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MONOMORPHISMS OF COALGEBRAS

BY

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Abstract. We prove new necessary and sufficient conditions for a morphism of coalgebras to be a monomorphism, different from the ones already available in the literature. More precisely, $\varphi : C \to D$ is a monomorphism of coalgebras if and only if the first cohomology groups of the coalgebras C and D coincide if and only if $\sum_{i\in I} \varepsilon(a^i)b^i = \sum_{i\in I} a^i\varepsilon(b^i)$ for all $\sum_{i\in I} a^i \otimes b^i \in C \,\square_D\, C$. In particular, necessary and sufficient conditions for a Hopf algebra map to be a monomorphism are given.

Introduction. In any concrete category $\mathcal{C}$ the natural problem of whether epimorphisms are surjective maps arises, as well as the dual problem of whether monomorphisms are injective maps. This type of problems have already been studied before in several well known categories: for example in [9] it is shown that the property of epimorphisms of being surjective holds in the categories of von Neumann algebras, C^*-algebras, groups, finite groups, Lie algebras, compact groups, while it fails to be true in the categories of finite-dimensional Lie algebras, semisimple finite-dimensional Lie algebras, locally compact groups and unitary rings (see [7], [10]). The more recent paper [4] deals with the same problem in the context of Hopf algebras: several examples of non-injective monomorphisms and non-surjective epimorphisms are given. It turns out that the above problem is also intimately related to Kaplansky's first conjecture in the sense that every non-surjective epimorphism of Hopf algebras provides a counterexample to Kaplansky's problem.

In [8] the problems mentioned above are studied in the category of coalgebras. The problem of whether epimorphisms of coalgebras are surjective maps is easily settled in the positive using the existence of a cofree coalgebra on every vector space. The dual problem, on the other hand, is more interesting: an example of a non-injective monomorphism is given and several characterizations of monomorphisms are proved in [8, Theorem 3.5]. In this note we complete the above characterization with two new equivalences. Our interest in this problem comes also from the fact that a morphism of Hopf algebras is a monomorphism if and only if it is a monomorphism when viewed as a morphism of coalgebras ([4, Proposition 2.5]).

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A detailed discussion regarding the theory of coalgebras can be found in [5] and [3].

1. Preliminaries. Throughout this paper, k is an arbitrary field. Unless specified otherwise, all vector spaces, homomorphisms, algebras, coalgebras, tensor products, comodules and so on are over k. For the standard categories we use the following notations: ${}_k\mathcal{M}$ (k-vector spaces), k-Coalg (coalgebras over k), $\mathcal{M}^C$ (right C-comodules), ${}^C\mathcal{M}^D$ ((C,D)-bicomodules).

For a coalgebra C, we use Sweedler's Σ-notation, that is, $\Delta(c)=c_{(1)}\otimes c_{(2)}$, $(I\otimes\Delta)\Delta(c)=c_{(1)}\otimes c_{(2)}\otimes c_{(3)}$, etc. We also use the Sweedler notation for left and right C-comodules: $\rho^r_M(m)=m_{[0]}\otimes m_{[1]}$ for any $m\in M$ if (M,ρ^r_M) is a right C-comodule, and $\rho^l_N(n)=n_{\langle -1\rangle}\otimes n_{\langle 0\rangle}$ for any $n\in N$ if (N,ρ^l_N) is a left C-comodule. For further details regarding the theory of comodules we refer to [1].

If M is a right C-comodule with structure map ρ^r_M and N a left C-comodule with structure map ρ^l_N, the *cotensor product* $M\,\square_C\,N$ is the kernel of the k-linear map

$$\rho^r_M\otimes I - I\otimes\rho^l_N : M\otimes N\to M\otimes C\otimes N.$$

Given comodule maps $f:M\to M'$ and $g:N\to N'$, the k-linear map $f\otimes g: M\otimes N\to M'\otimes N'$ induces a k-linear map $f\,\square_C\,g: M\,\square_C\,N\to M'\,\square_C\,N'$. A left C-comodule M induces a functor $-\,\square_C\,M:\mathcal{M}^C\to{}_k\mathcal{M}$.

If $\varphi: C\to D$ is a coalgebra map then every right (left) C-comodule (M,ρ^r_M) can be made into a right (left) D-comodule with structure map $\tau^r_M: M\to M\otimes D$ given by $\tau^r_M(m)=m_{[0]}\otimes\varphi(m_{[1]})$. This association defines a functor $\varphi^*_{C,D}:\mathcal{M}^C\to\mathcal{M}^D$ usually called the *corestriction functor*.

If $M\in{}^C\mathcal{M}^D$ we obtain a functor $-\,\square_C\,M:\mathcal{M}^C\to\mathcal{M}^D$. In particular, C becomes a left D-comodule via φ and we obtain a functor $-\,\square_D\,C:\mathcal{M}^D\to\mathcal{M}^C$ called the *coinduction functor* which is a right adjoint to the corestriction functor $\varphi^*_{C,D}$ (see [1, 22.12]). Recall that (F,G) is a pair of adjoint functors, with $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ if and only if there exist two natural transformations $\eta: 1_{\mathcal{C}}\to GF$ and $\varepsilon: FG\to 1_{\mathcal{D}}$, called the *unit* and the *counit* of the adjunction, such that $G(\varepsilon_D)\circ\eta_{G(D)}=I_{G(D)}$ and $\varepsilon_{F(C)}\circ F(\eta_C)=I_{F(C)}$ for all $C\in\mathcal{C}$ and $D\in\mathcal{D}$.

Recall from [2] the construction of the *trivial coextension* of a coalgebra C by a (C,C)-bicomodule N. We define a comultiplication and a counit on the space $C\rtimes N := C\oplus N$ by

$$\Delta(c,n)=(c_{(1)},0)\otimes(c_{(2)},0)+(n_{\langle -1\rangle},0)\otimes(0,n_{\langle 0\rangle})+(0,n_{[0]})\otimes(n_{[1]},0),$$
$$\varepsilon(c,n)=\varepsilon(c)$$

for all $(c,n)\in C\oplus N$. In this way $C\rtimes N$ becomes a coalgebra, known as the *trivial coextension* of C and N.

Define the map $\pi_C : C \rtimes N \to C$ by $\pi_C(c, n) = c$ for all $(c, n) \in C \oplus N$. For every $(c, n) \in C \rtimes N$ we have

$$\varepsilon_C \circ \pi_C(c, n) = \varepsilon_C(c) = \varepsilon(c, n)$$

and

$$\begin{aligned}(\pi_C \otimes \pi_C)(\Delta(c, n)) &= (\pi_C \otimes \pi_C)\big((c_{(1)}, 0) \otimes (c_{(2)}, 0) + (n_{\langle -1\rangle}, 0) \otimes (0, n_{\langle 0\rangle}) \\ &\qquad + (0, n_{[0]}) \otimes (n_{[1]}, 0)\big) \\ &= c_{(1)} \otimes c_{(2)} + n_{\langle -1\rangle} \otimes 0 + 0 \otimes n_{[0]} \\ &= c_{(1)} \otimes c_{(2)} = \Delta_C \circ \pi_C(c, n).\end{aligned}$$

Thus π_C is a coalgebra map.

We also require some notions related to homological coalgebra. For basic definitions and properties we refer to [6]. We just recall here, for further reference, the description of the zeroth cohomology group of a coalgebra C with coefficients in a (C, C)-bicomodule N:

$$\begin{aligned}H^0(N, C) &= \{\gamma \in N^* \mid (I \otimes \gamma)\rho_N^l = (\gamma \otimes I)\rho_N^r\} \\ &= \{\gamma \in N^* \mid n_{\langle -1\rangle}\gamma(n_{\langle 0\rangle}) = \gamma(n_{[0]})n_{[1]},\ \forall n \in N\}.\end{aligned}$$

2. Characterizations of monomorphisms of coalgebras. As mentioned before, a characterization of monomorphisms in k-Coalg is given in [8, Theorem 3.5] and the equivalences (1)–(5) in our Theorem 2.1 are proved there. In what follows we complete the description of monomorphisms in k-Coalg with two more characterizations: the first one indicates a cohomological description of monomorphisms while the other is an elementary one involving the cotensor product $C \,\square_D\, C$.

THEOREM 2.1. *Let $\varphi : C \to D$ be a coalgebra map. The following statements are equivalent:*

(1) *φ is a monomorphism in the category k-*Coalg.
(2) *The functor $\varphi^*_{C,D}$ is full.*
(3) $C \,\square_D\, \mathrm{Ker}(\varphi) = 0$.
(4) *The map $\eta_C = \Delta_C : C \to C \,\square_D\, C$ is surjective (hence bijective).*
(5) *The unit of the adjunction $(\varphi^*_{C,D}, - \,\square_D\, C)$,*

$$\eta : 1_{\mathcal{M}^C} \to (- \,\square_D\, C) \circ \varphi^*_{C,D},$$

is a natural isomorphism.
(6) *$H^0(N, C) = H^0(N, D)$ for any (C, C)-bicomodule N.*
(7) *$\sum_{i\in I} \varepsilon(a^i)b^i = \sum_{i\in I} a^i\varepsilon(b^i)$ for all $\sum_{i\in I} a^i \otimes b^i \in C \,\square_D\, C$.*

Proof. (1)⇒(6). Suppose φ is a monomorphism of coalgebras. It is easy to see that $H^0(N, C) \subset H^0(N, D)$. Now let $\gamma \in H^0(N, D)$. Define the map

$\beta : C \rtimes N \to C$ by

$$\beta(c,n) = c - n_{\langle -1\rangle}\gamma(n_{\langle 0\rangle}) + \gamma(n_{[0]})n_{[1]}$$

for all $(c,n) \in C \rtimes N$. We prove that β is a coalgebra map. For all $(c,n) \in C \rtimes N$ we have

$$\begin{aligned}\varepsilon_C \circ \beta(c,n) &= \varepsilon_C(c - n_{\langle -1\rangle}\gamma(n_{\langle 0\rangle}) + \gamma(n_{[0]})n_{[1]})\\ &= \varepsilon_C(c) - \varepsilon_C(n_{\langle -1\rangle})\gamma(n_{\langle 0\rangle}) + \gamma(n_{[0]})\varepsilon_C(n_{[1]})\\ &= \varepsilon_C(c) - \gamma(\varepsilon_C(n_{\langle -1\rangle})n_{\langle 0\rangle}) + \gamma(n_{[0]}\varepsilon_C(n_{[1]})\\ &= \varepsilon_C(c) - \gamma(n) + \gamma(n)\\ &= \varepsilon(c,n)\end{aligned}$$

and

$$\begin{aligned}(\beta\otimes\beta)\circ \Delta(c,n) &= (\beta\otimes\beta)\big((c_{(1)},0)\otimes(c_{(2)},0) + (m_{\langle -1\rangle},0)\otimes(0,m_{\langle 0\rangle})\\ &\quad + (0,m_{[0]})\otimes(m_{[1]},0)\big)\\ &= \beta((c_{(1)},0))\otimes\beta((c_{(2)},0)) + \beta((m_{\langle -1\rangle},0))\otimes\beta((0,m_{\langle 0\rangle}))\\ &\quad + \beta((0,m_{[0]}))\otimes\beta((m_{[1]},0))\\ &= c_{(1)}\otimes c_{(2)} + n_{\langle -1\rangle}\otimes(-n_{\langle 0\rangle\langle -1\rangle}\gamma(n_{\langle 0\rangle\langle 0\rangle}) + \gamma(n_{\langle 0\rangle[0]})n_{\langle 0\rangle[1]})\\ &\quad + (-n_{[0]\langle -1\rangle}\gamma(n_{[0]\langle 0\rangle}) + \gamma(n_{[0][0]})n_{[0][1]})\otimes n_{[1]}\\ &= c_{(1)}\otimes c_{(2)} - n_{\langle -1\rangle}\otimes n_{\langle 0\rangle\langle -1\rangle}\gamma(n_{\langle 0\rangle\langle 0\rangle}) + \gamma(n_{[0][0]})n_{[0][1]}\otimes n_{[1]}\\ &= c_{(1)}\otimes c_{(2)} - n_{\langle -1\rangle(1)}\otimes n_{\langle 0\rangle(2)}\gamma(n_{\langle 0\rangle}) + \gamma(n_{[0]})n_{[1](1)}\otimes n_{[1](2)}\\ &= c_{(1)}\otimes c_{(2)} - \Delta_C(n_{\langle -1\rangle}\gamma(n_{\langle 0\rangle})) + \Delta_C(\gamma(n_{[0]})n_{[1]})\\ &= \Delta_C(c - n_{\langle -1\rangle}\gamma(n_{\langle 0\rangle}) + \gamma(n_{[0]})n_{[1]})\\ &= \Delta_C\circ\beta(c,n).\end{aligned}$$

Hence, β is a coalgebra map. Furthermore, it is easy to see that $\varphi\circ\pi_C = \varphi\circ\beta$ and $\pi_C = \beta$ because φ is a monomorphism. Thus $n_{\langle -1\rangle}\gamma(n_{\langle 0\rangle}) = \gamma(n_{[0]})n_{[1]}$, which implies that $\gamma \in H^0(N,C)$.

(6)⇒(7). $C \,\square_D\, C$ is a (C,C)-bicomodule with left and right structures given by

$$\psi^l_{C\square_D C}\Big(\sum_{i\in I} a^i\otimes b^i\Big) = \sum_{i\in I} a^i_{(1)}\otimes(a^i_{(2)}\otimes b^i),$$

$$\psi^r_{C\square_D C}\Big(\sum_{i\in I} a^i\otimes b^i\Big) = \sum_{i\in I}(a^i\otimes b^i_{(1)})\otimes b^i_{(2)}$$

for all $\sum_{i\in I} a^i\otimes b^i \in C\,\square_D\, C$. We define the k-linear map $T : C\,\square_D\, C \to k$ by

$$T\Big(\sum_{i\in I} a^i\otimes b^i\Big) = \sum_{i\in I}\varepsilon(a^i)\varepsilon(b^i)$$

for all $\sum_{i\in I} a^i \otimes b^i \in C \,\square_D\, C$. Now let $\sum_{i\in I} a^i \otimes b^i \in C \,\square_D\, C$, that is,

$$\sum_{i\in I} a^i_{(1)} \otimes \varphi(a^i_{(2)}) \otimes b^i = \sum_{i\in I} a^i \otimes \varphi(b^i_{(1)}) \otimes b^i_{(2)}.$$

By applying $\varepsilon \otimes I \otimes \varepsilon$ in the above identity we obtain $\sum_{i\in I} \varphi(a^i)\varepsilon(b^i) = \sum_{i\in I} \varepsilon(a^i)\varphi(b^i)$. Thus $T \in H^0(C \,\square_D\, C, D) = H^0(C \,\square_D\, C, C)$ and it follows that $\sum_{i\in I} a^i\varepsilon(b^i) = \sum_{i\in I} \varepsilon(a^i)b^i$.

(7)⇒(1). Let $M \in \mathcal{M}^C$. Define $\nu_M : M \,\square_D\, C \to M$ by

$$\nu_M\Big(\sum_{i\in I} m^i \otimes c^i\Big) = \sum_{i\in I} m^i\varepsilon(c^i).$$

For any $\sum_{i\in I} m^i \otimes c^i \in M \,\square_D\, C$ we have

$$\begin{aligned}
&(\nu_M \circ I) \circ \rho^r_{M\square_D C}\Big(\sum_{i\in I} m^i \otimes c^i\Big)\\
&\quad= \sum_{i\in I} (\nu_M \circ I)(m^i \otimes c^i_{(1)} \otimes c^i_{(2)}) = \sum_{i\in I} \nu_M(m^i \otimes c^i_{(1)}) \otimes c^i_{(2)}\\
&\quad= \sum_{i\in I} m^i\varepsilon(c^i_{(1)}) \otimes c^i_{(2)} = \sum_{i\in I} m^i_{[0]}\varepsilon(m^i_{[1]}) \otimes c^i\\
&\quad= \sum_{i\in I} m^i_{[0]} \otimes \varepsilon(m^i_{[1]})c^i = \sum_{i\in I} m^i_{[0]} \otimes m^i_{[1]}\varepsilon(c^i)\\
&\quad= \sum_{i\in I} \rho^r_M(m^i\varepsilon(c^i)) = (\rho^r_M \circ \nu_M)\Big(\sum_{i\in I} m^i \otimes c^i\Big),
\end{aligned}$$

where we use the fact that $\sum_{i\in I} m^i_{[1]} \otimes c^i \in C \,\square_D\, C$ for all $\sum_{i\in I} m^i \otimes c^i \in M \,\square_D\, C$. Thus ν_M is a morphism of right C-comodules. Moreover, in the computations above we also prove $(\rho^r_M \circ \nu_M)(\sum_{i\in I} m^i \otimes c^i) = \sum_{i\in I} m^i \otimes c^i$ for all $\sum_{i\in I} m^i \otimes c^i \in M \,\square_D\, C$. It follows that for all $M \in \mathcal{M}^C$ there exists a morphism of right C-comodules such that $\rho^r_M \circ \nu_M = I$. Since $\eta_M = \rho^r_M$, it follows that $\varphi^*_{C,D}$ is a full functor. Now, in the light of [8, Theorem 3.5] we conclude that φ is a monomorphism in k-Coalg. ■

In view of a remark of A. Chirvăsitu ([4]) that a morphism of Hopf algebras is a monomorphism if and only if it is a monomorphism when viewed as a morphism of coalgebras, we obtain the following useful fact:

Corollary 2.2. *Let $\varphi : K \to L$ be a Hopf algebra map. The following are equivalent:*

(1) *$\varphi : K \to L$ is a Hopf algebra monomorphism.*
(2) *The map $\eta_K = \Delta_K : K \to K \,\square_L\, K$ is surjective (hence bijective).*
(3) *$H^0(N, K) = H^0(N, L)$ for any (K, K)-bicomodule N.*
(4) *$\sum_{i\in I} \varepsilon(x^i)y^i = \sum_{i\in I} x^i\varepsilon(y^i)$ for all $\sum_{i\in I} x^i \otimes y^i \in K \,\square_L\, K$.*

EXAMPLE 2.3. Let $\pi : \mathcal{M}^2(k) \to \mathcal{M}^2(k)/I$ be the canonical projection, where k is a field and I is the coideal of the comatrix coalgebra $\mathcal{M}^2(k)$ generated by the elements c_{21}. It is proved in [8], using a result on epimorphisms of finite-dimensional algebras [8, Theorem 3.2], that π is a non-injective monomorphism of coalgebras. However, this can be easily shown by a simple computation using (7) of Theorem 2.1.

REMARK 2.4. The equivalence of the statements (1) and (4) in Theorem 2.1 can be alternatively proved by applying the categorical duality $(k\text{-Coalg})^{\mathrm{op}} \cong \mathcal{PC}_k$, $C \mapsto C^*$, between the category k-Coalg of k-coalgebras and the category $\mathcal{PC}_k$ of pseudocompact k-algebras described in [11, Section 3]. One should apply the isomorphism $(C \,\square\, C)^* \cong C^* \,\widehat{\otimes}\, C^*$ and the results of Knight ([7]), where $\widehat{\otimes}$ is the complete tensor product (see the monograph [5]).

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