SUBFIELDS OF HENSELIAN VALUED FIELDS

BY

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Abstract. Let (K, v) be a henselian valued field of arbitrary rank which is not separably closed. Let k be a subfield of K of finite codimension and v_k be the valuation obtained by restricting v to k. We give some necessary and sufficient conditions for (k, v_k) to be henselian. In particular, it is shown that if k is dense in its henselization, then (k, v_k) is henselian. We deduce some well known results proved in this direction through other considerations.

1. Introduction. Let v be a valuation of a field K. It is well known that if (K, v) is henselian (respectively complete of rank one), then every finite extension of (K, v) is henselian (respectively complete). In 2006, Bevelacqua and Motley [1] characterized those complete rank one valued fields (K, v) whose each subfield of finite-codimension is complete. They proved that if K is not an algebraically closed field, then every finite-codimensional subfield of K is complete in the v-adic topology if and only if either the characteristic of K is zero or the characteristic is $p > 0$ and $[K : K^p]$ is finite. This has led us to consider the following analogous question:

Let (K, v) be a henselian valued field of arbitrary rank. Is it true that every finite-codimensional subfield of (K, v) is henselian with respect to the valuation obtained by restricting v? If not, how can we characterize those finite-codimensional subfields of (K, v) which are henselian?

It is known that the answer to the first question is "yes" when K is not a separably closed field with rank v one (cf. [4, Theorem 4.4.4]) and is "no" in general (see Example 3.3 or [3, §3]). As regards the second question, Theorem 4.4.4 of [4] also provides a sufficient condition for each finite-codimensional subfield of (K, v) to be henselian when v is a valuation of arbitrary rank and K is not a separably closed field. If K is separably closed, then each valuation of K is henselian and if k is a finite-codimensional subfield of K, then either k is a separably closed field or it is a real closed field and $K = k(\sqrt{-1})$ (cf. [4, Theorem 4.3.5]). In the latter case, k is not a henselian subfield of (K, v) with respect to any non-trivial valuation v provided the ordering of k is

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archimedean (see Lemma 3.2 below). In this paper, we give some necessary and sufficient conditions for any finite-codimensional subfield of (K, v) to be henselian, when K is not separably closed. These conditions generate examples of non-henselian finite-codimensional subfields of henselian valued fields as shown in Example 3.3. Some results of Endler [2], Engler [3] and Engler and Prestel [4, Chapter 4] are also deduced in the course of proof of the main theorem.

In what follows, R_v will denote the valuation ring of a valuation v defined on K, and $\overline{K}_v$ the residue field of v. A valuation w of K is called a *coarsening* of v if $R_v \subseteq R_w$. If $R_v \neq R_w$, then the coarsening w of v is called *proper*. In this situation, the valuation v of K is composed of w and the valuation $\overline{v}$ induced by v on the residue field $\overline{K}_w$ of w; we will write v as $w \circ \overline{v}$.

With the above notations, we prove

THEOREM 1.1. *Let v be a henselian valuation of arbitrary rank of a field K which is not separably closed. Let k be a subfield of K of finite codimension with valuation denoted by v_k obtained by restricting the given valuation to k. Then the following statements are equivalent:*

(i) *(k, v_k) is henselian.*
(ii) *k is dense in the henselization k^h of (k, v_k).*
(iii) *For each valuation w of K which is a proper coarsening of $v = w \circ \overline{v}$, the residue field $\overline{k}_w$, of w restricted to k, is henselian with respect to the restriction of $\overline{v}$ to $\overline{k}_w$.*
(iv) *Whenever w is a proper coarsening of $v = w \circ \overline{v}$ such that the residue field of w is an algebraically closed field of characteristic zero, then the restriction of $\overline{v}$ to $\overline{k}_w$ has a unique prolongation to $\overline{K}_w$.*

The corollary stated below is an immediate consequence of statement (iv) of the theorem.

COROLLARY 1.2. *Let (K, v) be as in Theorem 1.1. If there exists no proper coarsening w of v such that the residue field of w is an algebraically closed field of characteristic zero, then each finite-codimensional subfield of (K, v) is henselian.*

In particular, the above corollary yields the following

COROLLARY 1.3. *Let (K, v) be as above. If the characteristic of K is $p > 0$ or the rank of v is one, then every finite-codimensional subfield of (K, v) is henselian.*

Recall that the residue field of a non-trivial valuation of a separably closed field is algebraically closed (cf. [4, Theorem 3.2.11]). Therefore with notations as in Theorem 1.1, if the residue field of a valuation v of K is not algebraically closed, then for any proper coarsening w of $v = w \circ \overline{v}$, the

residue field $\overline{K}_w$ of w is not separably closed, for otherwise the residue field of the non-trivial valuation $\overline{v}$ of $\overline{K}_w$ (which is same as the residue field of v) would be algebraically closed. Thus Corollary 1.2 yields the following result, which is proved in [4, Theorem 4.4.4] through other considerations.

COROLLARY 1.4. *Let (K, v) be as in Theorem* 1.1. *If the residue field of v is not algebraically closed, then any finite-codimensional subfield k of K is henselian with respect to the valuation obtained by restricting v to k.*

2. Preliminary results. For a valued field (K, v) of arbitrary rank, recall that the *v-adic topology* on K is the one for which a basis of neighbourhoods at each element a of K consists of the sets $\{b \in K \mid v(b-a) > \lambda\}$ where λ is in the value group of v.

LEMMA 2.1. *If K is not separably closed and (K, v) is henselian, then every field automorphism of K is continuous with respect to the v-adic topology.*

Proof. If σ is any field automorphism of K, then clearly the valuation $v \circ \sigma$ of K is henselian. Since K is not separably closed, it follows from Schmidt's Theorem [4, Theorem 4.4.1] that v and $v \circ \sigma$ are dependent valuations. Therefore v and $v \circ \sigma$ induce the same topology on K and so σ is continuous with respect to the v-adic topology.

PROPOSITION 2.2. *Let (K, v) be as in Lemma* 2.1 *and k be a subfield of K such that either k is the fixed field of the group G of automorphisms of K/k or K/k is a finite separable extension. Then k is a closed subset of K with respect to the v-adic topology.*

Proof. Suppose first that k is the fixed field of G. By Lemma 2.1, each σ in G is continuous with respect to the v-adic topology. Therefore the function $f_\sigma : K \mapsto K$ given by $f_\sigma(a) = \sigma(a) - a$ is continuous on K. Thus $k = \bigcap_{\sigma \in G} f_\sigma^{-1}(0)$ is closed because each $f_\sigma^{-1}(0)$ is closed in the Hausdorff space K and hence the result is proved in this case.

Suppose now that K/k is a finite separable extension. We first show that if K is a real closed field, then $K = k$. By Artin's Theorem [5, A.17] the algebraic closure $\widetilde{K}$ of K has degree two over K and any proper finite-codimensional subfield k' of $\widetilde{K}$ will be real closed with $\widetilde{K} = k'(\sqrt{-1})$, which proves that $K = k$. So it may be assumed that K is not real closed. Let E/k be the normal closure of K/k. Then E/k is a finite Galois extension. Note that E is not separably closed because otherwise K will be either a separably closed field or a real closed field in view of [4, Theorem 4.3.5] and these are not the cases under consideration. Since E is henselian with respect to the prolongation of v to E, it follows from the first case proved above that k is closed in E and hence in K as desired.

LEMMA 2.3. *Let K_1 be a purely inseparable extension of a valued field (K, v) and v_1 be the unique prolongation of v to K_1. If (K_1, v_1) is henselian, then so is (K, v).*

Proof. If (K, v) is not henselian, then there exists β in the algebraic closure of K such that v has more than one prolongation to $K(\beta)$ and hence v_1 has more than one prolongation to $K_1(\beta)$, which contradicts the henselian property of (K_1, v_1).

Keeping in mind that a rank one valued field is dense in its henselization, we see that the following well known result due to Endler [2] is an immediate consequence of Proposition 2.2 and Lemma 2.3.

COROLLARY 2.4. *Let v be a henselian rank one valuation of a field K which is not separably closed. If K/k is a normal extension (finite or infinite), then k is henselian.*

LEMMA 2.5. *Let (K, v) be as in Lemma 2.1 and k be a finite-codimensional subfield of K with valuation obtained by restricting v. Suppose that k is dense in the henselization k^h. Then k is henselian.*

Proof. As (K, v) is henselian, $k \subseteq k^h \subseteq K$. If K/k is a finite separable extension, then k is closed in K by Proposition 2.2 and hence closed in k^h. The hypothesis that k is dense in k^h now implies $k = k^h$ in this case.

Suppose now that K/k is not a separable extension. Let k^s denote the separable closure of k in K. Then K/k^s is a purely inseparable extension. So there exists an i such that $K^{p^i} \subseteq k^s$. By Lemma 2.3, K^{p^i} is henselian and hence k^s is henselian with respect to the restriction of v. Note that k^s is not a separably closed field, for otherwise its finite extension K will be separably closed, which is contrary to the hypothesis. By Proposition 2.2 applied to the extension k^s/k, k is closed in k^s.

Since k^h/k is a separable extension, we have

$$(1) \qquad\qquad k^h \subseteq k^s \subseteq K.$$

Using the hypothesis that k is dense in k^h and the fact that k is closed in k^s proved in the preceding paragraph, it follows from (1) that $k = k^h$ as desired.

3. Proof of Theorem 1.1. Clearly (i) implies (ii). Also (ii) implies (i) in view of Lemma 2.5. We shall prove that (i)⇒(iii)⇒(iv)⇒(i).

(i)⇒(iii). Let w be a proper coarsening of v with $v = w \circ \overline{v}$. Let w_k and $\overline{v}_k$ denote respectively the restrictions of w to k and of $\overline{v}$ to the residue field $\overline{k}_w$ of w_k, so that $v_k = w_k \circ \overline{v}_k$. Since (k, v_k) is henselian, $(\overline{k}_w, \overline{v}_k)$ must be henselian as stated in (iii).

Obviously (iii) implies (iv). Suppose now that (iv) holds and suppose to the contrary that (k, v_k) is not henselian. Then v_k has at least two prolongations v and v' to K because (K, v) is henselian. Let $\widetilde{v}, \widetilde{v'}$ be prolongations of v, v' respectively to the algebraic closure $\widetilde{K}$ of K (the prolongation of v' may not be unique). Since both $\widetilde{v}, \widetilde{v'}$ extend v_k, they are conjugates over k. So there exists an automorphism σ of $\mathrm{Gal}(\widetilde{K}/k)$ such that $\widetilde{v'} = \widetilde{v} \circ \sigma^{-1}$. As (K, v) is henselian, so is the isomorphic image $(\sigma(K), \widetilde{v} \circ \sigma^{-1}) = (\sigma(K), \widetilde{v'})$. Taking $L = K.\sigma(K)$ as the field compositum of K and $\sigma(K)$, we see that L is henselian with respect to v_L and v'_L, where v_L and v'_L are the restrictions of $\widetilde{v}, \widetilde{v'}$ respectively to L. Note that both v_L and v'_L being prolongations of v_k to L, none is a coarsening of the other. Let w be the smallest common coarsening of v_L and v'_L with valuation ring $R_w = R_{v_L}.R_{v'_L}$. Write $v_L = w \circ \overline{v}_L$, $v'_L = w \circ \overline{v'}_L$. We first show that $\overline{v}_L$ and $\overline{v'}_L$ are independent valuations on the residue field $\overline{L}_w$ of w, i.e.,

$$(2) \qquad \overline{L}_w = R_{\overline{v}_L}.R_{\overline{v'}_L}.$$

Let $a + \mathcal{M}_w \in \overline{L}_w = R_w/\mathcal{M}_w$, where $\mathcal{M}_w$ is the maximal ideal of R_w. Then a is in R_w. Therefore $a = \sum a_i a'_i$, $a_i \in R_{v_L}, a'_i \in R_{v'_L}$ and hence the w-residue $\overline{a}$ of a can be written as $\sum \overline{a}_i \overline{a'}_i$, which belongs to $R_{\overline{v}_L}.R_{\overline{v'}_L}$ proving (2). Therefore the residue field $\overline{L}_w$ of w, being henselian with respect to the independent valuations $\overline{v}_L$ and $\overline{v'}_L$, is separably closed by Schmidt's Theorem [4, Theorem 4.4.1]. Now, we have $\overline{k}_w \subseteq \overline{K}_w \subseteq \overline{L}_w$ and $\overline{L}_w$ is a separably closed field. Therefore by [4, Theorem 4.3.5], either $\overline{k}_w$ is a separably closed field or $\overline{k}_w$ is a real closed field with $\overline{L}_w$ an algebraically closed field of characteristic zero. The first possibility cannot occur because

$$(3) \qquad \overline{v} \text{ and } \overline{v}' \text{ are distinct prolongations of } \overline{v}_k \text{ to } \overline{K}_w.$$

If $\overline{L}_w$ is an algebraically closed field of characteristic zero, then by Artin's Theorem [5, A.17], either $\overline{k}_w = \overline{K}_w$ or $\overline{K}_w = \overline{L}_w$. But $\overline{k}_w = \overline{K}_w$ is not possible in view of (3). The case $\overline{K}_w = \overline{L}_w$ with $\overline{L}_w$ an algebraically closed field of characteristic zero cannot occur by virtue of condition (iv) of the theorem and (3). This contradiction completes the proof of the theorem.

The proof that (iv) implies (i) can be carried over verbatim to show that if (K, v) is henselian and K/k is a normal extension with (k, v_k) not henselian, then there exists a proper coarsening w of v such that the residue field $\overline{K}_w$ is a separably closed field of characteristic zero, and consequently the residue field of v will be algebraically closed by virtue of [4, Theorem 3.2.11]. Thus we obtain the following result of Engler proved in [3, Corollary 3.5].

Theorem 3.1. *Let v be a henselian non-trivial valuation of a field K whose residue field is not algebraically closed. Let K/k be a normal extension (finite or infinite). Then k is henselian with respect to the restriction of v to k.*

The following lemma will be used to construct examples of henselian valued fields having non-henselian subfields of finite-codimension. It can be deduced from Lemma 4.3.6 and Theorem 4.3.7 of [4]. For the reader's convenience, we give below a simple proof of the lemma.

Lemma 3.2. *Let R be a real closed field with respect to an archimedean ordering and $R' = R(\sqrt{-1})$ be its algebraic closure. Then R is not henselian with respect to any non-trivial valuation v' of R'.*

Proof. The proof is split in two cases.

Case 1: v' extends the p-adic valuation v_p of $\mathbb{Q}$ for some prime number p. Define $f(x) = x^2 - x + 2$ when $p = 2$, and if $p \neq 2$ then set $f(x) = x^2 + c$ where c is a positive integer such that $-c$ is a quadratic residue modulo p. Then $f(x)$ has distinct roots modulo p and consequently distinct roots in the residue field of v'. If R were henselian with respect to the restriction of v', then by Hensel's Lemma [4, Theorem 4.1.3], $f(x)$ would have a root in R, say α, which is impossible as neither $\alpha(1 - \alpha)$ can be 2 nor α^2 can be negative in view of R being a real field.

Case 2: v' is trivial on $\mathbb{Q}$. Let y belonging to R be a positive element in the maximal ideal $\mathcal{M}_{v'}$ of v'. Since the ordering is archimedean, there exists a positive integer r with $ry > 1$. Let n be a positive integer such that $n < ry < n + 1$. Define $f(x) = x^2 - (ry - n)(ry - n - 1)$. Then the polynomial $\overline{f}(x)$ obtained by replacing the coefficients of $f(x)$ modulo $\mathcal{M}_{v'}$ has two distinct roots in the residue field of v' restricted to R, viz. $\pm\sqrt{n(n+1)}$. But $f(x)$ has no root in R as $(ry - n)(ry - n - 1)$ is negative and thus R is not henselian.

Example 3.3. Let R be any real closed field with an archimedean ordering and $R' = R(\sqrt{-1})$ be its algebraic closure. Let $K = R'((t))$ be the field of Laurent series in one variable t over R'. Let v_t be the t-adic valuation of K trivial on R' characterized by $v_t(t) = 1$. Its residue field is R'. Let v' be any non-trivial valuation on R' and v be the valuation of K given by $v = v_t \circ v'$. Let $k = R((t))$ be the subfield of K of codimension two. By Lemma 3.2, R is not henselian with respect to the restriction of v' to R, which implies that k is not henselian with respect to the restriction of the henselian valuation $v = v_t \circ v'$ of K.

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