. 38 (2006), 1316–1328.

[10] F. Bouchut, F. James, and S. Mancini, *Uniqueness and weak stability for multi-dimensional transport equations with one-sided Lipschitz coefficient*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) 4 (2005), 1–25.

[11] F. Colombini, T. Luo, and J. Rauch, *Uniqueness and nonuniqueness for nonsmooth divergence free transport*, Séminaire Équations aux Dérivées Partielles, École Polytech., Palaiseau, 2002–2003, exp. 23, 2003, 21 pp.

[12] G. Crippa, *The flow associated to weakly differentiable vector fields*, Tesi Scuola Norm. Sup. Pisa (N.S.) 12, Ed. della Normale, Pisa, 2009.

[13] R. Danchin, *On the well-posedness of the incompressible density-dependent Euler equations in the $L^p$ framework*, J. Differential Equations 248 (2010), 2130–2170.

[14] R. Danchin and P. B. Mucha, *A critical functional framework for the inhomogeneous Navier–Stokes equations in the half-space*, J. Funct. Anal. 256 (2009), 881–927.

[15] C. De Lellis, *Notes on hyperbolic systems of conservation laws and transport equations*, in: Handbook of Differential Equations: Evolutionary Equations. Vol. III, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007, 277–382.

[16] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. 98 (1989), 511–547.

[17] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.

[18] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in* BMO, *with application to the Euler equations*, Comm. Math. Phys. 214 (2000), 191–200.

[19] P. B. Mucha, *Transport equation: extension of classical results for* $\operatorname{div} b \in \mathrm{BMO}$, J. Differential Equations 249 (2010), 1871–1883.

[20] P. B. Mucha and W. M. Rusin, *Zygmund spaces, inviscid limit and uniqueness of Euler flows*, Comm. Math. Phys. 280 (2008), 831–841.

[21] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, NJ, 1993,

[22] T. Valkonen, *Transport equation and image interpolation with SBD velocity fields*, J. Math. Pures Appl. (9) 95 (2011), 459–494.

[23] A. Zygmund, *Trigonometric Series*, 2nd ed., Vols. I, II, Cambridge Univ. Press, New York, 1959.

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