COLLOQUIUM MATHEMATICUM

VOL. 120 2010 NO. 1

INVERSE ZERO-SUM PROBLEMS IN FINITE ABELIAN p-GROUPS

BY

BENJAMIN GIRARD (Palaiseau)

Abstract. We study the minimal number of elements of maximal order occurring in a zero-sumfree sequence over a finite Abelian p-group. For this purpose, and in the general context of finite Abelian groups, we introduce a new number, for which lower and upper bounds are proved in the case of finite Abelian p-groups. Among other consequences, our method implies that, if we denote by $\exp(G)$ the exponent of the finite Abelian p-group G considered, every zero-sumfree sequence S with maximal possible length over G contains at least $\exp(G)-1$ elements of order $\exp(G)$, which improves a previous result of W. Gao and A. Geroldinger.

1. Introduction. Let $\mathcal{P}$ be the set of prime numbers and let G be a finite Abelian group, written additively. We denote by $\exp(G)$ the exponent of G. If G is cyclic of order n, it will be denoted by C_n. In the general case, we can decompose G (see for instance [18]) as a direct product of cyclic groups $C_{n_1} \oplus \cdots \oplus C_{n_r}$ where $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$.

By a *sequence* over G with *length* ℓ, we mean a finite sequence of ℓ elements from G, where repetitions are allowed and the order of elements is disregarded. We use multiplicative notation for sequences.

Let

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)}$$

be a sequence over G, where, for all $g \in G$, $\mathsf{v}_g(S) \in \mathbb{N}$ is called the *multiplicity* of g in S. We say that $s \in G$ is a *subsum* of S when

$$s = \sum_{i \in I} g_i \quad \text{for some } \emptyset \subsetneq I \subseteq \{1, \ldots, \ell\}.$$

If 0 is not a subsum of S, we say that S is a *zero-sumfree sequence*. If $\sum_{i=1}^{\ell} g_i = 0$, then S is said to be a *zero-sum sequence*. If moreover $\sum_{i \in I} g_i \neq 0$ for all proper subsets $\emptyset \subsetneq I \subsetneq \{1, \ldots, \ell\}$, then S is called a *minimal zero-sum sequence*.

2010 *Mathematics Subject Classification*: 11R27, 11B75, 11P70, 20D60, 20K01, 05E99, 13F05.

Key words and phrases: zero-sumfree sequence, finite Abelian group, Davenport constant, cross number.

DOI: 10.4064/cm120-1-2

In a finite Abelian group G, the order of an element g will be written $\operatorname{ord}_G(g)$. Moreover, we denote by $\langle g \rangle$ the subgroup generated by g, and for every divisor d of $\exp(G)$, we denote by G_d the subgroup of G consisting of all elements of order dividing d:

$$G_d = \{x \in G \mid dx = 0\}.$$

For every sequence S over G, we denote by S_d the subsequence of S consisting of all elements of order d contained in S.

Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ be a finite Abelian group. We set

$$\mathsf{D}^*(G) = \sum_{i=1}^{r}(n_i - 1) + 1 \quad \text{and} \quad \mathsf{d}^*(G) = \mathsf{D}^*(G) - 1.$$

Let $\mathsf{D}(G)$ denote the smallest integer $t \in \mathbb{N}^*$ such that every sequence S over G with length $|S| \geq t$ contains a non-empty zero-sum subsequence. The number $\mathsf{D}(G)$ is called the *Davenport constant* of the group G.

We denote by $\mathsf{d}(G)$ the greatest length of a zero-sumfree sequence over G. It can be readily seen that

$$\mathsf{d}(G) = \mathsf{D}(G) - 1.$$

If $G \simeq C_{\nu_1} \oplus \cdots \oplus C_{\nu_s}$ with $\nu_i > 1$ for all $i \in [\![1, s]\!]$ is the longest possible decomposition of G into a direct product of cyclic groups, then we set

$$\mathsf{k}^*(G) = \sum_{i=1}^{s} \frac{\nu_i - 1}{\nu_i}.$$

The *cross number* of a sequence $S = g_1 \cdot \ldots \cdot g_\ell$, denoted by $\mathsf{k}(S)$, is then defined by

$$\mathsf{k}(S) = \sum_{i=1}^{\ell} \frac{1}{\operatorname{ord}_G(g_i)}.$$

The notion of cross number was introduced by U. Krause in [13] (see also [14]). Finally, we define the so-called *little cross number* $\mathsf{k}(G)$ of G:

$$\mathsf{k}(G) = \max\{\mathsf{k}(S) \mid S \text{ a zero-sumfree sequence over } G\}.$$

Given a finite Abelian group G, two elementary constructions (see Proposition 5.1.8 in [7]) give the following lower bounds:

$$\mathsf{d}^*(G) \leq \mathsf{d}(G) \quad \text{and} \quad \mathsf{k}^*(G) \leq \mathsf{k}(G).$$

The invariants $\mathsf{d}(G)$ and $\mathsf{k}(G)$ play a key role in the theory of non-unique factorization (see for instance Chapter 9 in [15], the book [7] which presents various aspects of the theory, and also the survey [8]). They have been extensively studied during the last decades and even if numerous results were proved (see Chapter 5 of the book [7], [3] and [5] for surveys with many references on the subject, and [12] for recent results on the cross number of finite Abelian groups), their exact values are known for very special types of groups only. We will need these values for finite Abelian p-groups, so we gather them in the following theorem (see [17] and [4]).

THEOREM 1.1. *Let $p \in \mathcal{P}$ and $G \simeq C_{p^{a_1}} \oplus \cdots \oplus C_{p^{a_r}}$ with $1 \leq a_1 \leq \cdots \leq a_r \in \mathbb{N}$. Then*

(i) $$\mathsf{d}(G) = \sum_{i=1}^{r} (p^{a_i} - 1) = \mathsf{d}^*(G),$$

(ii) $$\mathsf{k}(G) = \sum_{i=1}^{r} \frac{p^{a_i} - 1}{p^{a_i}} = \mathsf{k}^*(G).$$

In [17], J. Olson actually proved a more general result than Theorem 1.1(i), which will be useful in this article. To state this theorem, we need to introduce the following notation. For every element g in a finite Abelian p-group G, the *height* of g, denoted by $\alpha(g)$, is defined by

$$\alpha(g) = \max\{p^n \mid \exists h \in G \text{ with } g = p^n h\}.$$

We can now state Olson's result.

THEOREM 1.2. *Let G be a finite Abelian p-group and $S = g_1 \cdot \ldots \cdot g_\ell$ be a sequence over G. Then S is not a zero-sumfree sequence whenever*

$$\sum_{i=1}^{\ell} \alpha(g_i) > \mathsf{d}(G).$$

2. Inverse problems in zero-sum theory. What can be said about the structure of a "large" zero-sumfree sequence over a finite Abelian group? This type of problems has a long tradition in additive combinatorics (see for instance [16] and [20]), and an answer would provide a new insight into open problems in non-unique factorization theory (see Chapter 5 in [5]).

Yet, the already known results show that the exact structure of such sequences highly relies on the structure of the group itself, so that it seems difficult to obtain a complete characterization in general (see for instance [1]). Therefore, instead of an exhaustive description, research focused on the properties which have to be satisfied by such sequences, whatever the group is. The following inverse problems give an illustration of this idea.

PROBLEM 1. *What is the maximal cross number of a long zero-sumfree sequence?*

In [11], the author addressed the following general conjecture, which bears upon the distribution of orders occurring in a long zero-sumfree sequence over a finite Abelian group.

Conjecture 1. *Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ be a finite Abelian group. Let also S be a zero-sumfree sequence over G. Then*

$$|S| \geq \mathsf{d}^*(G) = \sum_{i=1}^{r}(n_i - 1) \quad \textit{implies} \quad \mathsf{k}(S) \leq \sum_{i=1}^{r} \frac{n_i - 1}{n_i}.$$

Conjecture 1 would imply two classical and long-standing conjectures related to the Davenport constant of finite Abelian groups of the form C_n^r (see Proposition 2.1 in [11]), and would also provide the general upper bound $\mathsf{D}(G) \leq rn$ for every finite Abelian group G of rank r and exponent n (see Proposition 2.2 in [11]).

The following theorem gathers what is currently known concerning this conjecture. Statements (i)–(iii) were proved by the author (see Proposition 2.3 and Theorem 2.4 in [11]), and statement (iv) was obtained by W. Schmid (see Corollary 4.5 in [19]).

Theorem 2.1. *Conjecture* 1 *holds whenever:*

(i) *G is a finite Abelian p-group.*
(ii) *G is a finite cyclic group.*
(iii) *G is a finite Abelian group of rank two.*
(iv) *$G \simeq C_2 \oplus C_2 \oplus C_{2n}$, where $n \in \mathbb{N}^*$.*

Problem 2. *What is the maximal length of a zero-sumfree sequence with large cross number?*

One can notice that Problem 2 is a somewhat dual version of Problem 1. Concerning this question, the author proposed the following conjecture (see Section 7 in [11]). The reader is also referred to [6] for a recent investigation on the order of elements in sequences with large cross number (see also [10]).

Conjecture 2. *Let G be a finite Abelian group, and $G \simeq C_{\nu_1} \oplus \cdots \oplus C_{\nu_s}$ with $\nu_i > 1$ for all $i \in [\![1, s]\!]$ be its longest possible decomposition into a direct product of cyclic groups. Let also S be a zero-sumfree sequence over G. Then*

$$\mathsf{k}(S) \geq \mathsf{k}^*(G) = \sum_{i=1}^{s} \frac{\nu_i - 1}{\nu_i} \quad \textit{implies} \quad |S| \leq \sum_{i=1}^{s}(\nu_i - 1).$$

It can readily be seen, using Theorem 1.1(i), that Conjecture 2 holds for finite Abelian p-groups, yet this conjecture remains widely open, even in the case of finite cyclic groups (see Theorem 7.2 in [11]).

Problem 3. *What is the order of elements in a zero-sumfree sequence?*

In this article, we study Problem 3 in two directions. First, we investigate the minimal number of elements of order $\exp(G)$ occurring in a zero-sumfree sequence over a finite Abelian p-group G. This question was raised and investigated, in the general context of finite Abelian groups, by W. Gao and A. Geroldinger (see Section 6 in [1], and Theorem 2.5 in [11] for recent progress).

In order to study this kind of inverse zero-sum problems, we introduce the following number. Given a finite Abelian group G and an integer $\delta \in [\![0, \mathsf{d}(G)-1]\!]$, we denote by $\Gamma_\delta(G)$ the minimal number of elements of order $\exp(G)$ contained in a zero-sumfree sequence S over G with $|S| \geq \mathsf{d}(G) - \delta$.

In Section 3, we present a general method which was introduced in [12] for the study of the cross number of finite Abelian groups. Then, using this method, we prove in Section 4 the following theorem, which gives a lower bound on $\Gamma_\delta(G)$ for finite Abelian p-groups.

Theorem 2.2. *Let $p \in \mathcal{P}$ and $G \simeq C_{p^{a_1}} \oplus \cdots \oplus C_{p^{a_r}}$ with $1 \leq a_1 \leq \cdots \leq a_r \in \mathbb{N}$. Let also $\delta \in [\![0, \mathsf{d}(G)-1]\!]$ and $j_0 = \min\{i \in [\![1,r]\!] \mid a_i = a_r\}$. Then*

$$\Gamma_\delta(G) \geq (p^{a_r}-1) + (r-j_0)(p-1)p^{a_r-1} - \delta - \left\lfloor \frac{\delta}{(r-j_0+1)(p-1)} \right\rfloor.$$

This lower bound improves significantly a previous result of W. Gao and A. Geroldinger (see Corollary 5.1.13 in [7]), stating that every zero-sumfree sequence with maximal possible length over a finite Abelian p-group contains at least *one* element of maximal order. Indeed, by specifying $\delta = 0$ in Theorem 2.2, one obtains the following corollary.

Corollary 2.3. *Let G be a finite Abelian p-group. Then every zero-sumfree sequence S over G with $|S| = \mathsf{d}(G)$ contains at least* $\exp(G) - 1$ *elements of order* $\exp(G)$.

It may be underlined that, as it stands, Corollary 2.3 cannot be generalized to a wider framework of finite Abelian groups. Indeed, as shown by W. Schmid (see Corollary 4.6 in [19]), there exist a finite Abelian group G and a zero-sumfree sequence S over G with $|S| = \mathsf{d}(G)$ such that S contains strictly less than $\exp(G)-1$ elements of order $\exp(G)$. In other words, there exist finite Abelian groups such that $\Gamma_0(G) < \exp(G) - 1$.

In Section 4 we also obtain, using some explicit constructions of zero-sumfree sequences, an upper bound of $\Gamma_\delta(G)$ for finite Abelian p-groups (see Proposition 4.1), which, combined with the lower bound of Theorem 2.2, implies the following result.

Theorem 2.4. *Let $p \in \mathcal{P}$ and let $G \simeq C_{p^{a_1}} \oplus \cdots \oplus C_{p^{a_r}}$ with $1 \leq a_1 \leq \cdots \leq a_r \in \mathbb{N}$ be such that $j_0 = \min\{i \in [\![1,r]\!] \mid a_i = a_r\} = r$. Then*

$$\Gamma_\delta(G) = \max\left(0, (p^{a_r}-1) - \delta - \left\lfloor \frac{\delta}{p-1} \right\rfloor\right) \quad \text{for all } \delta \in [\![0, \mathsf{d}(G)-1]\!].$$

In Section 5, we study another conjecture related to Problem 3, bearing upon the greatest common divisor of the orders of the elements occurring in a long zero-sumfree sequence over a finite Abelian group, and which is the following.

CONJECTURE 3. *Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ be a finite Abelian group. Then, given a zero-sumfree sequence S over G such that $|S| = \mathsf{d}(G)$,*

$$n_1 \mid \mathrm{ord}_G(g) \quad \textit{for all } g \in S.$$

At the beginning of Section 5, we state a result which shows that in Conjecture 3, the condition $|S| = \mathsf{d}(G)$ cannot be replaced by the weaker condition $|S| \geq \mathsf{d}^*(G)$ (see Lemma 5.1).

Conjecture 3 is known to be true in the trivial case of finite cyclic groups. This conjecture also holds for finite Abelian groups of rank two (see Proposition 6.3.1 in [1]), and we prove in Section 5 that it holds for finite Abelian p-groups as well, which is statement (i) in the following theorem. Statement (iv) can be easily deduced from Theorem 3.13 in [19].

THEOREM 2.5. *Conjecture* 3 *holds whenever:*

(i) *G is a finite Abelian p-group.*
(ii) *G is a finite cyclic group.*
(iii) *G is a finite Abelian group of rank two.*
(iv) *$G \simeq C_2 \oplus C_2 \oplus C_{2n}$, where $n \in \mathbb{N}^*$.*

Finally, in Section 6, we propose and discuss a general conjecture concerning the behaviour of $\mathit{\Gamma}_\delta(G)$ when G is a finite Abelian p-group.

3. Outline of the method. Let G be a finite Abelian group, and let S be a sequence over G. The general method that we will use in this paper (see also [11] and [12] for applications in two other contexts) consists in considering, for every $d', d \in \mathbb{N}$ such that $1 \leq d' \mid d \mid \exp(G)$, the following exact sequence:

$$0 \to G_{d/d'} \hookrightarrow G_d \xrightarrow{\pi_{(d',d)}} G_d/G_{d/d'} \to 0.$$

Now, let U be the subsequence of S consisting of all elements whose order divides d. If, for some $1 \leq d' \mid d \mid \exp(G)$, it is possible to find sufficiently many disjoint non-empty zero-sum subsequences in $\pi_{(d',d)}(U)$, that is, sufficiently many disjoint subsequences in U, the sum of each being an element of order dividing d/d', then S cannot be a zero-sumfree sequence over G.

To make this idea more precise, we introduced in [12] the following number, which can be seen as an extension of the classical Davenport constant.

Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ and $d', d \in \mathbb{N}$ be such that $1 \leq d' \mid d \mid \exp(G)$. We denote by $\mathsf{D}_{(d',d)}(G)$ the smallest $t \in \mathbb{N}^*$

such that every sequence S over G_d with $|S| \geq t$ contains a non-empty subsequence with sum in $G_{d/d'}$.

Using this definition, we can prove the following simple lemma, which is an illustration of our idea. This result will be useful in Section 4 and states that given a finite Abelian group, there exist strong constraints on the way the orders of elements are distributed within a zero-sumfree sequence.

LEMMA 3.1. *Let G be a finite Abelian group and let $d', d \in \mathbb{N}$ be such that $1 \leq d' \mid d \mid \exp(G)$. Given a sequence S over G, write T for the subsequence of S consisting of all elements whose order divides d/d', and write U for the subsequence of S consisting of all elements whose order divides d (in particular, $T \mid U$). Then S is not a zero-sumfree sequence whenever*

$$|T| + \left\lfloor \frac{|U| - |T|}{\mathsf{D}_{(d',d)}(G)} \right\rfloor \geq \mathsf{D}_{(d/d',d/d')}(G).$$

Proof. Set $\Delta = \mathsf{D}_{(d/d',d/d')}(G)$. The above inequality implies that there are Δ disjoint subsequences $S_1, \ldots, S_\Delta$ of S, the sum of each being an element of order dividing d/d'. Now, by the very definition of $\mathsf{D}_{(d/d',d/d')}(G)$, S has to contain a non-empty zero-sum subsequence. ∎

Now, in order to obtain effective inequalities from the symbolic constraints of Lemma 3.1, one can use a result proved in [12], which states that for any finite Abelian group G and every $1 \leq d' \mid d \mid \exp(G)$, the invariant $\mathsf{D}_{(d',d)}(G)$ is linked with the classical Davenport constant of a particular subgroup of G, which can be characterized explicitly. In order to define this subgroup properly, we introduce the following notation.

For all $i \in [\![1, r]\!]$, we set

$$A_i = \gcd(d', n_i), \qquad B_i = \frac{\operatorname{lcm}(d, n_i)}{\operatorname{lcm}(d', n_i)}, \qquad \upsilon_i(d', d) = \frac{A_i}{\gcd(A_i, B_i)}.$$

For instance, whenever d divides n_i, we have $\upsilon_i(d', d) = \gcd(d', n_i) = d'$, and in particular $\upsilon_r(d', d) = d'$. We can now state our result on $\mathsf{D}_{(d',d)}(G)$ (see Proposition 3.1 in [12]).

PROPOSITION 3.2. *Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ be a finite Abelian group, and let $d', d \in \mathbb{N}$ be such that $1 \leq d' \mid d \mid \exp(G)$. Then*

$$\mathsf{D}_{(d',d)}(G) = \mathsf{D}(C_{\upsilon_1(d',d)} \oplus \cdots \oplus C_{\upsilon_r(d',d)}).$$

4. On the quantity $\mathit{\Gamma}_\delta(G)$ for finite Abelian p-groups. Let G be a finite Abelian p-group. In this section, we show how the method presented in Section 3 can be used to study the number of elements of order $\exp(G)$ occurring in a zero-sumfree sequence over G. First, we prove Theorem 2.2, which gives a lower bound of $\mathit{\Gamma}_\delta(G)$ for all $\delta \in [\![0, \mathsf{d}(G) - 1]\!]$.

Proof of Theorem 2.2. Let S be a zero-sumfree sequence over $G \simeq C_{p^{a_1}} \oplus \cdots \oplus C_{p^{a_r}}$ with $|S| \geq \mathsf{d}(G) - \delta$. We set $d' = p$ and $d = p^{a_r}$, which leads to $d/d' = p^{a_r - 1}$. Let also T and U be the two subsequences of S defined in Lemma 3.1. In particular, $T \mid U = S$.

To start with, we determine the exact value of $\mathsf{D}_{(d',d)}(G)$. One has, for every $i \in [\![1, r]\!]$,

$$\upsilon_i(d', d) = \frac{p}{\gcd(p, p^{a_r}/p^{a_i})} = \begin{cases} 1 & \text{if } i < j_0, \\ p & \text{if } i \geq j_0. \end{cases}$$

Therefore, using Proposition 3.2 and Theorem 1.1(i), we obtain

$$\begin{aligned} \mathsf{D}_{(d',d)}(G) &= \mathsf{D}(C_{\upsilon_1(d',d)} \oplus \cdots \oplus C_{\upsilon_r(d',d)}) = \mathsf{D}(C_p^{r-j_0+1}) \\ &= (r - j_0 + 1)(p - 1) + 1. \end{aligned}$$

Now, for all $i \in [\![1, r]\!]$, set

$$\beta_i = \begin{cases} a_i & \text{if } i < j_0, \\ a_r - 1 & \text{if } i \geq j_0. \end{cases}$$

If we had the inequality

$$|T| > \sum_{i=1}^{r-1} (p^{\beta_i} - 1) + \frac{\delta}{(r - j_0 + 1)(p - 1)},$$

then it would imply that

$$\begin{aligned} |T| + \frac{|U| - |T|}{\mathsf{D}_{(d',d)}(G)} &\geq |T| + \frac{\sum_{i=1}^{r} (p^{a_i} - 1) - \delta - |T|}{(r - j_0 + 1)(p - 1) + 1} \\ &> \sum_{i=1}^{r-1} (p^{\beta_i} - 1) + \frac{\sum_{i=1}^{r} (p^{a_i} - 1) - \sum_{i=1}^{r-1} (p^{\beta_i} - 1)}{(r - j_0 + 1)(p - 1) + 1} \\ &= \sum_{i=1}^{r-1} (p^{\beta_i} - 1) + \frac{(p^{a_r} - 1) + (r - j_0)(p^{a_r} - p^{a_r - 1})}{(r - j_0 + 1)(p - 1) + 1} \\ &= \sum_{i=1}^{r-1} (p^{\beta_i} - 1) + \frac{((r - j_0 + 1)(p - 1) + 1)p^{a_r - 1} - 1}{(r - j_0 + 1)(p - 1) + 1} \\ &= \sum_{i=1}^{r} (p^{\beta_i} - 1) + 1 - \frac{1}{(r - j_0 + 1)(p - 1) + 1} \\ &= \mathsf{D}_{(d/d', d/d')}(G) - \frac{1}{\mathsf{D}_{(d',d)}(G)}, \end{aligned}$$

and, according to Lemma 3.1, S would contain a non-empty zero-sum subsequence, which is a contradiction. Thus, one obtains

$$|T| \leq \sum_{i=1}^{r-1}(p^{\beta_i}-1)+\left\lfloor\frac{\delta}{(r-j_0+1)(p-1)}\right\rfloor,$$

which gives the following lower bound for the number of elements of order $p^{a_r}=\exp(G)$ occurring in S:

$$\begin{aligned}
|S_{p^{a_r}}| &= |S|-|T| \\
&\geq \sum_{i=1}^{r}(p^{a_i}-1)-\delta-\sum_{i=1}^{r-1}(p^{\beta_i}-1)-\left\lfloor\frac{\delta}{(r-j_0+1)(p-1)}\right\rfloor \\
&= (r-j_0+1)(p^{a_r}-1)-(r-j_0)(p^{a_r-1}-1)-\delta-\left\lfloor\frac{\delta}{(r-j_0+1)(p-1)}\right\rfloor \\
&= (p^{a_r}-1)+(r-j_0)(p-1)p^{a_r-1}-\delta-\left\lfloor\frac{\delta}{(r-j_0+1)(p-1)}\right\rfloor,
\end{aligned}$$

and the proof is complete. ■

As announced in Section 2, we can also obtain, using some explicit constructions of zero-sumfree sequences, the following upper bound of $\Gamma_\delta(G)$ for finite Abelian p-groups.

PROPOSITION 4.1. *Let $p \in \mathcal{P}$ and $G \simeq C_{p^{a_1}} \oplus \cdots \oplus C_{p^{a_r}}$ with $1 \leq a_1 \leq \cdots \leq a_r \in \mathbb{N}$. Let also $\delta \in [\![0, \mathsf{d}(G)-1]\!]$ and $j_0 = \min\{i \in [\![1,r]\!] \mid a_i = a_r\}$. Then*

$$\Gamma_\delta(G) \leq \max(0,(r-j_0+1)(p^{a_r}-1)-\delta-f(\delta)),$$

where

$$f(\delta)=\min\left(\left\lfloor\frac{\delta}{p-1}\right\rfloor,(r-j_0+1)(p^{a_r-1}-1)\right).$$

Proof. Let $(e_1,\dots,e_r)$ be a basis of G with $\operatorname{ord}_G(e_i)=p^{a_i}$ for every $i \in [\![1,r]\!]$. One can distinguish the following three cases.

CASE 1. If $0 \leq \delta < (r-j_0+1)(p-1)(p^{a_r-1}-1)$, then write

$$\delta = \delta_1(p-1)(p^{a_r-1}-1)+\delta_2$$

with $\delta_1 \in [\![0, r-j_0]\!]$ and $\delta_2 \in [\![0,(p-1)(p^{a_r-1}-1)-1]\!]$. Thus, the sequence

$$S=\Big(\prod_{i=1}^{r-\delta_1-1} e_i^{p^{a_i}-1}\Big)\Big(\prod_{i=r-\delta_1}^{r-1}(e_i)^{p-1}(pe_i)^{p^{a_i-1}-1}\Big) \cdot (e_r)^{p^{a_r}-1-\delta_2-\lfloor\delta_2/(p-1)\rfloor}(pe_r)^{\lfloor\delta_2/(p-1)\rfloor}$$

is a zero-sumfree sequence over G. On the one hand, since $\delta_1 \leq r-j_0$, one obtains

$$
\begin{aligned}
|S| = & \sum_{i=1}^{r-\delta_1-1} (p^{a_i}-1) + \sum_{i=r-\delta_1}^{r-1} [(p-1)+(p^{a_r-1}-1)] \\
& + (p^{a_r}-1) - \delta_2 - \left\lfloor \frac{\delta_2}{p-1} \right\rfloor + \left\lfloor \frac{\delta_2}{p-1} \right\rfloor \\
= & \sum_{i=1}^{r}(p^{a_i}-1) + \sum_{i=r-\delta_1}^{r-1} [(p-1)+(p^{a_r-1}-1)-(p^{a_r}-1)] - \delta_2 \\
= & \sum_{i=1}^{r}(p^{a_i}-1) - \delta_1(p-1)(p^{a_r-1}-1) - \delta_2 = \mathsf{d}(G) - \delta.
\end{aligned}
$$

On the other hand, S contains the following number of elements of order $p^{a_r} = \exp(G)$:

$$
\begin{aligned}
|S_{p^{a_r}}| = & \sum_{i=j_0}^{r-\delta_1-1} (p^{a_r}-1) + \sum_{i=r-\delta_1}^{r-1} (p-1) + (p^{a_r}-1) - \delta_2 - \left\lfloor \frac{\delta_2}{p-1} \right\rfloor \\
= & (r-\delta_1-j_0+1)(p^{a_r}-1) + \delta_1(p-1) - \delta_2 - \left\lfloor \frac{\delta_2}{p-1} \right\rfloor \\
= & (r-j_0+1)(p^{a_r}-1) - \delta - \delta_1(p^{a_r-1}-1) - \left\lfloor \frac{\delta_2}{p-1} \right\rfloor \\
= & (r-j_0+1)(p^{a_r}-1) - \delta - \left\lfloor \frac{\delta}{p-1} \right\rfloor,
\end{aligned}
$$

and we are done.

Case 2. If $(r-j_0+1)(p-1)(p^{a_r-1}-1) \le \delta < (r-j_0+1)(p-1)p^{a_r-1}$, then write

$$\delta' = \delta - (r-j_0+1)(p-1)(p^{a_r-1}-1),$$

and

$$\delta' = \delta_1'(p-1) + \delta_2', \qquad \text{with } \delta_1' \in [\![0, r-j_0]\!] \text{ and } \delta_2' \in [\![0, p-2]\!].$$

Thus, the sequence

$$
\begin{aligned}
S = & \Big(\prod_{i=1}^{j_0-1} e_i^{p^{a_i}-1} \Big) \Big(\prod_{i=j_0}^{r-\delta_1'-1} (e_i)^{p-1} (pe_i)^{p^{a_r-1}-1} \Big) \Big(\prod_{i=r-\delta_1'}^{r-1} (pe_i)^{p^{a_r-1}-1} \Big) \\
& \cdot (e_r)^{p-1-\delta_2'} (pe_r)^{p^{a_r-1}-1}
\end{aligned}
$$

is a zero-sumfree sequence over G. On the one hand, since $\delta_1' \le r-j_0$, one obtains

$$\begin{aligned}|S| &= \sum_{i=1}^{j_0-1}(p^{a_i}-1)+(r-\delta_1'-j_0)(p-1)\\ &\quad +(r-j_0+1)(p^{a_r-1}-1)+(p-1)-\delta_2'\\ &= \sum_{i=1}^{j_0-1}(p^{a_i}-1)+(r-j_0+1)(p-1)+(r-j_0+1)(p^{a_r-1}-1)-\delta'\\ &= \sum_{i=1}^{j_0-1}(p^{a_i}-1)\\ &\quad +(r-j_0+1)[(p-1)+(p^{a_r-1}-1)+(p-1)(p^{a_r-1}-1)]-\delta\\ &= \sum_{i=1}^{j_0-1}(p^{a_i}-1)+(r-j_0+1)(p^{a_r}-1)-\delta=\mathsf{d}(G)-\delta.\end{aligned}$$

On the other hand, S contains the following number of elements of order $p^{a_r}=\exp(G)$:

$$\begin{aligned}|S_{p^{a_r}}| &= (r-\delta_1'-j_0)(p-1)+(p-1)-\delta_2'\\ &= (r-j_0+1)(p-1)-\delta'\\ &= (r-j_0+1)(p^{a_r}-1)-\delta-(r-j_0+1)(p^{a_r-1}-1),\end{aligned}$$

and we are done.

Case 3. If $(r-j_0+1)(p-1)p^{a_r-1}\le\delta\le\mathsf{d}(G)-1$, then

$$\begin{aligned}&(r-j_0+1)(p^{a_r}-1)-\delta-f(\delta)\\ &\qquad\le(r-j_0+1)[(p^{a_r}-1)-(p-1)p^{a_r-1}-(p^{a_r-1}-1)]\le 0,\end{aligned}$$

as well as

$$\begin{aligned}\mathsf{d}(G)-\delta &\le \sum_{i=1}^{r}(p^{a_i}-1)-(r-j_0+1)(p-1)p^{a_r-1}\\ &= \sum_{i=1}^{j_0-1}(p^{a_i}-1)+(r-j_0+1)[(p^{a_r}-1)-(p-1)p^{a_r-1}]\\ &= \sum_{i=1}^{j_0-1}(p^{a_i}-1)+(r-j_0+1)(p^{a_r-1}-1).\end{aligned}$$

Now, consider the zero-sumfree sequence

$$S=\Big(\prod_{i=1}^{j_0-1}e_i^{p^{a_i}-1}\Big)\Big(\prod_{i=j_0}^{r}(pe_i)^{p^{a_r-1}-1}\Big),$$

which does not contain any element of order $p^{a_r} = \exp(G)$. Thus, choosing any subsequence of S with length $\mathsf{d}(G) - \delta$, we conclude that $\Gamma_\delta(G) = 0$, which is the desired result. ∎

It is now easy, using Theorem 2.2 and Proposition 4.1, to derive Theorem 2.4, which gives, when $j_0 = r$, the exact value of $\Gamma_\delta(G)$ for all $\delta \in [\![0, \mathsf{d}(G) - 1]\!]$.

Proof of Theorem 2.4. Specifying $j_0 = r$ in Theorem 2.2, one readily obtains

$$\Gamma_\delta(G) \geq (p^{a_r} - 1) - \delta - \left\lfloor \frac{\delta}{p-1} \right\rfloor.$$

Now, one can distinguish the following three cases.

Case 1. If $0 \leq \delta < (p-1)(p^{a_r - 1} - 1)$, then the upper bound of Proposition 4.1 gives

$$\Gamma_\delta(G) = (p^{a_r} - 1) - \delta - \left\lfloor \frac{\delta}{p-1} \right\rfloor.$$

Case 2. If $(p-1)(p^{a_r-1} - 1) \leq \delta < (p-1)p^{a_r-1}$, then Proposition 4.1 implies that

$$\Gamma_\delta(G) \leq (p^{a_r} - 1) - \delta - (p^{a_r-1} - 1),$$

and since

$$p^{a_r-1} - 1 = \left\lfloor \frac{\delta}{p-1} \right\rfloor,$$

one obtains the desired equality

$$\Gamma_\delta(G) = (p^{a_r} - 1) - \delta - \left\lfloor \frac{\delta}{p-1} \right\rfloor.$$

Case 3. If $(p-1)p^{a_r-1} \leq \delta \leq \mathsf{d}(G) - 1$, then Proposition 4.1 implies that $\Gamma_\delta(G) = 0$, and the proof is complete. ∎

5. On Conjecture 3. First, we show that the condition $|S| = \mathsf{d}(G)$ in Conjecture 3 is essential, and cannot be replaced by the weaker condition $|S| \geq \mathsf{d}^*(G)$.

Lemma 5.1. *There exist $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$, a zero-sumfree sequence S over G with $|S| \geq \mathsf{d}^*(G)$ and an element g in S such that* $\mathrm{ord}_G(g) < n_1$.

Proof. By Theorem 3 in [9], there exists a finite Abelian group $H \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$, where $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ and n_1 is not a prime, such that $\mathsf{d}(H) \geq \mathsf{d}^*(H) + 1$. Now, let p be a prime divisor of n_1. We set

$$G = C_{n_1} \oplus H^p,$$

and pick an element e_0 in G with $\mathrm{ord}_G(e_0) = n_1$ and $G = \langle e_0 \rangle \oplus H^p$. By Proposition 5.1.11.1 in [7], we have $\mathsf{d}(H^p) \geq p\mathsf{d}(H) \geq p(\mathsf{d}^*(H) + 1)$. We now pick a zero-sumfree sequence U over H^p with $|U| = \mathsf{d}(H^p)$. Clearly, the sequence

$$S = (pe_0)e_0^{n_1-p-1}U$$

is a zero-sumfree sequence such that $\mathrm{ord}_G(pe_0) = n_1/p < n_1$ and

$$|S| = n_1 - p + |U| \geq n_1 - p + p\mathsf{d}^*(H) + p = \mathsf{d}^*(G) + 1. \ \blacksquare$$

We now prove the following lemma, which can be seen as a slightly more general version of Proposition 4.3 in [2].

Lemma 5.2. *Let G be a finite Abelian p-group and let S be a zero-sumfree sequence over G with $|S| \geq \mathsf{d}(G) - p + 2$. Then every element of S has height* 1.

Proof. Let $S = g_1 \cdot \ldots \cdot g_\ell$. Assume that there exists an element in S, say g_1, such that $\alpha(g_1) > 1$. Then $\alpha(g_1) \geq p$, and setting $T = g_1^{-1}S$, we deduce that

$$\begin{aligned}\sum_{i=1}^{\ell} \alpha(g_i) &\geq p + |T| \geq p + (\mathsf{d}(G) - p + 1) \\ &= \mathsf{d}(G) + 1 > \mathsf{d}(G).\end{aligned}$$

Thus, by Theorem 1.2, S cannot be a zero-sumfree sequence, which is a contradiction. $\blacksquare$

Now, using Lemma 5.2, we can prove Theorem 2.5(i) as a simple corollary of the following stronger theorem.

Theorem 5.3. *Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ be a finite Abelian p-group. Then, given a zero-sumfree sequence S over G such that $|S| \geq \mathsf{d}(G) - p + 2$,*

$$n_1 \mid \mathrm{ord}_G(g) \quad \textit{for all } g \in S.$$

Proof. By Lemma 5.2, every element of S has height 1. Now, let $(e_1, \ldots, e_r)$ be a basis of G with $\mathrm{ord}_G(e_i) = n_i$ for all $i \in [\![1, r]\!]$, and let $g = a_1e_1 + \cdots + a_re_r$ be an element of S with $a_i \in [\![0, n_i - 1]\!]$ for all $i \in [\![1, r]\!]$. The equality $\alpha(g) = 1$ implies that there exists $i_0 \in [\![1, r]\!]$ such that p does not divide a_{i_0}. Therefore, $\mathrm{ord}_{C_{n_{i_0}}}(a_{i_0}) = n_{i_0}$, and we obtain

$$\begin{aligned}\mathrm{ord}_G(g) &= \max_{i \in [\![1,r]\!]} \mathrm{ord}_{C_{n_i}}(a_i) \geq \mathrm{ord}_{C_{n_{i_0}}}(a_{i_0}) \\ &= n_{i_0} \geq n_1,\end{aligned}$$

which completes the proof. $\blacksquare$

6. A concluding remark. In this section, we propose a conjecture supported by Theorem 2.2, which states that the upper bound of Proposition 4.1 is actually the right value of $\Gamma_\delta(G)$ for finite Abelian p-groups.

CONJECTURE 4. *Let $p \in \mathcal{P}$ and $G \simeq C_{p^{a_1}} \oplus \cdots \oplus C_{p^{a_r}}$ with $1 \leq a_1 \leq \cdots \leq a_r \in \mathbb{N}$. Let also $\delta \in [\![0, \mathsf{d}(G)-1]\!]$ and $j_0 = \min\{i \in [\![1, r]\!] \mid a_i = a_r\}$. Then*

$$\Gamma_\delta(G) = \max(0, (r - j_0 + 1)(p^{a_r} - 1) - \delta - f(\delta)),$$

where

$$f(\delta) = \min\left(\left\lfloor \frac{\delta}{p-1} \right\rfloor, (r - j_0 + 1)(p^{a_r - 1} - 1)\right).$$

By our Theorem 2.4, this conjecture holds true in the case where $j_0 = r$. One can also notice that, when $j_0 = 1$, Theorem 5.3 implies that Conjecture 4 holds for every $\delta \in [\![0, p-2]\!]$.

Acknowledgments. I am grateful to my Ph.D. advisor Alain Plagne for his help during the preparation of this paper. I would also like to thank Alfred Geroldinger, Wolfgang Schmid and the referee for their precious comments, as well as the Centre de Recerca Matemàtica in Barcelona, where this work was begun.

REFERENCES

[1] W. Gao and A. Geroldinger, *On long minimal zero sequences in finite abelian groups*, Period. Math. Hungar. 38 (1999), 179–211.

[2] —, —, *On the order of elements in long minimal zero-sum sequences*, ibid. 44 (2002), 63–73.

[3] —, —, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. 24 (2006), 337–369.

[4] A. Geroldinger, *The cross number of finite abelian groups*, J. Number Theory 48 (1994), 219–223.

[5] —, *Additive group theory and non-unique factorizations*, in: A. Geroldinger and I. Ruzsa, Combinatorial Number Theory and Additive Group Theory, Adv. Courses in Math., CRM Barcelona, Birkhäuser, 2009, 1–86.

[6] A. Geroldinger and D. Grynkiewicz, *On the structure of minimal zero-sum sequences with maximal cross number*, J. Combin. Number Theory 1 (2009), no. 2, 9–26.

[7] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure Appl. Math. 278, Chapman & Hall/CRC, 2006.

[8] —, —, *Non-unique factorizations: a survey*, in: Multiplicative Ideal Theory in Commutative Algebra, Springer, New York, 2006, 207–226.

[9] A. Geroldinger and R. Schneider, *On Davenport's constant*, J. Combin. Theory Ser. A 61 (1992), 147–152.

[10] —, —, *On minimal zero sequences with large cross number*, Ars Combin. 46 (1997), 297–303.

[11] B. Girard, *Inverse zero-sum problems and algebraic invariants*, Acta Arith. 135 (2008), 231–246.
[12] —, *A new upper bound for the cross number of finite Abelian groups*, Israel J. Math. 172 (2009), 253–278.
[13] U. Krause, *A characterization of algebraic number fields with cyclic class group of prime power order*, Math. Z. 186 (1984), 143–148.
[14] U. Krause and C. Zahlten, *Arithmetic in Krull monoids and the cross number of divisor class groups*, Mitt. Math. Ges. Hamburg 12 (1991), 681–696.
[15] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, 3rd ed., Springer, 2004.
[16] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.
[17] J. E. Olson, *A combinatorial problem on finite abelian groups I*, J. Number Theory 1 (1969), 8–10.
[18] P. Samuel, *Théorie algébrique des nombres*, Hermann, 2003.
[19] W. A. Schmid, *The inverse problem associated to Davenport's constant for $C_2 \oplus C_2 \oplus C_{2n}$ and applications to the arithmetical characterization of class groups*, submitted.
[20] T. Tao and V. Vu, *Additive Combinatorics*, Cambridge Stud. Adv. Math. 105, Cambridge Univ. Press, 2006.

Benjamin Girard
Centre de Mathématiques Laurent Schwartz
UMR 7640 du CNRS
École polytechnique
91128 Palaiseau Cedex, France
E-mail: benjamin.girard@math.polytechnique.fr

Received 5 February 2009;
revised 20 July 2009 (5156)