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A GENERALIZATION OF A THEOREM OF SCHINZEL

BY

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Abstract. We give lower bounds for the Mahler measure of totally positive algebraic integers. These bounds depend on the degree and the discriminant. Our results improve earlier ones due to A. Schinzel. The proof uses an explicit auxiliary function in two variables.

1. Introduction. Let $P = a_0X^d + \cdots + a_d$ be a polynomial in $\mathbb{C}[X]$ of degree $d \geq 1$. The *Mahler measure* of P is defined by

$$M(P) = |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|) \tag{1}$$

where $\alpha_1, \ldots, \alpha_d$ are the roots of P in $\mathbb{C}$. The *Mahler measure* of an algebraic number α is the Mahler measure of its minimal polynomial in $\mathbb{Z}[X]$. In this paper we prove a lower bound for the Mahler measure of some algebraic numbers. Kronecker's theorem implies that if $M(\alpha) = 1$ then α is a root of unity or 0. Lehmer's question: "does there exist a constant $c > 1$ such that, if $M(\alpha) > 1$, then $M(\alpha) > c$?" is open. Using the Cantor and Straus method [1], P. Voutier has proved [9] that if $M(\alpha) > 1$ and $d = \deg(\alpha) \geq 3$ then

$$M(\alpha) \geq 1 + \frac{1}{4}\left(\frac{\log\log d}{\log d}\right)^3. \tag{2}$$

Let Ω be the set of nonzero algebraic integers α of degree d such that $\deg(\alpha) = \deg(\alpha^p)$ for all primes p. For $\varepsilon > 0$, let $d^\star = \max(\delta(\alpha), \delta_0(\varepsilon))$ where $\delta(\alpha) = d/|\mathrm{disc}(\alpha)|^{1/d}$ and $\delta_0(\varepsilon) > 0$ depends only on ε. E. M. Matveev [5] proved that if $\alpha \in \Omega$, the inequality (2) is still valid if the constant $1/4$ is replaced by $2 - \varepsilon$ and d is replaced by $d^\star$.

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In 1973, A. Schinzel [7] proved that if α is a nonzero totally real algebraic integer, different from ± 1, of degree d then

$$M(\alpha) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}, \tag{3}$$

and if α is totally positive (i.e. all the conjugates of α are positive) then

$$M(\alpha) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d}. \tag{4}$$

We remark that for $\alpha = (3+\sqrt{5})/2$, which is a Pisot number of degree 2, the previous inequality becomes an equality. C. J. Smyth [8] has studied the set of $M(\alpha)^{1/\deg(\alpha)}$ for α totally real and for α totally positive. These results have been extended by V. Flammang [3].

Here we will restrict ourselves to totally positive algebraic integers and give lower bounds for their Mahler measure in terms of their discriminant. Using Hadamard's inequality it is easy to prove that, in this case, if $d = \deg(\alpha) \geq 2$ then

$$M(\alpha) \geq |\mathrm{disc}(\alpha)|^{1/2(d-1)}. \tag{5}$$

T. Zaïmi [10] has proved an inequality which implies

$$M(\alpha) \geq (|\mathrm{disc}(\alpha)|^{1/2\sqrt{5}(d-1)})\left(\frac{1+\sqrt{5}}{2}\right)^{d/2}. \tag{6}$$

We will prove the following:

THEOREM 1. *Let α be a totally positive algebraic integer of degree $d \geq 2$. Then*

$$M(\alpha) \geq \left(\frac{\delta+\sqrt{\delta^2+4}}{2}\right)^{d/2} \tag{7}$$

where $\delta = |\mathrm{disc}(\alpha)|^{1/d(d-1)}$.

REMARK. Inequality (7) becomes an equality when α is a totally positive Pisot number of degree 2 which is a unit.

The method of proof uses an auxiliary function in two variables. For a short proof of Schinzel's result by means of an auxiliary function in one variable see G. Höhn and N. P. Skoruppa [4]. For an inequality using also this method and giving a lower bound involving the norm of α see V. Flammang [2]. These results have been improved by L. Panaitopol [6]. Theorem 1 is a consequence of the more general Theorem 2:

THEOREM 2. *Let c be a real number $0 < c < 1$ and P be a monic polynomial of degree $d \geq 2$ with real coefficients whose roots are distinct* *positive real numbers. Put* $\delta = |\mathrm{disc}(P)|^{1/d(d-1)}$. *Then*

$$M(P)^{2/d} \geq \left(\delta\,\frac{1+c}{2c}\right)^{c}\left(|P(0)|^{2/d}\,\frac{1+c}{1-c}\right)^{(1-c)/2}. \tag{8}$$

Theorem 2 is proved in Section 2. In Section 3 we deduce Theorem 1 from Theorem 2 and prove that inequality (7) is better than (5) and (6) and that it is better than (4) when $\delta > \sqrt{5}$.

I am indebted to Chris Smyth for interesting discussions about the results of this paper.

2. Proof of Theorem 2. For $x > 0$, $y > 0$ and $x \neq y$ we consider the auxiliary function

$$g(x,y) = \log_+ x + \log_+ y - c\log|x-y| - \frac{1-c}{2}\log xy$$

where $\log_+ x = \log(\max(1,x))$. We need the following lemma:

LEMMA. *The function* g *satisfies the three properties*:

(i) $g(y,x) = g(x,y)$.

(ii) $g(1/x,1/y) = g(x,y)$.

(iii) $\displaystyle\min_{x>0,\,y>0,\,x\neq y} g(x,y) = c\log\frac{1+c}{2c} + \frac{1-c}{2}\log\frac{1+c}{1-c}$.

The definition of g implies (i), and (ii) is obtained by a straightforward computation. To prove (iii), it is sufficient to compute the minimum of $g(x,y)$ in the two regions

$$R_1 := \{(x,y) : 0 < y < x \leq 1\}, \qquad R_2 := \{(x,y) : 0 < y < 1 < x\}.$$

In the open region $R_1 \setminus \{x=1\}$ the conditions $\frac{\partial g}{\partial x}(x,y) = \frac{\partial g}{\partial y}(x,y) = 0$ give the equations

$$\frac{c}{x-y} + \frac{1-c}{2x} = 0, \qquad \frac{c}{y-x} + \frac{1-c}{2y} = 0.$$

We multiply these equations by $2x(x-y)$ and $2y(y-x)$ respectively to obtain

$$(1+c)x - (1-c)y = 0, \qquad (1-c)x - (1+c)y = 0.$$

This implies that $x = y$. The minimum of g in R_1 is then taken on the line $x = 1$, because $g(x,y)$ tends to ∞ when y or $|x-y|$ tends to 0. The function

$$g(1,y) = -c\log(1-y) - \frac{1-c}{2}\log y$$

attains its minimum

$$m = c\log\frac{1+c}{2c} + \frac{1-c}{2}\log\frac{1+c}{1-c}$$

at $y = (1-c)/(1+c)$.

In the region R_2 the conditions $\frac{\partial g}{\partial x}(x,y) = \frac{\partial g}{\partial y}(x,y) = 0$ give the equations

$$\frac{1}{x} - \frac{c}{x-y} - \frac{1-c}{2x} = 0, \qquad \frac{c}{y-x} + \frac{1-c}{2y} = 0.$$

This gives

$$(1+c)(x-y) - 2cx = 0, \qquad 2cy + (1-c)(y-x) = 0,$$

which is equivalent to

$$(1-c)x - (1+c)y = 0.$$

So, in R_2, the minimum of the function g is taken on the interval

$$\left\{ \left(t, \frac{1-c}{1+c}\, t \right) : 1 \le t \le \frac{1+c}{1-c} \right\}$$

because g tends to ∞ when y or $|x-y|$ tends to 0 and when x tends to infinity. A direct computation shows that for all t with $1 \le t \le (1+c)/(1-c)$,

$$g\left(t, \frac{1-c}{1+c}\, t \right) = m.$$

Now we can prove Theorem 2. Let $(\alpha_i)_{1\le i\le d}$ be the roots of P. Then

$$\sum_{\substack{1\le i,j\le d\\ i\ne j}} g(\alpha_i, \alpha_j) \ge d(d-1)m$$

and

$$2(d-1)\log M(P) - c\log|\operatorname{disc}(P)| - (d-1)(1-c)\log|P(0)| \ge d(d-1)m.$$

We divide both sides by $d(d-1)$ and take the exponential; then

$$M(P)^{2/d} \ge |\operatorname{disc}(P)|^{c/d(d-1)} |P(0)|^{(1-c)/d} \left(\frac{1+c}{2c} \right) \left(\frac{1+c}{1-c} \right)^{(1-c)/2}$$

and

$$M(P)^{2/d} \ge \left(\frac{1+c}{2c}\, \delta \right)^{c} \left(\frac{1+c}{1-c}\, |P(0)|^{2/d} \right)^{1-c/2}.$$

3. Proof of Theorem 1. We suppose now that P is the minimal polynomial of a totally positive algebraic number of degree $d \ge 2$. Then we may apply Theorem 2 to P. Put

$$\varrho = |P(0)|^{2/d}, \qquad c = \frac{\delta}{\sqrt{\delta^2 + 4\varrho}}, \qquad \omega = \frac{\delta + \sqrt{\delta^2 + 4\varrho}}{2}.$$

Then

$$\varrho \ge 1, \qquad 0 < c < 1, \qquad \frac{1+c}{2c} = \frac{\omega}{\delta}, \qquad \frac{1+c}{1-c} = \frac{\omega^2}{\varrho}.$$

Inequality (8) gives

$$M(P)^{2/d} \geq \omega^c(\omega^2)^{(1-c)/2} = \omega \geq \frac{\delta + \sqrt{\delta^2+4}}{2}.$$

Now we prove that Theorem 1 implies (5), (6) and (4) when $\delta \geq \sqrt{5}$. Since $(\delta + \sqrt{\delta^2+4})/2 > \delta$, Theorem 1 implies (5). To prove that Theorem 1 implies (6) it is sufficient to prove that

$$\frac{\delta + \sqrt{\delta^2+4}}{2} > \delta^{1/\sqrt{5}}\,\frac{1+\sqrt{5}}{2}.$$

But one may obtain inequality (6) (but not a strict inequality) directly from Theorem 2 with $c = 1/\sqrt{5}$ and $|P(0)| \geq 1$. The term $(\delta + \sqrt{\delta^2+4})/2$ is an increasing function of $\delta \geq 1$ which is equal to $((1+\sqrt{5})/2)^2$ when $\delta = \sqrt{5}$. So inequality (7) is better than (4) when $\delta > \sqrt{5}$.

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