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PRODUCT OF THREE NUMBERS BEING A SQUARE AS A RAMSEY PROPERTY

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Abstract. For any partition of a set of squarefree numbers with relative density greater than 3/4 into two parts, at least one part contains three numbers whose product is a square. Also generalizations to partitions into more than two parts are discussed.

We start with the following result:

THEOREM 1. *If a set $\mathcal{A} \subseteq \mathbb{N}$ consists entirely of squarefree numbers and has lower asymptotic density*

$$\delta^*(\mathcal{A}) = \liminf_{x\to\infty} \frac{1}{x} \sum_{n\in\mathcal{A},\, n\le x} 1 > \frac{3}{4}\cdot\frac{6}{\pi^2}$$

then for any partition $\mathcal{A}_1, \mathcal{A}_2$ *of* $\mathcal{A}$ (*i.e.*, $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ *and* $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$) *there exist* $j \in \{1,2\}$ *and* $a,b,c \in \mathcal{A}_j$ *such that*

$$abc = \square.$$

REMARK. The constant $(3/4)\cdot(6/\pi^2)$ is optimal as the following example shows. Let $\Omega(n)$ count the number of prime factors of n. Define

$$\mathcal{A}_1 = \{n \in \mathbb{N} \mid n \text{ squarefree and } \Omega(n) \equiv 1 \pmod 2\},$$
$$\mathcal{A}_2 = \{n \in \mathbb{N} \mid n \text{ squarefree and } \Omega(n) \equiv n \equiv 0 \pmod 2\}.$$

It is well known that

$$\delta^*(\mathcal{A}_1) = \frac{1}{2}\cdot\frac{6}{\pi^2} \quad\text{and}\quad \delta^*(\mathcal{A}_2) = \frac{1}{4}\cdot\frac{6}{\pi^2},$$

and obviously no three numbers in either $\mathcal{A}_1$ or $\mathcal{A}_2$ give a square product.

For the proof of the theorem we need a lemma.

LEMMA 1. *Let* $\mathbf{V}$ *be a finite-dimensional vector space over* $\mathbb{F}_2$ *and* $\mathcal{B}$ *a subset of* $\mathbf{V}$ *not containing the vector* $\vec{0}$. *If*

$$|\mathcal{B}| > \left(1 - \frac{1}{2^k}\right)\cdot|\mathbf{V}| \tag{1}$$

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then there exist $\beta_1, \ldots, \beta_{k+1}$ *satisfying the condition*

$$\forall_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} \quad \sum_{j \in J} \beta_j \in \mathcal{B}. \tag{2}$$

Proof. For $k = 0$ there is nothing to prove. Assuming that the lemma is proved for k we consider $\mathcal{B} \subset \mathbf{V}$ with $\vec{0} \notin \mathcal{B}$ such that

$$|\mathcal{B}| > \left(1 - \frac{1}{2^{k+1}}\right) \cdot |\mathbf{V}|.$$

By the inductive assumption there exist $\beta_1, \ldots, \beta_{k+1}$ satisfying (2). Now define

$$\mathcal{B}_S := \mathcal{B} + \sum_{j \in S} \beta_j \quad \text{for all } S \subseteq \{1, \ldots, k+1\},$$

($\mathcal{B}_\emptyset := \mathcal{B}$) and observe that

$$\Big| \bigcup_{S \subseteq \{1,\ldots,k+1\}} \mathbf{V} \setminus \mathcal{B}_S \Big| < \sum_{S \subseteq \{1,\ldots,k+1\}} |\mathbf{V} \setminus \mathcal{B}_S| < 2^{k+1} \cdot \frac{|\mathbf{V}|}{2^{k+1}},$$

which implies

$$\bigcap_{S \subseteq \{1,\ldots,k+1\}} \mathcal{B}_S \neq \emptyset.$$

Now choose β_{k+2} from the above set and observe that it works.

Proof of Theorem 1. For $\mathcal{C} \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ let

$$\tau(n, \mathcal{C}) = \sum_{d|n,\, d \in \mathcal{C}} 1.$$

Moreover let $\mathcal{SF}$ denote the set of all squarefree positive integers. By Corollary 2 of [1] there exists $m \in \mathbb{N}$ such that

$$\tau(m, \mathcal{A}) > \frac{3}{4} \cdot \tau(m, \mathcal{SF}).$$

Let n be the greatest squarefree divisor of m. Then every squarefree divisor of m is a divisor of n and we obtain

$$\tau(n, \mathcal{A}) > \frac{3}{4} \tau(n). \tag{3}$$

Now let $\mathbf{V}$ be the set of all positive divisors of n equipped with the structure of a vector space over $\mathbb{F}_2$ as follows. For $d_1, d_2, d_3 \in \mathbf{V}$ we have $d_1 \oplus d_2 = d_3$ if and only if $\mathrm{sf}(d_1 d_2) = d_3$, where $\mathrm{sf}(r)$ stands for the squarefree kernel of r. We can apply Lemma for $k = 2$ to the set $\mathcal{B} := \mathbf{V} \cap \mathcal{A}$ in virtue of the inequality (3). Hence, there are positive divisors $\beta_1, \beta_2, \beta_3$ of n such that

$$\mathcal{C} := \{\beta_1, \beta_2, \beta_3, \mathrm{sf}(\beta_1\beta_2), \mathrm{sf}(\beta_1\beta_3), \mathrm{sf}(\beta_2\beta_3), \mathrm{sf}(\beta_1\beta_2\beta_3)\} \subset \mathcal{B}. \tag{4}$$

Put additionally

$$\mathbf{V}' = \mathcal{C} \cup \{\vec{0}\}$$

and

$$\mathcal{B}' = \begin{cases} \mathcal{A}_1 \cap \mathcal{C} & \text{if } |\mathcal{A}_1 \cap \mathcal{C}| \geq 4, \\ \mathcal{A}_2 \cap \mathcal{C} & \text{if } |\mathcal{A}_2 \cap \mathcal{C}| \geq 4. \end{cases}$$

Now we distinguish three cases. If $|\mathcal{B}'| = 4$ and $\mathcal{B}'$ contains a triple of "vectors" $\gamma_1, \gamma_2, \gamma_3$ that is linearly dependent then $\gamma_1\gamma_2\gamma_3 = \Box$ and we are done. If $|\mathcal{B}'| = 4$ but all triples of $\mathcal{B}'$ are linearly independent then necessarily $\prod_{\gamma\in\mathcal{B}'} = \Box$, hence $\prod_{\gamma\in\mathcal{C}\setminus\mathcal{B}'} = \Box$ and $|\mathcal{C} \setminus \mathcal{B}'| = 7 - 4 = 3$ and we are happy again. Finally, if $|\mathcal{B}'| \geq 5$ then we apply the Lemma for $k = 1$ to $\mathbf{V}'$ and $\mathcal{B}'$ and obtain $\gamma_1, \gamma_2 \in \mathcal{B}'$ such that also $\operatorname{sf}(\gamma_1\gamma_2) \in \mathcal{B}'$. Now $\gamma_1\gamma_2 \operatorname{sf}(\gamma_1\gamma_2) = \Box$.

A direct generalization of Theorem 1 to three components $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ of $\mathcal{A} \subseteq \mathcal{SF}$ having sufficiently high density is somehow problematic. Using the above method the following result can be easily proved:

Theorem 2. *If a set $\mathcal{A} \subseteq \mathbb{N}$ consists entirely of squarefree numbers and has lower asymptotic density*

$$\delta^*(\mathcal{A}) = \liminf_{x\to\infty} \frac{1}{x} \sum_{n\in\mathcal{A}, n\leq x} 1 > \frac{7}{8} \cdot \frac{6}{\pi^2}$$

then for any partition $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ of $\mathcal{A}$ there exist $j \in \{1, 2, 3\}$ and $a, b, c, d, f \in \mathcal{A}_j$ such that

$$abcdf = \Box.$$

But if we insist on three factors giving a square, a direct generalization of our argument does fail. Namely, take $\mathbf{V} = \mathbb{F}_{16}^* = \langle g \rangle$ with g a generator. It has a unique subgroup $\mathcal{H}_1$ of index 3. Put further $\mathcal{H}_2 = g\mathcal{H}_1$ and $\mathcal{H}_3 = g^2\mathcal{H}_1$. For each fixed $j \in \{1, 2, 3\}$ all the elements of $\mathcal{H}_j$ sum up to $\vec{0}$, but no three have vanishing sum. We use this idea to prove the following negative result:

Theorem 3. *There exists a set $\mathcal{A} \subseteq \mathbb{N}$ consisting entirely of squarefree numbers with natural asymptotic density*

$$\delta^*(\mathcal{A}) = \frac{15}{16} \cdot \frac{6}{\pi^2}$$

and its partition $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ such that for each $j \in \{1, 2, 3\}$ and $a, b, c \in \mathcal{A}_j$,

$$abc \neq \Box.$$

Proof. We will work with $\mathbb{F}_{16} = \mathbb{F}_2[x]/(x^4 + x^3 + 1)$ and use the table on p. 435 of [2] which gives the coordinates of the powers of the generator x in the basis $1, x, x^2, x^3$. So we define

$$\begin{aligned}\mathcal{H}_1 &= \{1, x^3, x^6, x^9, x^{12}\} \\ &= \{(1,0,0,0), (0,0,0,1), (1,1,1,1), (1,0,1,0), (1,1,0,0)\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_2 &= \{x, x^4, x^7, x^{10}, x^{13}\}\\ &= \{(0,1,0,0),(1,0,0,1),(1,1,1,0),(0,1,0,1),(0,1,1,0)\},\\ \mathcal{H}_3 &= \{x^2, x^5, x^8, x^{11}, x^{14}\}\\ &= \{(0,0,1,0),(1,1,0,1),(0,1,1,1),(1,0,1,1,),(0,0,1,1)\}.\end{aligned}$$

Moreover we define completely multiplicative functions

$$f_1, f_2, f_3, f_4 : \mathcal{SF} \to \{-1, 1\}$$

by giving their values for primes p:

$$\begin{aligned} f_1(p) &= -1 \qquad \text{for all } p \in \mathbb{P},\\ f_2(p) &= \begin{cases} -1 & \text{for } p = 2,\\ 1 & \text{for } p \neq 2,\end{cases}\\ f_3(p) &= \begin{cases} -1 & \text{for } p \equiv 3 \pmod 4,\\ 1 & \text{for } p \not\equiv 3 \pmod 4,\end{cases}\\ f_4(p) &= \begin{cases} -1 & \text{for } p \equiv \pm 3 \pmod 8,\\ 1 & \text{for } p \not\equiv \pm 3 \pmod 8.\end{cases}\end{aligned}$$

After these preparations we put

$$\mathcal{A}_j = \left\{ n \in \mathcal{SF} \;\middle|\; \left(\frac{1 - f_k(n)}{2}\right)_{1 \le k \le 4} \in \mathcal{H}_j \right\}$$

and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. The set $\mathcal{B} = \mathcal{SF} \setminus \mathcal{A}$ is easily identifiable as

$$\begin{aligned}\mathcal{B} &= \{n \in \mathcal{SF} \mid \forall 1 \le k \le 4,\ f_k(n) = 1\}\\ &= \{n \in \mathcal{SF} \mid \Omega(n) \equiv 0 \pmod 2 \text{ and } n \equiv 1 \pmod 8\}.\end{aligned}$$

and so has the relative density $1/16$. Therefore $\mathcal{A}$ has the required density $15/16$ and its partition defined above has the property of *omitting triples with square product.*

We finish our note with the corresponding positive result:

THEOREM 4. *If a set $\mathcal{A} \subseteq \mathbb{N}$ consists entirely of squarefree numbers and has lower asymptotic density*

$$\delta^*(\mathcal{A}) > \frac{15}{16} \cdot \frac{6}{\pi^2}$$

then for any partition $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ of $\mathcal{A}$ there exist $j \in \{1, 2, 3\}$ and $a, b, c \in \mathcal{A}_j$ such that

$$abc = \square.$$

Proof. Again, by Corollary 2 of [1] there exists $m \in \mathbb{N}$ such that

$$\tau(m, \mathcal{A}) > \frac{15}{16} \cdot \tau(m, \mathcal{SF}).$$

Let n be the greatest squarefree divisor of m. Then every squarefree divisor of m is a divisor of n and we obtain

$$\tau(n,\mathcal{A}) > \frac{15}{16}\tau(n). \tag{5}$$

Using similarly Lemma 1 we find five distinct divisors $\beta_1,\dots,\beta_5$ of n such that

$$\gamma_J := \mathrm{sf}\Big(\prod_{j\in J}\beta_j\Big) \in \mathcal{A}\setminus\{1\} \quad \text{for } \emptyset \neq J \subseteq \{1,\dots,5\}.$$

We now consider the graph with 31 vertices γ_J and colour the edge $\gamma_J\gamma_K$ using colour j ($j \in \{1,2,3\}$) if and only if $\mathrm{sf}(\gamma_J\gamma_K) \in \mathcal{A}_j$. By the Ramsey theorem and the classical result $R(3,3,3) = 17 < 31$ we obtain at least one monochromatic triangle $\gamma_J\gamma_K\gamma_L$. This means that if we put

$$a = \mathrm{sf}(\gamma_J\gamma_K), \quad b = \mathrm{sf}(\gamma_K\gamma_L), \quad c = \mathrm{sf}(\gamma_J\gamma_L),$$

then $a,b,c \in \mathcal{A}_j$ and obviously $abc = \square$.

Using the relatively new estimate [3]

$$R(3,3,3,3) < 63$$

and reasoning along the same lines one can prove

THEOREM 5. *If a set* $\mathcal{A} \subseteq \mathbb{N}$ *consists entirely of squarefree numbers and has lower asymptotic density*

$$\delta^*(\mathcal{A}) > \frac{31}{32}\cdot\frac{6}{\pi^2}$$

then for any partition $\{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4\}$ *of* $\mathcal{A}$ *there exist* $j \in \{1,2,3,4\}$ *and* $a,b,c \in \mathcal{A}_j$ *such that*

$$abc = \square.$$

The constant 31/32 is probably not optimal but we do not know any example.

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