PRODUCT OF THREE NUMBERS BEING A SQUARE
AS A RAMSEY PROPERTY

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Abstract. For any partition of a set of squarefree numbers with relative density greater than $3/4$ into two parts, at least one part contains three numbers whose product is a square. Also generalizations to partitions into more than two parts are discussed.

We start with the following result:

**Theorem 1.** If a set $\mathcal{A} \subseteq \mathbb{N}$ consists entirely of squarefree numbers and has lower asymptotic density

$$\delta^*(\mathcal{A}) = \liminf_{x \to \infty} \frac{1}{x} \sum_{n \in \mathcal{A}, n \leq x} 1 > \frac{3}{4} \cdot \frac{6}{\pi^2},$$

then for any partition $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{A}$ (i.e., $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$) there exist $j \in \{1, 2\}$ and $a, b, c \in \mathcal{A}_j$ such that

$$abc = \square.$$

**Remark.** The constant $(3/4) \cdot (6/\pi^2)$ is optimal as the following example shows. Let $\Omega(n)$ count the number of prime factors of $n$. Define

$$\mathcal{A}_1 = \{n \in \mathbb{N} \mid n \text{ squarefree and } \Omega(n) \equiv 1 \pmod{2}\},$$
$$\mathcal{A}_2 = \{n \in \mathbb{N} \mid n \text{ squarefree and } \Omega(n) \equiv n \equiv 0 \pmod{2}\}.$$

It is well known that

$$\delta^*(\mathcal{A}_1) = \frac{1}{2} \cdot \frac{6}{\pi^2} \quad \text{and} \quad \delta^*(\mathcal{A}_2) = \frac{1}{4} \cdot \frac{6}{\pi^2},$$

and obviously no three numbers in either $\mathcal{A}_1$ or $\mathcal{A}_2$ give a square product.

For the proof of the theorem we need a lemma.

**Lemma 1.** Let $\mathbf{V}$ be a finite-dimensional vector space over $\mathbb{F}_2$ and $\mathcal{B}$ a subset of $\mathbf{V}$ not containing the vector $\vec{0}$. If

$$|\mathcal{B}| > \left(1 - \frac{1}{2^k}\right) \cdot |\mathbf{V}|$$

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then there exist \( \beta_1, \ldots, \beta_{k+1} \) satisfying the condition
\[
\forall \emptyset \neq J \subseteq \{1, \ldots, k+1\} \quad \sum_{j \in J} \beta_j \in \mathcal{B}.
\]

**Proof.** For \( k = 0 \) there is nothing to prove. Assuming that the lemma is proved for \( k \) we consider \( \mathcal{B} \subseteq \mathcal{V} \) with \( \mathbf{0} \notin \mathcal{B} \) such that
\[
|\mathcal{B}| > \left(1 - \frac{1}{2^{k+1}}\right) \cdot |\mathcal{V}|.
\]
By the inductive assumption there exist \( \beta_1, \ldots, \beta_{k+1} \) satisfying (2). Now define
\[
\mathcal{B}_S := \mathcal{B} + \sum_{j \in S} \beta_j \quad \text{for all } S \subseteq \{1, \ldots, k+1\},
\]
(\( \mathcal{B}_0 := \mathcal{B} \)) and observe that
\[
\left| \bigcup_{S \subseteq \{1, \ldots, k+1\}} \mathcal{V} \setminus \mathcal{B}_S \right| < \sum_{S \subseteq \{1, \ldots, k+1\}} |\mathcal{V} \setminus \mathcal{B}_S| < 2^{k+1} \cdot \frac{|\mathcal{V}|}{2^{k+1}},
\]
which implies
\[
\bigcap_{S \subseteq \{1, \ldots, k+1\}} \mathcal{B}_S \neq \emptyset.
\]
Now choose \( \beta_{k+2} \) from the above set and observe that it works.

**Proof of Theorem 1.** For \( \mathcal{C} \subseteq \mathbb{N} \) and \( n \in \mathbb{N} \) let
\[
\tau(n, \mathcal{C}) = \sum_{d \mid n, d \in \mathcal{C}} 1.
\]
Moreover let \( \mathcal{SF} \) denote the set of all squarefree positive integers. By Corollary 2 of [1] there exists \( m \in \mathbb{N} \) such that
\[
\tau(m, \mathcal{A}) > \frac{3}{4} \cdot \tau(m, \mathcal{SF}).
\]
Let \( n \) be the greatest squarefree divisor of \( m \). Then every squarefree divisor of \( m \) is a divisor of \( n \) and we obtain
(3) \[
\tau(n, \mathcal{A}) > \frac{3}{4} \tau(n).
\]
Now let \( \mathcal{V} \) be the set of all positive divisors of \( n \) equipped with the structure of a vector space over \( \mathbb{F}_2 \) as follows. For \( d_1, d_2, d_3 \in \mathcal{V} \) we have \( d_1 \oplus d_2 = d_3 \) if and only if \( \text{sf}(d_1d_2) = d_3 \), where \( \text{sf}(r) \) stands for the squarefree kernel of \( r \). We can apply Lemma for \( k = 2 \) to the set \( \mathcal{B} := \mathcal{V} \cap \mathcal{A} \) in virtue of the inequality (3). Hence, there are positive divisors \( \beta_1, \beta_2, \beta_3 \) of \( n \) such that
(4) \[
\mathcal{C} := \{\beta_1, \beta_2, \beta_3, \text{sf}(\beta_1 \beta_2), \text{sf}(\beta_1 \beta_3), \text{sf}(\beta_2 \beta_3), \text{sf}(\beta_1 \beta_2 \beta_3)\} \subseteq \mathcal{B}.
\]
Put additionally
\[
\mathcal{V'} = \mathcal{C} \cup \{0\}
\]
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Now we distinguish three cases. If $|B'| = 4$ and $B'$ contains a triple of “vectors” $\gamma_1, \gamma_2, \gamma_3$ that is linearly dependent then $\gamma_1\gamma_2\gamma_3 = \square$ and we are done. If $|B'| = 4$ but all triples of $B'$ are linearly independent then necessarily $\prod_{\gamma \in B'} = \square$, hence $\prod_{\gamma \in C \setminus B'} = \square$ and $|C \setminus B'| = 7 - 4 = 3$ and we are happy again. Finally, if $|B'| \geq 5$ then we apply the Lemma for $k = 1$ to $V'$ and $B'$ and obtain $\gamma_1, \gamma_2 \in B'$ such that also $sf(\gamma_1\gamma_2) \in B'$. Now $\gamma_1\gamma_2 sf(\gamma_1\gamma_2) = \square$.

A direct generalization of Theorem 1 to three components $A_1, A_2, A_3$ of $A \subseteq SF$ having sufficiently high density is somehow problematic. Using the above method the following result can be easily proved:

**Theorem 2.** If a set $A \subseteq \mathbb{N}$ consists entirely of squarefree numbers and has lower asymptotic density

$$\delta^*(A) = \liminf_{x \to \infty} \frac{1}{x} \sum_{n \in A, n \leq x} 1 > \frac{7}{8} \cdot \frac{6}{\pi^2}$$

then for any partition $\{A_1, A_2, A_3\}$ of $A$ there exist $j \in \{1, 2, 3\}$ and $a, b, c, d, f \in A_j$ such that

$$abcdf = \square.$$

But if we insist on three factors giving a square, a direct generalization of our argument does fail. Namely, take $V = F_{16}^* = \langle g \rangle$ with $g$ a generator. It has a unique subgroup $H_1$ of index 3. Put further $H_2 = gH_1$ and $H_3 = g^2H_1$. For each fixed $j \in \{1, 2, 3\}$ all the elements of $H_j$ sum up to $\vec{0}$, but no three have vanishing sum. We use this idea to prove the following negative result:

**Theorem 3.** There exists a set $A \subseteq \mathbb{N}$ consisting entirely of squarefree numbers with natural asymptotic density

$$\delta^*(A) = \frac{15}{16} \cdot \frac{6}{\pi^2}$$

and its partition $\{A_1, A_2, A_3\}$ such that for each $j \in \{1, 2, 3\}$ and $a, b, c \in A_j$,

$$abc \neq \square.$$

**Proof.** We will work with $F_{16} = \mathbb{F}_2[x]/(x^4 + x^3 + 1)$ and use the table on p. 435 of [2] which gives the coordinates of the powers of the generator $x$ in the basis $1, x, x^2, x^3$. So we define

$$H_1 = \{1, x^3, x^6, x^9, x^{12}\} = \{(1, 0, 0, 0), (0, 0, 0, 1), (1, 1, 1, 1), (1, 0, 1, 0), (1, 1, 0, 0)\},$$
\[ \mathcal{H}_2 = \{ x, x^4, x^7, x^{10}, x^{13} \} \]
\[ = \{ (0, 1, 0, 0), (1, 0, 0, 1), (1, 1, 1, 0), (0, 1, 0, 1), (0, 1, 1, 0) \}, \]
\[ \mathcal{H}_3 = \{ x^2, x^5, x^8, x^{11}, x^{14} \} \]
\[ = \{ (0, 0, 1, 0), (1, 1, 0, 1), (0, 1, 1, 1), (1, 0, 1, 1), (0, 0, 1, 1) \}. \]

Moreover we define completely multiplicative functions
\[ f_1, f_2, f_3, f_4 : SF \to \{-1, 1\} \]
by giving their values for primes \( p \):
\[
\begin{align*}
  f_1(p) &= -1 & \text{for all } p \in \mathbb{P}, \\
  f_2(p) &= \begin{cases} -1 & \text{for } p = 2, \\ 1 & \text{for } p \neq 2, \end{cases} \\
  f_3(p) &= \begin{cases} -1 & \text{for } p \equiv 3 \pmod{4}, \\ 1 & \text{for } p \not\equiv 3 \pmod{4}, \end{cases} \\
  f_4(p) &= \begin{cases} -1 & \text{for } p \equiv \pm 3 \pmod{8}, \\ 1 & \text{for } p \not\equiv \pm 3 \pmod{8}. \end{cases}
\end{align*}
\]

After these preparations we put
\[ A_j = \left\{ n \in SF \left| \left( \frac{1 - f_k(n)}{2} \right)_{1 \leq k \leq 4} \in \mathcal{H}_j \right. \right\} \]
and \( \mathcal{A} = A_1 \cup A_2 \cup A_3 \). The set \( \mathcal{B} = SF \setminus \mathcal{A} \) is easily identifiable as
\[
\begin{align*}
  \mathcal{B} &= \{ n \in SF \mid \forall 1 \leq k \leq 4, f_k(n) = 1 \} \\
  &= \{ n \in SF \mid \Omega(n) \equiv 0 \pmod{2} \text{ and } n \equiv 1 \pmod{8} \}.
\end{align*}
\]

and so has the relative density 1/16. Therefore \( \mathcal{A} \) has the required density 15/16 and its partition defined above has the property of *omitting triples with square product*.

We finish our note with the corresponding positive result:

**Theorem 4.** If a set \( \mathcal{A} \subseteq \mathbb{N} \) consists entirely of squarefree numbers and has lower asymptotic density
\[ \delta^*(\mathcal{A}) > \frac{15}{16} \cdot \frac{6}{\pi^2} \]
then for any partition \( \{ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \} \) of \( \mathcal{A} \) there exist \( j \in \{1, 2, 3\} \) and \( a, b, c \in \mathcal{A}_j \) such that
\[ abc = \square. \]

**Proof.** Again, by Corollary 2 of [1] there exists \( m \in \mathbb{N} \) such that
\[ \tau(m, \mathcal{A}) > \frac{15}{16} \cdot \tau(m, SF). \]
Let $n$ be the greatest squarefree divisor of $m$. Then every squarefree divisor of $m$ is a divisor of $n$ and we obtain

\[ \tau(n, A) > \frac{15}{16} \tau(n). \]

Using similarly Lemma 1 we find five distinct divisors $\beta_1, \ldots, \beta_5$ of $n$ such that

\[ \gamma_J := \text{sf} \left( \prod_{j \in J} \beta_j \right) \in A \setminus \{1\} \quad \text{for } \emptyset \neq J \subseteq \{1, \ldots, 5\}. \]

We now consider the graph with 31 vertices $\gamma_J$ and colour the edge $\gamma_J \gamma_K$ using colour $j$ ($j \in \{1, 2, 3\}$) if and only if $\text{sf}(\gamma_J \gamma_K) \in A_j$. By the Ramsey theorem and the classical result $R(3, 3, 3) = 17 < 31$ we obtain at least one monochromatic triangle $\gamma_J \gamma_K \gamma_L$. This means that if we put

\[ a = \text{sf}(\gamma_J \gamma_K), \quad b = \text{sf}(\gamma_K \gamma_L), \quad c = \text{sf}(\gamma_J \gamma_L), \]

then $a, b, c \in A_j$ and obviously $abc = \square$.

Using the relatively new estimate [3]

\[ R(3, 3, 3, 3) < 63 \]

and reasoning along the same lines one can prove

**Theorem 5.** If a set $A \subseteq \mathbb{N}$ consists entirely of squarefree numbers and has lower asymptotic density

\[ \delta^*(A) > \frac{31}{32} \cdot \frac{6}{\pi^2} \]

then for any partition $\{A_1, A_2, A_3, A_4\}$ of $A$ there exist $j \in \{1, 2, 3, 4\}$ and $a, b, c \in A_j$ such that

\[ abc = \square. \]

The constant $31/32$ is probably not optimal but we do not know any example.

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