A NOTE ON THE SONG–ZHANG THEOREM FOR HAMILTONIAN GRAPHS

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Abstract. An independent set $S$ of a graph $G$ is said to be essential if $S$ has a pair of vertices that are distance two apart in $G$. In 1994, Song and Zhang proved that if for each independent set $S$ of cardinality $k + 1$, one of the following condition holds:

(i) there exist $u \neq v \in S$ such that $d(u) + d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha(G)$;

(ii) for any distinct $u$ and $v$ in $S$, $|N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\}$,

then $G$ is Hamiltonian. We prove that if for each essential independent set $S$ of cardinality $k + 1$, one of conditions (i) or (ii) holds, then $G$ is Hamiltonian. A number of known results on Hamiltonian graphs are corollaries of this result.

1. Introduction. We consider only finite simple graphs in this paper; undefined notation and terminology can be found in [1]. In particular, we use $V(G)$, $E(G)$, $k(G)$, $\alpha(G)$ and $\delta(G)$ to denote the vertex set, edge set, connectivity, independence number and minimum degree of $G$, respectively. If $G$ is a graph and $u, v \in V(G)$, then a path in $G$ from $u$ to $v$ is called a $(u,v)$-path of $G$. If $v \in V(G)$ and $H$ is a subgraph of $G$, then $N_H(v)$ denotes the set of vertices in $H$ that are adjacent to $v$ in $G$. Thus, $d_H(v)$, the degree of $v$ relative to $H$, is $|N_H(v)|$. We also write $d(v) = d_G(v)$ and $N(v) = N_G(v)$ when the graph in use is clear. If $C$ and $H$ are subgraphs of $G$, then $N_C(H) = \bigcup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of $G$ induced by $V(G) - V(C)$. For vertices $u, v \in V(G)$, the distance between $u$ and $v$, denoted $d(u, v)$, is the length of a shortest $(u,v)$-path in $G$, or $\infty$ if no such path exists.

Let $C_m = x_0x_1\ldots x_{m-1}x_0$ denote a cycle of order $m$. Define $N^+_C(u) = \{x_{i+1} : x_i \in N_C(u)\}$, $N^-_C(u) = \{x_{i-1} : x_i \in N_C(u)\}$ and $N^\pm_C(u) = N^+_C(u) \cup N^-_C(u)$, where subscripts are taken modulo $m$. Let $S \subseteq V(G)$, and define $\Delta(S) = \max\{d(x) : x \in S\}$.

A subset $S \subseteq V(G)$ is said to be an essential independent set (EIS) if $S$ is an independent set in $G$ and there exist two distinct vertices $x, y \in S$ with $d(x, y) = 2$.

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Three classical results on Hamiltonian graphs are:

**Theorem 1.1** (Dirac, [4]). If $\delta(G) \geq n/2$, then $G$ is Hamiltonian.

**Theorem 1.2** (Ore, [11]). If $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is Hamiltonian.

**Theorem 1.3** (Chvátal and Erdős, [3]). If $G$ is a graph with $\kappa(G) \geq \alpha(G)$, then $G$ is Hamiltonian.

Theorem 1.2 was generalized by Fan [5] who showed that only pairs of vertices at distance 2 are essential. In 1996, Chen et al. [2] proved a Dirac-type result for essential independent sets with $k$ vertices.

**Theorem 1.4** (Chen et al., [2]). Let $G$ be a $k$-connected $(k \geq 2)$ graph on $n \geq 3$ vertices. If $\max\{d(u) : u \in S\} \geq n/2$ for any essential independent set $S$ with $k$ vertices in $G$, then $G$ is Hamiltonian.

In 1997, Liu and Wei [10] considered essential independent sets with $k + 1$ vertices in the following:

**Theorem 1.5** (Liu and Wei, [10]). Let $G$ be a $k$-connected $(k \geq 2)$ graph on $n \geq 3$ vertices. If $\max\{d(u) : u \in S\} \geq n/2$ for any essential independent set $S$ with $k + 1$ vertices in $G$, then $G$ is Hamiltonian or is in one of three exceptional classes of graphs.

In 2002, Hirohata [9] considered essential independent sets $S$ with $k$ vertices and showed that the length of a longest cycle depends on $\max\{d(u) : u \in S\}$. Recently, in [8] Theorem 1.5 as well as some other results were generalized.

Neighborhood unions have already been shown to be very useful in studying Hamiltonian graphs. The first use of this generalized degree condition was to provide another generalization of Dirac’s theorem by Faudree et al. [7] in 1989.

**Theorem 1.6** (Faudree et al. [7]). If $G$ is a 2-connected graph and if $|N(u) \cup N(v)| \geq (2n-1)/3$ for each pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is Hamiltonian.

In 1991, Faudree et al. [6] considered the effect of $\delta(G)$.

**Theorem 1.7** (Faudree et al., [6]). If $G$ is a 2-connected graph and if $|N(u) \cup N(v)| \geq n - \delta(G)$ for each pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is Hamiltonian.

In 1994, Song and Zhang [12] considered independent sets with $k + 1$ vertices and proved the following theorem.

**Theorem 1.8** (Song and Zhang, [12]). Let $G$ be a $k$-connected graph $(k \geq 2)$ with independence number $\alpha$. If for each independent set $S$ of cardinality $k + 1$, one of the following conditions holds:
(i) there exist \( u \neq v \in S \) such that \( d(u) + d(v) \geq n \) or \( |N(u) \cap N(v)| \geq \alpha \); 
(ii) for any distinct \( u \) and \( v \) in \( S \), \( |N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\} \),
then \( G \) is Hamiltonian.

The purpose of this paper is to unify and extend the theorems above through the use of essential independent sets by proving the following result.

**Theorem 1.9.** Let \( G \) be a \( k \)-connected graph \((k \geq 2)\) with independence number \( \alpha \). If for each essential independent set \( S \) of cardinality \( k + 1 \), one of the following conditions holds:

(i) there exist \( u \neq v \in S \) such that \( d(u) + d(v) \geq n \) or \( |N(u) \cap N(v)| \geq \alpha \); 
(ii) for any distinct \( u \) and \( v \) in \( S \), \( |N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\} \),
then \( G \) is Hamiltonian.

Obviously, Theorem 1.9 generalizes Theorems 1.1, 1.2, 1.3, 1.6, 1.7 and 1.8.

Next we present an example that shows that Theorem 1.9 is stronger than Theorem 1.8.

Let \( k \geq 2 \) and \( n \geq (k + 1)(k + 3) + k + 2 + 1 = k^2 + 5k + 6 \). Let \( H = K_{n-(k+2)} \) and build a graph \( G \) as follows. Take \( H \) along with a disjoint set of vertices \( S = \{x_1, \ldots, x_{k+2}\} \). Now join each \( x_i \in S \), \( 1 \leq i \leq k + 1 \), to a distinct set of \( k + 3 \) vertices of \( H \). That is, make the neighborhoods of these vertices of \( S \) disjoint. Next join \( x_{k+2} \) to a set of \( k \) vertices of \( H \) in such a way that \( N(x_{k+2}) \cap N(x_i) = \emptyset \) for \( 1 \leq i \leq k + 1 \).

Now the resulting graph \( G \) is clearly \( k \)-connected. Also \( \alpha(G) = k + 3 \). If we consider the independent vertex set \( S' = \{x_1, \ldots, x_{k+1}\} \) we see that \( d(x_i) + d(x_j) = 2k + 6 < n \).

Also, for two vertices in \( S' \) we have \( |N(x_i) \cap N(x_j)| = \emptyset \). Thus condition (i) of the Song–Zhang Theorem fails to hold. Further,
\[
|N(x_i) \cup N(x_j)| = 2k + 6 < n - \max\{d(x) : x \in S\} = n - (k + 3)
\]
(uses the bound on \( n \)). Thus, condition (ii) of the Song–Zhang Theorem also fails to hold. Hence, Theorem 1.8 cannot be applied to \( G \).

However, the only essential independent sets of order \( k + 1 \) contain a vertex \( y \) in \( H \) and \( k \) vertices from \( S = \{x_1, \ldots, x_{k+2}\} \). For any such set, there exists some vertex \( x_i \) such that \( d(y, x_i) = 2 \) and
\[
d(y) + d(x_i) = n - (k + 2) - 1 + k + 3 = n
\]
Therefore, Theorem 1.9 does apply.

2. **Proof of Theorem 1.9** Before we begin the proof of Theorem 1.9 we need to establish a few basic facts. Within these facts we will also establish some useful inequalities.

For a cycle \( C_m = x_0x_1 \ldots x_{m-1}x_0 \), we write \([x_i, x_j] \) to denote the subpath \( x_i, x_{i+1}, \ldots, x_j \) of the cycle \( C_m \), where subscripts are taken modulo \( m \). For
notational convenience, \([x_i, x_j]\) will denote the \((x_i, x_j)\)-path of \(C_m\) as well as the vertex set of this path.

**Claim.** Let \(G\) be a 2-connected non-Hamiltonian graph. Let \(C_m = x_0x_1 \ldots x_{m-1}x_0\) be a longest cycle of \(G\), \(H\) a component of \(G - C_m\), \(x \in V(H)\). Suppose that \(x_i \in N_{C_m}(x)\) and \(x_j \in N_{C_m}(H)\) satisfy \(\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} \cap N_{C_m}(H) = \emptyset\). Then Facts (I)–(III) and inequalities (1)–(5) below hold.

**Proof of Claim.** Let \(P\) be a path in \(H\) with end-vertices adjacent to \(x_i, x_j \in V(C_m)\), respectively.

**Fact (I).** Suppose \(x_h \in \{x_{i+1}, x_{i+2}, \ldots, x_{j-2}\} - \{x_j, x_{j-1}\}\) is adjacent to \(x_{j+1}\). Then \(x_{h+1}\) is adjacent to neither \(x_{i+1}\) nor \(x\).

First, since \(\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} \cap N_{C_m}(H) = \emptyset\), it follows that \(x_{h+1}\) is not adjacent to \(x\).

If \(x_{h+1}\) is adjacent to \(x_{i+1}\) we obtain the cycle
\[
C^* = x_{i}Px_{j}x_{j-1} \ldots x_{h+1}x_{i+1}x_{i+2} \ldots x_{h}x_{j+1}x_{j+2} \ldots x_{i}
\]
which is longer than \(C_m\), a contradiction.

**Fact (II).** Suppose \(x_h \in \{x_{j+1}, x_{j+2}, \ldots, x_i\}\) is adjacent to \(x_{j+1}\). Then \(x_{h-1}\) is adjacent to neither \(x_{i+1}\) nor \(x\).

Otherwise, if \(x_{h-1}\) is adjacent to \(x_{i+1}\), then the cycle
\[
C^* = x_{i}Px_{j}x_{j-1} \ldots x_{i+1}x_{h-1}x_{h-2} \ldots x_{j+1}x_{h}x_{h+1} \ldots x_{i}
\]
is longer than \(C_m\), a contradiction. Also, suppose \(x_{h-1}\) is adjacent to \(x\). Let \(P'\) be a path in \(H\) with end-vertices adjacent to \(x_{h-1}, x_j\), respectively. Then
\[
C^* = x_{h-1}P'x_{j}x_{j-1} \ldots x_{h}x_{j+1}x_{j+2} \ldots x_{h-1}
\]
is a cycle longer than \(C_m\), a contradiction.

**Fact (III).** Suppose \(y \in V(G - C_m)\) is adjacent to \(x_{j+1}\). Then \(y\) is adjacent to neither \(x_{i+1}\) nor \(x\).

Clearly, \(y\) is not in \(H\), so \(y\) is not adjacent to \(x\). If \(y\) is adjacent to \(x_{i+1}\), then the cycle
\[
C^* = x_{i}Py_{j}x_{j-1} \ldots x_{i+1}yx_{j+1}x_{j+2} \ldots x_{i}
\]
is longer than \(C_m\), a contradiction. In Fact (I) above, we do not assume that the two vertices \(\{x_j, x_{j-1}\}\) are adjacent to \(x_{j+1}\). Hence, we have
\[
|N(x_{i+1}) \cup N(x)| \leq n - (d(x_{j+1}) - |\{x_{j-1}, x_j\}|) - |\{x_{i+1}, x\}|
\]
\[
\leq n - d(x_{j+1}).
\]
Clearly, all vertices in \( N_{C_m}^+ (H) \cup V(H) \) are nonadjacent to \( x_{i+1} \) and nonadjacent to \( x_{j+1} \). Hence, we also have
\[
(2) \quad |N(x_{i+1}) \cup N(x_{j+1})| \leq n - |N_{C_m}^+ (H) \cup V(H)| \\
\leq n - |N_{C_m}^+ (C) - |V(H)|. 
\]

Moreover, if \( x_{j-1} \) is adjacent to \( x_{j+1} \), then \( x_j \) is not adjacent to \( x_{i+1} \). Combining this with the discussion in Facts (I), (II) and (III), there are at least \((d(x_{j+1}) - |\{x_j\}| - |\{x_{i+1}\}| - |V(H)|) \) vertices not adjacent to \( x_{i+1} \). Hence,
\[
d(x_{i+1}) \leq n - (d(x_{j+1}) - |\{x_j\}| - |\{x_{i+1}\}| - |V(H)|),
\]
which implies that
\[
(3) \quad d(x_{i+1}) + d(x_{j+1}) \leq n - |V(H)|. 
\]

Similarly, all vertices in \( N_{C_m}^+ (x_{j+1}) \cup N_{C_m}^- (x_{j+1}) \cup \{x\} \) are nonadjacent to \( x \), thus we have
\[
d(x) \leq n - |N_{C_m}^+ (x_{j+1}) \cup N_{C_m}^- (x_{j+1}) \cup \{x\}|,
\]
which implies
\[
(4) \quad d(x) + d(x_{j+1}) \leq n - 1. 
\]

Clearly, the common neighbors of \( x_{i+1} \) and \( x \) are all on \( C_m \). Hence, we also have
\[
(5) \quad |N(x_{i+1}) \cap N(x)| \leq \alpha - 1. 
\]

**Proof of Theorem 1.9** Assume that \( G \) is not Hamiltonian. Let \( C_m = x_0x_1 \ldots x_{m-1}x_0 \) be a longest cycle of \( G \), and \( H \) a component of \( G - C_m \). Since \( G \) is \( k \)-connected, we have \(|N_{C_m}(H)| \geq k \). Let \( P \) be a path in \( H \) whose end-vertices \( x^*, y^* \) are adjacent to \( x_i \) and \( x_j \) on \( C_m \) respectively. Let \( x \in V(H) \) and \( x_i \in N_{C_m}(x) \). Let \( S^* \) denote \( k \) vertices of \( N_{C_m}^+ (H) \) containing \( x_{i+1} \), and let \( S = S^* \cup \{x\} \). Clearly, \( S \) is an EIS. Now, \( G \) satisfies conditions (i) or (ii) of the Theorem.

Suppose (i) holds, that is, there exist \( u, v \in S \) with \( u \neq v \) such that
\[
d(u) + d(v) \geq n \text{ or } |N(u) \cap N(v)| \geq \alpha. 
\]

Since \( S \) is an EIS, we have \( \alpha(G) \geq k+1 \). Then, by inequalities (3) and (4),
\[
d(u) + d(v) \geq n \text{ is impossible. Together with (i), this implies } |N(u) \cap N(v)| \geq \alpha. \]

By (5), if \(|N(u) \cap N(v)| \geq \alpha(G)\), then \( u, v \in N_{C_m}^+ (H) \). Without loss of generality, assume \( \{u, v\} = \{x_{i+1}, x_{j+1}\} \).

Since \( C_m \) is a longest cycle of \( G \), the vertices of \( N(x_{i+1}) \cap N(x_{j+1}) \) are not in \( G - C_m \), for otherwise a cycle longer than \( C_m \) is easily found. Thus, \( N(x_{i+1}) \cap N(x_{j+1}) \subseteq V(C_m) \), which implies \(|N(x_{i+1}) \cap N(x_{j+1})| = |N_{C_m}(x_{i+1}) \cap N_{C_m}(x_{j+1})| = |N_{C_m}^-(x_{i+1}) \cap N_{C_m}^-(x_{j+1})| \). Since \( C_m \) is a longest
cycle of $G$, $N^{-}_{C_{m}}(x_{i+1}) \cap N^{-}_{C_{m}}(x_{j+1})$ is an independent set (or again, using $P$, a longer cycle is easily found). Let $w$ be a vertex of $H$. Then $\{w\} \cup (N^{-}_{C_{m}}(x_{i+1}) \cap N^{-}_{C_{m}}(x_{j+1}))$ is also an independent set. By condition (i) of the Theorem, $|N^{-}_{C_{m}}(x_{i+1}) \cap N^{-}_{C_{m}}(x_{j+1})| \geq \alpha$. This implies that $|\{w\} \cup (N^{-}_{C_{m}}(x_{i+1}) \cap N^{-}_{C_{m}}(x_{j+1}))| \geq \alpha + 1$, contradicting the fact that the independence number of $G$ is $\alpha$.

Now suppose (ii) holds, that is, for any distinct $u, v \in S$, $|N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\} = n - \Delta(S)$.

If $\Delta(S) = d(x)$, then by inequality (2), we have $|N(x_{i+1}) \cup N(x_{j+1})| \leq n - d(x) - 1$, while condition (ii) says that $|N(u) \cup N(v)| \geq n - \Delta(S)$, a contradiction.

We now consider the following two cases.

**Case 1:** $k \geq 3$ and $|N_{C_{m}}(H)| \geq k$. In this case, let $x_{i1}, \ldots, x_{ik} \in N_{C_{m}}(H)$ be such that there are no neighbors of $H$ in the intervals $[x_{i(t-1)+1}, \ldots, x_{it}]$ for $t = 1, \ldots, k - 1$. Let $z \in V(H)$ be adjacent to some vertex of $\{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ and let $S = \{z, x_{i1+1}, \ldots, x_{ik+1}\}$.

Clearly, $S$ is an EIS. Without loss of generality, $\Delta(S) = d(x_{ik+1})$.

If $x_{ik-1}x_{ik+1} \notin E(G)$, then by (1), there exist

$$(d(x_{ik+1}) - |\{x_{ik}\}|) + |\{x_{i(k-1)+1}, z\}|$$

vertices that are nonadjacent to $x_{i(k-1)+1}$ and nonadjacent to $z$, hence we have

$$|N(x_{i(k-1)+1}) \cup N(z)| \leq n - (d(x_{ik+1}) - |\{x_{ik}\}|) - |\{x_{i(k-1)+1}, z\}|$$

$$\leq n - d(x_{ik+1}) - 1,$$

a contradiction.

Suppose $x_{ik-1}x_{ik+1} \in E(G)$. Without loss of generality, $x_{ik+t}$ is not adjacent to $x_{ik-1}$, and all of $\{x_{ik+1}, x_{ik+2}, \ldots, x_{ik+t-2}, x_{ik+t-1}\}$ are adjacent to $x_{ik-1}$ (clearly, $x_{ik+t}$ must exist in the set $\{x_{ik+1}, x_{ik+2}, \ldots, x_{i(k+1)-1}\}$, since $x_{i(k+1)-1}$ is not adjacent to $x_{ik-1}$). Then, without loss of generality, $x \in V(H)$ is adjacent to some vertex of $\{x_{i1}, x_{i2}, \ldots, x_{i(k-1)}\}$. Let $S^{*} = \{x, x_{i1+1}, x_{i2+1}, \ldots, x_{i(k-1)+1}, x_{ik+t}\}$. Clearly, $S^{*}$ is an EIS (for otherwise a longer cycle clearly exists, a contradiction). For the EIS $S^{*}$ we will prove that conditions (i) and (ii) of the Theorem fail to hold.

First, when $w, y \in \{x, x_{i1+1}, x_{i2+1}, \ldots, x_{i(k-1)+1}, x_{ik+t}\} \setminus \{x_{ik+t}\}$, we can easily check that $d(w) + d(y) \leq n - 1$ and $|N(w) \cap N(y)| \leq \alpha - 1$.

Next, suppose $w = x_{ik+t}$ and $y = x$. Clearly we also have $d(w) + d(y) \leq n - 1$ and $|N(w) \cap N(y)| \leq \alpha - 1$. 


Now suppose \( w = x_{ik+t} \) and \( y \in \{x_{i1+1}, x_{i2+1}, \ldots, x_{i(k-1)+1}\} \).

Since \( x_{ik-1} x_{ik+1} \in E(G) \) and each vertex of \( \{x_{ik+1}, x_{ik+2}, \ldots, x_{ik+t-1}\} \) is adjacent to \( x_{ik-1} \), it follows that each vertex of \( \{x, x_{i1+1}, x_{i2+1}, \ldots, x_{ik+t}\} \) \( \{x_{ik+t}\} \) is nonadjacent to any vertex of \( \{x_{ik}, x_{ik+1}, \ldots, x_{ik+t-2}, x_{ik+t-1}\} \) (otherwise, we easily get a longer cycle). Then clearly, for any \( x_{i(k-r)+1} \) (\( 1 \leq r \leq k-1 \)),

(F1) if \( x_h \in \{x_{i(k-r)+1}, x_{i(k-r)+2}, \ldots, x_{ik-1}\} \) is adjacent to \( x_{i(k-r)+1} \),
then \( x_{h-1} \) is not adjacent to \( x_{ik+t} \)

(otherwise, the cycle

\[ x_{ik} x_{ik+1} x_{ik+2} \ldots x_{ik+t-1} x_{ik-1} x_{ik-2} \ldots x_h x_{i(k-r)+1} x_{i(k-r)+2} \ldots x_{h-1} x_{ik+t} x_{ik+t+1} \ldots x_{i(k-r)} Px_{ik} \]

is longer than \( C_M \), a contradiction). Similarly,

(F2) If \( x_h \in \{x_{ik+t+1}, x_{ik+t+2}, \ldots, x_{i(k-r)}\} \) \( \{x_{i(k-r)}\} \) is adjacent to \( x_{i(k-r)+1} \), then \( x_{h+1} \) is not adjacent to \( x_{ik+t} \).

If there exist \( p \) vertices of \( C_M \) \( \{x_{i(k-i)}\} \) adjacent to \( x_{i(k-r)+1} \) or \( x_{ik+t} \), then there must also exist \( p \) vertices of \( C_M \) \( \{x_{i(k-r)}\} \) not adjacent to \( x_{ik+t} \) or \( x_{i(k-r)+1} \). Moreover, no vertex of \( H \) is adjacent to both \( x_{i(k-r)+1} \) and \( x_{ik+t} \), and every vertex of \( G - C_M - H \) is adjacent to at most one of \( \{x_{i(k-r)+1}, x_{ik+t}\} \) and \( x_{i(k-r)+1} \) \( \{x_{ik+t}\} \) are not adjacent to both \( x_{i(k-r)+1} \) and \( x_{ik+t} \). Hence, we have \( d(x_{i(k-r)+1}) + d(x_{ik+t}) \leq n - 1 \).

It follows that \( |N(x_{i(k-r)+1}) \cap N(x_{ik+t})| \leq \alpha - 1 \) (otherwise, by a proof similar to case (i), we must get a longer cycle). Thus, \( |N(x_{i(k-r)+1}) \cap N(x_{ik+t})| \leq \alpha(G) - 1 \).

Therefore, when \( w = x_{ik+t} \), \( y \in \{x_{i1+1}, x_{i2+1}, \ldots, x_{i(k-1)+1}\} \), we also have \( d(w) + d(y) \leq n - 1 \) and \( |N(w) \cap N(y)| \leq \alpha(G) - 1 \).

Now, we consider condition (ii) of the Theorem.

Suppose \( d(x_{ik+t}) \leq \max\{d(x_{ih+1}) : h = 1, \ldots, k-1\} \). Without loss of generality, assume \( \Delta(S^*) = d(x_{ih+1}) \), where \( h \in \{1, \ldots, k-1\} \). Clearly \( x_{ih+1} \) is not adjacent to \( x_{ik+2} \) (otherwise, the cycle \( C^* = x_{ih} P x_{ik} x_{ik+1} x_{ik-1} x_{ik-2} \ldots x_{ih+1} x_{ik+2} x_{ik+3} \ldots x_{ih} \) is longer than \( C_M \)). Further, \( x_{i(h-1)+1} \) and \( x \) are not both adjacent to \( x_{ik+1} \) (otherwise, we must get a longer cycle). Thus, by (1), we have

\[
|N(x_{i(h-1)+1}) \cup N(x)| \leq n - (d(x_{ih+1}) - |\{x_{ih-1}, x_{ih}\}|) - |\{x_{i(h-1)+1}, x\}| - |\{x_{ik+1}\}|
\leq n - d(x_{ih+1}) - 1,
\]

a contradiction.

Suppose \( d(x_{ik+t}) > \max\{d(x_{ih+1}) : h = 1, \ldots, k-1\} \).
In this case, clearly, none of \( \{x_{i(k-1)+1}, x_{i(k-1)+2}, \ldots, x_{ik-1}\} \) is adjacent to \( x \). By the choice of \( x_{ik+t} \), none of \( \{x_{ik+1}, x_{ik+2}, \ldots, x_{ik+t}\} \) is adjacent to \( x_{i(k-1)+1} \) and none to \( x \) (otherwise, we obtain a cycle longer than \( C_m \)).

Since \( C_m \) is a longest cycle of \( G \), we have:

- If \( x_{i(k-1)+r} \in \{x_{i(k-1)+2}, x_{i(k-1)+3}, \ldots, x_{ik-2}\} \) is adjacent to \( x_{ik+t} \), then \( x_{i(k-1)+r+1} \) is adjacent to neither \( x_{i(k-1)+1} \) nor \( x \).
- \( x_{ik+r} \in \{x_{ik+t}, x_{ik+t+1}, \ldots, x_{i(k-1)}\} \) is adjacent to \( x_{ik+t} \), then \( x_{ik+r-1} \) is adjacent to neither \( x_{i(k-1)+1} \) nor \( x \).
- \( x_{ik+r} \in \{x_{ik}, x_{ik+1}, \ldots, x_{ik+t-1}\} \setminus \{x_{ik}\} \) is adjacent to \( x_{ik+t} \), then \( x_{ik+r} \) is adjacent to neither \( x_{i(k-1)+1} \) nor \( x \).

Similar to the discussion of inequality (1), we have

\[
|N(x_{i(k-1)+1}) \cup N(x)| \leq n - (d(x_{ik+t}) - |\{x_{ik}\}|) - |\{x_{i(k-1)+1}, x\}|
\]

\[
\leq n - d(x_{ik+t}) - 1,
\]
a contradiction.

**Case 2:** \( |N_{C_m}(H)| = |\{x_i, x_j\}| = 2 \). In this case, without loss of generality, assume \( d(x_{i+1}) \leq d(x_{j+1}) \).

**Claim (a).** Let \( x, y \) be two vertices of \( H \) which are adjacent to \( x_i, x_j \), respectively. If \( d(x_{i+1}, x_{j+1}) = 2 \), then there is a Hamilton path in the subgraph \( H \) with end-vertices \( x, y \).

**Proof of Claim (a).** Let \( P' \) be a longest path of \( H \) with end-vertices \( x, y \). If \( P' \) is not a Hamilton path of \( H \), let \( w \) be a vertex of \( H - P' \) which is adjacent to some vertex of \( P' \). Clearly, \( \{x_{i+1}, x_{j+1}, w\} \) is an EIS. Further, we know that condition (i) of the Theorem does not hold. Thus, (ii) must hold. Then we can check that \( w \) must be adjacent to every vertex of \( H - \{w\} \), for otherwise, by (1), we again reach a contradiction. Thus, we get a path in \( H \) longer than \( P' \) with end-vertices \( x, y \), a contradiction.

**Claim (b).** If \( u \in V(H) \) is adjacent to \( x_i \), then \( u \) must be adjacent to \( x_j \).

**Proof of Claim (b).** If \( u \) is not adjacent to \( x_j \), then, by a proof similar to that of (1), \( |N(x_{i+1}) \cup N(x)| \leq n - (d(x_{j+1}) - |\{x_j\}| - |\{x_{i+1}, x\}| \leq n - d(x_{j+1}) - 1 \), a contradiction.

**Subcase 2.1:** \( |V(H)| \geq 2 \). Let \( \{x_i, x_j\} = N_{C_m}(H) \), and let \( x, y \in V(H) \) be adjacent to \( x_i, x_j \), respectively. Moreover, let \( |V(H)| = h \).

**Subcase 2.1.1:** \( d(x_{i+1}, x_{j+1}) \geq 3 \).

**Subcase 2.1.1.1:** \( d(x) \geq \max\{d(x_{i+1}), d(x_{j+1})\} \) or \( d(y) \geq \max\{d(x_{i+1}), d(x_{j+1})\} \). Without loss of generality, assume \( d(x) \geq \max\{d(x_{i+1}), d(x_{j+1})\} \).
Clearly \( \{x,x_{i+1},x_{j+1}\} \) is an EIS. Further, we know that condition (i) of the Theorem does not hold. Thus, (ii) must hold. But we can check that

\[
|N(x_{i+1}) \cup N(x_{j+1})| \leq n - |N(x)| \leq n - \max\{d(x) : x \in S\} - 1,
\]

counter to (ii).

**SUBCASE 2.1.1.2:** Subcase 2.1.1 fails to hold. Without loss of generality, \( d(x_{j+1}) = \max\{d(x_{i+1}),d(x_{j+1}),d(x),d(y)\} \). Since \( d(x_{i+1},x_{j+1}) \geq 3 \), let \( x_r \in \{x_{j+1},x_{j+2},\ldots,x_i\} \) be adjacent to \( x_{j+1} \) with \( r \) as large as possible. Then \( x_r \) is not adjacent to \( x_{i+1} \). Let \( x_h \in \{x_{i+1},x_{i+2},\ldots,x_{j-1}\} \) be adjacent to \( x_{j+1} \) with \( h \) as small as possible. Then \( x_h \) is not adjacent to \( x_{i+1} \). Hence, one can check that

\[
|N(x_{i+1}) \cup N(x)| \leq n - (d(x_{j+1}) - \{|x_j,x_{j-1}\}) - \{|x_{i+1},x\} - \{|x_k,x_h\|
\]

\[
\leq n - d(x_{j+1}) - 2,
\]
a contradiction.

**SUBCASE 2.1.2:** \( d(x_{i+1},x_{j+1}) = 2 \). By Claim (a), \( H \) has a Hamilton path with end-vertices \( x,y \). Suppose that

\((*)\) \( x_f \in \{x_{j+1},x_{j+2},\ldots,x_i\} \) is adjacent to \( x_{i+1} \) and \( x_{f+r} \in \{x_{j+1},x_{j+2},\ldots,x_i\} \) is adjacent to \( x_{j+1} \) (where \( r \geq 1 \) and \( x_{f+1} \) is not adjacent to \( x_{i+1} \)).

Then none of \( \{x_{f+1},x_{f+2},\ldots,x_{f+h}\} \) is adjacent to \( x_{j+1} \) (otherwise, together with Claim (a) that \( H \) has a Hamilton path with end-vertices \( x,y \), we get a cycle longer than \( C_m \)). Hence, we have

\[
|N(x_{i+1}) \cup N(x)| \leq n - (d(x_{j+1}) - \{|x_{j-1},x_j\}) - \{|x_{i+1},x\|
\]

\[
- (\{|x_{f+1},x_{f+2},\ldots,x_{f+h}\}| - 1)
\]

\[
\leq n - d(x_{j+1}) - 1,
\]
a contradiction.

Similarly, if \( x_f \in \{x_{i+1},x_{i+2},\ldots,x_{j-1}\} \) is adjacent to \( x_{j+1} \), and \( x_{f+r} \) is adjacent to \( x_{i+1} \), we also get a contradiction. Now suppose that (*) fails to hold. Namely, when \( x_f \in \{x_{j+1},x_{j+2},\ldots,x_i\} \) is adjacent to \( x_{j+1} \), then none of \( \{x_{j+1},x_{j+2},\ldots,x_f\} \) is adjacent to \( x_{i+1} \). If \( x_f \in \{x_{i+1},x_{i+2},\ldots,x_{j-1}\} \) is adjacent to \( x_{j+1} \), then none of \( \{x_{f+1},x_{f+2},\ldots,x_j\} \) is adjacent to \( x_{i+1} \). In this case, under the conditions of the Theorem, if \( x_f \in \{x_{j+1},x_{j+2},\ldots,x_i\} \) is adjacent to \( x_{j+1} \), and none of \( \{x_{f+1},x_{f+2},\ldots,x_i\} \) is adjacent to \( x_{j+1} \), then all of \( \{x_{j+1},x_{j+2},\ldots,x_f\} \) are adjacent to \( x_{j+1} \), and every vertex of \( \{x_f,x_{f+1},\ldots,x_i\} \) is adjacent to \( x_{i+1} \). Similarly, when \( x_t \in \{x_{i+1},x_{i+2},\ldots,x_j\} \) is adjacent to \( x_{i+1} \), and none of \( \{x_{t+1},x_{t+2},\ldots,x_j\} \) is adjacent to \( x_{i+1} \), then every vertex of \( \{x_{i+1},x_{i+2},\ldots,x_{j-1}\} \) is adjacent to \( x_{i+1} \) (otherwise we obtain the contradiction that \( |N(x_{i+1}) \cup N(x)| \leq n - d(j+1) - 1 \)). Clearly \( x_{f-1} \) is not adjacent to any of \( \{x_f,x_{f+1},\ldots,x_t\} - \{x_f,x_t\} \).
Thus, if $d(x_{i+1}) \leq d(x_{f-1})$, we have
\[
|N(x_{i+1}) \cup N(x)| \leq n - (d(x_{f-1}) - |\{x_f, x_t\}|) - |\{x_{i+1}, x, x_j\}|
\leq n - d(x_{f-1}) - 1,
\]
a contradiction. If $d(x_{i+1}) > d(x_{f-1})$, we have
\[
|N(x_{f-1}) \cup N(x)| \leq n - (d(x_{i+1}) - |\{x_f, x_t\}|) - |\{x_{i-1}, x, x_j\}| \leq n - d(x_{i+1}) - 1,
\]
a contradiction.

**Subcase 2.2:** $|V(H)| = 1$. Let $V(H) = \{x\}$ and $|NC_m(x)| = 2 = |\{x_1, x_f\}|$. In this case, we have $C_m = C_{n-1}$, since otherwise, as we are not in Subcase 2.1, any component $H'$ of $G - C_m - H$ has $|V(H')| = 1$. Without loss of generality, let $V(H') = \{y\}$, so $|NC_m(y)| = 2$. This implies that $|N(x) \cup N(y)| \leq 4$. Since $C_m$ is a longest cycle, $y$ is not adjacent to at least one of $\{x_2, x_{f+1}\}$. Without loss of generality, $y$ is not adjacent to $x_2$. Then $d(x_2) \leq n - |\{x, y, x_2, x_{f+1}\}| = n - 4$. Clearly, $S = \{x, y, x_2\}$ is an EIS. By condition (ii) of the Theorem, for any distinct $u$ and $v$ in $S$, $|N(u) \cup N(v)| \geq n - \Delta(S)$. Together with $d(x_2) \leq n - 4$, we see that $|N(x) \cup N(y)| = 4$ implies $m \geq 4$ and $d(x_2) = n - 4$. Since $C_m$ is a longest cycle we can easily check that $m \geq 6$ and $n \geq 8$. By inequality (3), we have $d(x_{f+1}) \leq n - |V(H)| - d(x_2) = 3$. If $y$ is not adjacent to $x_{f+1}$, then $S = \{x, y, x_{f+1}\}$ is an EIS. Then condition (ii) of the Theorem implies that $|N(x) \cup N(y)| \geq n\Delta(S)$ fails, a contradiction. Suppose $y$ is adjacent to $x_{f+1}$. Let $N(y) = \{x_i, x_j\}$, say $i < j$. Since $C_m$ is a longest cycle of $G$, if $x_{h+1} = x_i$, then $x_{j-1}$ is not adjacent to $x_2$. Thus we have $d(x_2) < n - 4$, which contradicts the above result that $d(x_2) = n - 4$. If $x_{f+1} = x_j$ then since $C_m$ is a longest cycle of $G'$; it follows that $x_{i+1}$ is not adjacent to $x_2$ and we have $d(x_2) < n - 4$, again contradicting $d(x_2) = n - 4$. Therefore, $C_m = C_{n-1}$.

Now, without loss of generality, assume $d(x_2) \leq d(x_{f+1})$. Let $S = \{x, x_2, x_{f+1}\}$.

**Claim (I).** The vertex $x_2$ is not adjacent to $x_f$.

For otherwise, we have $|N(x_2) \cup N(x)| = d(x_2)$. By condition (ii) of the Theorem, that implies $|N(x_2) \cup N(x)| \geq n - \Delta(S) = n - d(x_{f+1})$, and we have $d(x_2) \geq n - d(x_{f+1})$. This contradicts inequality (3).

**Claim (II).** The vertex $x_{f-1}$ is adjacent to $x_{f+1}$.

For otherwise, by inequality (3) and Claim (I), we have
\[
d(x_2) \leq n - (d(x_{f+1}) - |\{x_f\}|) - |\{x_2\}| - |\{x_f\}| - |V(H)| \leq n - d(x_{f+1}) - 2.
\]
But by condition (ii) of the Theorem and Claim (I), we have
\[ d(x_2) = |N(x_2) \cup N(x)| - 1 \geq n - \Delta(S) - 1 = n - d(x_{f+1}) - 1, \]
a contradiction.

**Claim (III).** The vertex \( x_2 \) is adjacent to \( x_{n-1} \).

For otherwise, if \( x_2 \) is not adjacent to \( x_{n-1} \), when \( d(x_2) = d(x_{f+1}) \), we can apply Claim (II) to deduce that \( x_2 \) is adjacent to \( x_{n-1} \), a contradiction. Suppose \( d(x_2) < d(x_{f+1}) \). If \( d(x_{n-1}) \geq d(x_{f+1}) \), Claim (II) implies that \( x_2 \) is adjacent to \( x_{n-1} \), again a contradiction. If \( d(x_{n-1}) < d(x_{f+1}) \), together with inequality (3) we have \( \max\{d(x_2), d(x_{n-1})\} < (n-1)/2 \). Let \( S = \{x, x_2, x_{n-1}\} \). Clearly, this contradicts condition (ii) of the Theorem.

**Claim (IV).** Let \( x_t \in \{x_{f+1}, x_{f+2}, \ldots, x_{n-1}\} \) be adjacent to \( x_2 \) and suppose none of \( \{x_{f+1}, x_{f+2}, \ldots, x_{t-2}, x_{t-1}\} \) is adjacent to \( x_2 \). Let \( x_t \in \{x_2, x_3, \ldots, x_{f-1}\} \) be adjacent to \( x_{f+1} \) and none of \( \{x_2, x_3, \ldots, x_{k-1}\} \) be adjacent to \( x_{f+1} \). Then \( d(x_{t-1}) + d(x_{k-1}) \leq n - 3 \).

In this case, \( x_t \) is adjacent to \( x_{f+1} \) (otherwise, by inequality (1), we have
\[ |N(x_2) \cup N(x)| \leq n - (d(x_{f+1}) - |\{x_{f-1}, x_f\}| - |\{x_2, x\}| - |\{x_{t-1}\}|)
\[ = n - d(x_{f+1}) - 1, \]
contradicting condition (ii) of the Theorem.

Since \( C_{n-1} \) is a longest cycle of \( G \), when \( x_r \in \{x_2, x_3, \ldots, x_{f-1}\} \) is adjacent to \( x_2 \), then \( x_{r-1} \) is not adjacent to \( x_{t-1} \). If \( x_r \in \{x_t, x_{t+1}, \ldots, x_{n-1}, x_1\} \) then \( x_{n-1}, x_1 \) is adjacent to \( x_2 \), then \( x_{t+1} \) is not adjacent to \( x_{t-1} \). Clearly, \( x_2 \) is adjacent to neither \( x_f \) nor \( x_{f+1} \), and \( x_{t-1} \) is adjacent to neither \( x_{f-1} \) nor \( x_f \). Hence, we have
\[ d(x_{t-1}) \leq n - (d(x_2) - |\{x_{n-1}, x_1\}| - |\{x_{f-1}, x_f, x_{t-1}, x\}| = n - d(x_2) - 2. \]
Similarly, \( d(x_{k-1}) \leq n - d(x_{f+1}) - 2 \). This implies
\[ d(x_{t-1}) + d(x_{k-1}) \leq [n - d(x_2) - 2] + [n - d(x_{f+1}) - 2]. \tag{6} \]

Without loss of generality, assume \( d(x_2) \geq d(x_{f+1}) \). By condition (ii) of the Theorem, we have \( d(x_2) + 1 = |N(x_2) \cup N(x)| \geq n - d(x_{f+1}) \), which implies that \( d(x_2) + d(x_{f+1}) \geq n - 1 \). By inequality (3), we have \( d(x_2) + d(x_{f+1}) \leq n - 1 \). This implies \( d(x_2) + d(x_{f+1}) = n - 1 \). Together with (6), we have
\[ d(x_{t-1}) + d(x_{k-1}) \leq [n - d(x_2) - 2] + [n - d(x_{f+1}) - 2] = n - 3. \tag{7} \]

In what follows we will show that \( d(x_{k-1}) + d(x_{t-1}) \geq n - 2 \), which contradicts the above inequality. First we must establish the following claims.

**Claim (A).** If \( x_{k-1}x_f \in E(G) \), then \( d(x_{k-1}) \geq d(x_{f-1}) \). If \( x_{k-1}x_{f-1} \notin E(G) \), then \( d(x_{k-1}) \geq d(x_{f+1}) - 1 \).
Clearly, \( \{x, x_{n-1}, x_{k-1}\} \) is an EIS. If the Claim fails to hold, then by condition (ii) of the Theorem, we have
\[
\text{(8)} \quad d(x_{n-1}) + 1 = |N(x_{n-1}) \cup N(x)| \geq n - \Delta\{x, x_{n-1}, x_{k-1}\}.
\]

If \( d(x_{k-1}) \geq d(x_{n-1}) \), then inequality (8) becomes
\[
d(x_{n-1}) + 1 = |N(x_{n-1}) \cup N(x)| \geq n - \Delta\{x, x_{n-1}, x_{k-1}\} = n - d(x_{k-1}).
\]

Since Claim (A) fails to hold,
\[
n - d(x_{f-1}) > n - d(x_{f-1}).
\]
Thus, \( d(x_{n-1}) + 1 > n - d(x_{h-1}) \), which implies that \( d(x_{n-1}) + d(x_{h-1}) > n - 1 \), which contradicts inequality (3).

If \( d(x_{k-1}) < d(x_{n-1}) \), then combined with the above hypothesis that \( d(x_{n-1}) \leq d(x_{f-1}) \), and that Claim (A) fails, we find that when \( x_{k-1}x_f \in E(G) \),
\[
\text{(9)} \quad d(x_{f-1}) + 1 > d(x_{k-1}) + 1 \geq |N(x_{k-1}) \cup N(x)|.
\]

When \( x_{k-1}x_f \notin E(G) \), we get
\[
\text{(10)} \quad d(x_{f-1}) + 1 > d(x_{k-1}) + 2 \geq |N(x_{k-1}) \cup N(x)|.
\]

Now \( |N(x_{k-1}) \cup N(x)| \geq n - \Delta\{x, x_{n-1}, x_{k-1}\} = n - d(x_{n-1}) \), which implies that \( d(x_{n-1}) + d(x_{f-1}) > n - 1 \). However, this contradicts (3). Thus, Claim (A) is proved.

**CLAIM (B).** If \( x_{t-1}x_n \in E(G) \), then \( d(x_{t-1}) \geq d(x_{n-1}) \). If \( x_{t-1}x_n \notin E(G) \), then \( d(x_{t-1}) \geq d(x_{n-1}) - 1 \).

The proof of Claim (B) is similar to that of (A) and is omitted.

Thus, when \( x_{k-1} \) is adjacent to \( x_f \) or \( x_{t-1} \) is adjacent to \( x_n \), we have
\[
d(x_{k-1}) + d(x_{t-1}) \geq d(x_{n-1}) + d(x_{f-1}) - 1 = n - 2.
\]
This contradicts (7).

When \( x_{k-1} \) is not adjacent to \( x_f \) and \( x_{t-1} \) is not adjacent to \( x_n \), we have
\[
d(x_{k-1}) + d(x_{t-1}) \geq d(x_{n-1}) + d(x_{f-1}) - 2 = n - 3.
\]

Together with inequality (7), we have \( d(x_{k-1}) + d(x_{t-1}) = n - 3 \). Then clearly \( x_{k-1}x_{t-1} \notin E(G) \) and \( x_{k-1}x_1 \notin E(G) \) and \( x_{k-1}x_f \notin E(G) \). Since \( x_{k-1}x_f \in E(G) \) and \( x_{t-1}x_n \notin E(G) \), none of the vertices of \( \{x_{k-1}, x_{t-1}, x_1, x_f\} \) is adjacent to both \( x_{k-1} \) and \( x_{t-1} \). Together with the fact that \( d(x_{k-1}) + d(x_{t-1}) = n - 3 \), we see that \( x_{k-1} \) and \( x_{t-1} \) must have at least one common neighbor. Thus, \( \{x, x_{t-1}, x_{k-1}\} \) is an EIS. Without loss of generality, \( d(x_{t-1}) \geq d(x_{k-1}) \).

By condition (ii) of the Theorem, we have
\[
d(x_{k-1}) + 2 = |N(x_{k-1}) \cup N(x)| \geq n - \Delta\{x, x_{t-1}, x_{k-1}\} = n - d(x_{t-1}),
\]

which implies $d(x_{k-1}) + d(x_{t-1}) \geq n - 2$, a contradiction to inequality (7), completing the proof of the Theorem.

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