ON MULTIPLE SOLUTIONS OF THE NEUMANN PROBLEM 
INVOLVING THE CRITICAL SOBOLEV EXPONENT

BY

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Abstract. We consider the Neumann problem involving the critical Sobolev exponent and a nonhomogeneous boundary condition. We establish the existence of two solutions. We use the method of sub- and supersolutions, a local minimization and the mountain-pass principle.

1. Introduction. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with a smooth boundary \( \partial \Omega \). We consider the Neumann problem

\[
\begin{align*}
-\Delta u + \lambda u &= Q(x)u^{2^*-1} \quad \text{in } \Omega, \\
\partial u/\partial v &= \phi(x) \quad \text{on } \partial \Omega, \quad u > 0 \quad \text{on } \Omega,
\end{align*}
\]

where \( \lambda > 0 \) is a parameter and \( 2^* = 2N/(N-2), \ N \geq 3, \) is the critical Sobolev exponent. We assume that \( Q(x) > 0 \) on \( \overline{\Omega} \), \( \phi(x) \geq 0 \) and \( \phi(x) \neq 0 \) on \( \partial \Omega \) and moreover \( Q \in C^\alpha (\overline{\Omega}) \) and \( \phi \in C^\alpha (\partial \Omega) \).

In the case where \( Q \equiv 1 \) and \( \phi \equiv 0 \), problem (1.1) has an extensive literature. We refer to papers [2], [3], [8] and [9], where further references can be found. In this case solutions of (1.1) have been obtained as minimizers of the constrained variational problem

\[
m_\lambda = \inf_{u \in H^1(\Omega) - \{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx}{(\int_\Omega |u|^2 \, dx)^{2/2^*}}.
\]

A suitable multiple of a minimizer \( u \) for \( m_\lambda \) is a solution of (1.1) and is called the least energy solution of this problem. The main ingredient in the proof of the existence of the least energy solution is the inequality \( m_\lambda < S/2^{2/N} \), which is valid for every \( \lambda \), provided \( \Omega \) is smooth and bounded. Here \( S \) is the best Sobolev constant. This inequality allows us to show that every minimizing sequence for \( m_\lambda \) is relatively compact in \( H^1(\Omega) \). These results have been extended to the case \( Q \neq \text{const} \) and \( \phi \equiv 0 \) (see [8] and [9]). In this situation the existence of least energy solutions depends on the relationship between the global maximum \( Q_M = \max_{x \in \Omega} Q(x) \) and \( Q_m = \max_{x \in \partial \Omega} Q(x) \).

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The authors of these papers studied two cases: (i) \( Q_M \leq 2^{2/(N-2)}Q_m \) and (ii) \( Q_M > 2^{2/(N-2)}Q_m \). In the first case problem (1.1\( _\lambda \)) has the least energy solution for every \( \lambda > 0 \), provided \( Q_m \) is achieved at a point \( x_0 \in \partial\Omega \) with a positive mean curvature. In case (ii), the least energy solutions exist only for \( \lambda \in (0, \Lambda] \), \( \Lambda > 0 \). For \( \lambda > \Lambda \) problem (1.1\( _\lambda \)) does not have the least energy solutions.

The main purpose of this paper is to establish an existence result for problem (1.1\( _\lambda \)) which involves a nonzero boundary data \( \phi \). We show that the presence of \( \phi \neq 0 \) generates the existence of at least two solutions. Results of this nature are known in the cases where a nonhomogeneous term appears in the nonlinear equation ([7], [6] and [13]).

Under an additional assumption on \( Q \) we establish the existence of a constant \( \lambda_* > 0 \) such that for \( \lambda > \lambda_* \) problem (1.1\( _\lambda \)) has at least two solutions, at least one solution for \( \lambda = \lambda_* \) and no solution for \( \lambda < \lambda_* \). In the case where \( \lambda > \lambda_* \) the existence of one solution will be established through the method of sub- and supersolutions. A second solution will be obtained via the mountain-pass principle. These existence results are presented in Sections 2, 3 and 4. In these sections we do not impose any restriction on \( \|\phi\|_{L^2(\partial\Omega)} \). In Section 5 we show that if \( \|\phi\|_{L^2(\partial\Omega)} \) is of order \( \lambda \) (as small as \( \lambda \)), then problem (1.1\( _\lambda \)) has at least two solutions. Section 6 is devoted to the case \( \lambda = 0 \).

In this paper we use standard notations. In a given Banach space \( X \) we denote strong convergence by “\( \rightharpoonup \)” and weak convergence by “\( \rightharpoonup^\ast \)”. We recall that a \( C^1 \)-functional \( \Phi : X \to \mathbb{R} \) on a Banach space \( X \) satisfies the *Palais–Smale condition at level \( c \) (\( (PS)_c \) condition for short) if each sequence \( \{x_m\} \) such that

\[
(*) \quad \Phi(x_m) \to c \quad \text{and} \quad (**) \quad \Phi'(x_m) \to 0 \quad \text{in} \quad X^* 
\]

is relatively compact in \( X \). Finally, any sequence satisfying (*) and (**) is called a *Palais–Smale sequence at level \( c \) (a \( (PS)_c \) sequence for short).

The norms in the Lebesgue spaces \( L^q(\Omega) \) will be denoted by \( \| \cdot \|_q \).

### 2. Sub- and supersolutions.

To construct a supersolution to problem (1.1\( _\lambda \)) we need the solution of the problem

\[
(2.1\lambda) \quad \begin{cases} -\Delta v + \lambda v = 0 & \text{in} \ \Omega, \\ \partial v/\partial n = \phi(x) & \text{on} \ \partial\Omega. \end{cases}
\]

This problem has a unique positive solution \( v_\lambda \in C^{1,\alpha}(\overline{\Omega}) \). Let \( v_1 \) be a solution of (2.1\( _\lambda \)) with \( \lambda = 1 \). We set

\[
\lambda_0 = \max_{x \in \Omega} Q(x)v_1(x)^{2^*-2} + 1.
\]

We then have
\[- \Delta v_1 + \lambda_0 v_1 - Q(x)v_1^{2^*-1} = - \Delta v_1 + (\lambda_0 - Q(x)v_1^{2^*-2})v_1 \]
\[= - \Delta v_1 + \max_{x \in \Omega} Q(x)v_1(x)^{2^*-2} + 1 - Q(x)v_1(x)^{2^*-2})v_1 \]
\[\geq - \Delta v_1 + v_1 = 0.\]

Hence \( \overline{u} = v_1 \) is a supersolution for (1.1\(\lambda_0 \)). Since \( u = 0 \) is a subsolution for (1.1\(\lambda_0 \)), there exists a minimal solution \( u_{\lambda_0} \) of (1.1\(\lambda_0 \)) satisfying
\[ u < u_{\lambda_0} < \overline{u} \text{ on } \Omega.\]

Let
\[ S = \{ \lambda; (1.1\lambda) \text{ has a positive solution} \} \]

If \( \lambda > \lambda_0 \), then \( u_{\lambda_0} \) is a supersolution to (1.1\(\lambda \)). Indeed, we have
\[ \left\{ \begin{array}{l}
- \Delta u_{\lambda_0} + \lambda u_{\lambda_0} > - \Delta u_{\lambda_0} + \lambda_0 u_{\lambda_0} = Q(x)u_{\lambda_0}^{2^*-1} \quad \text{in } \Omega, \\
\partial u_{\lambda_0} / \partial \nu = \phi(x) \quad \text{on } \partial \Omega.
\end{array} \right. \]

As before, since \( u = 0 \) is a subsolution, there exists a minimal solution \( u_\lambda \) satisfying
\[ u < u_\lambda < \overline{u} = u_{\lambda_0}.\]

This argument shows that \((\lambda_0, \infty) \subset S\). We set
\[ (2.1) \quad \lambda_* = \inf_{\lambda \in S} \lambda. \]

Repeating the above argument we show that for every \( \lambda > \lambda_* \) problem (1.1\(\lambda \)) has a solution. If \( u_\lambda > 0 \) is a solution of (1.1\(\lambda \)), then
\[ \int_\Omega \lambda u_\lambda \, dx - \int_{\partial \Omega} \phi(x) \, dS_x = \int_\Omega Q(x)u_\lambda^{2^*-1} \, dx. \]

This yields \( \lambda > 0 \) and consequently \( \lambda_* \geq 0 \).

Let \( \lambda > \lambda_* \) and let \( u_\lambda \) be a positive solution of (1.1\(\lambda \)). We now consider the variational problem
\[ (2.2) \quad \mu_\lambda = \inf \left\{ \int_\Omega (|\nabla v|^2 + \lambda v^2) \, dx; \ v \in H^1(\Omega), \right\} \]
\[ (2^*-1) \int_\Omega Q(x)u_\lambda^{2^*-2} v^2 \, dx = 1. \]

**Proposition 2.1.** If \( \lambda > \lambda_* \), then the constant \( \mu_\lambda \) defined by (2.2) satisfies \( \mu_\lambda > 1 \). Moreover, problem (2.2) has a minimizer \( V_\lambda \) which is the first eigenfunction of the problem
\[ (2.3) \quad \left\{ \begin{array}{l}
- \Delta v + \lambda v = \mu_\lambda(2^*-1)Q(x)u_\lambda^{2^*-2} v \quad \text{in } \Omega, \\
\partial v / \partial \nu = 0 \quad \text{on } \partial \Omega.
\end{array} \right. \]
Proof. Since the functional \( v \mapsto \int_\Omega Q(x) u_\lambda^{2^*_v-2} v^2 \, dx \) is completely continuous on \( H^1(\Omega) \), the existence of a minimizer easily follows. We show that \( \mu_\lambda > 0 \). Let \( \overline{\lambda} > \lambda \) and let \( u_\overline{\lambda} \) and \( u_\lambda \) be the corresponding minimal solutions of (1.1_\overline{\lambda}) and (1.1_\lambda), respectively. It follows from the construction of \( \{u_\lambda\} \) that \( u_\lambda > u_\overline{\lambda} > 0 \). We then have

\[
(2.4) \quad -\Delta(u_\lambda - u_\overline{\lambda}) + \overline{\lambda}(u_\lambda - u_\overline{\lambda})
\]

for some \( 0 < \theta < 1 \). Let \( V_\overline{\lambda} \) be the first eigenfunction of problem (2.3) with \( \lambda = \overline{\lambda} \). Since

\[
\frac{\partial}{\partial \nu}(u_\lambda - u_\overline{\lambda}) = 0 \quad \text{on} \; \partial \Omega ,
\]

testing (2.4) with \( V_\overline{\lambda} \) and integrating by parts gives

\[
\int _\Omega (u_\lambda - u_\overline{\lambda})(-\Delta V_\overline{\lambda} + \overline{\lambda} V_\overline{\lambda}) \, dx > (2^* - 1) \int _\Omega Q(x) u_\overline{\lambda}^{2^*_v-2}(u_\lambda - u_\overline{\lambda}) V_\overline{\lambda} \, dx .
\]

Hence

\[
\mu_\lambda(2^* - 1) \int _\Omega Q(x)(u_\lambda - u_\overline{\lambda})u_\overline{\lambda}V_\overline{\lambda} \, dx > (2^* - 1) \int _\Omega Q(x) u_\overline{\lambda}^{2^*_v-2}(u_\lambda - u_\overline{\lambda})V_\overline{\lambda} \, dx
\]

and the assertion follows. \( \blacksquare \)

Let \( Q_* = \min _{x \in \Omega} Q(x) \).

**Lemma 2.2.** Let \( u_\lambda \) be a solution of problem (1.1_\lambda) for some \( \lambda > 0 \). Then

\[
\lambda^{(N+2)/4} \geq Q_*^{(N+2)/4} \int _\Omega Q(x) u_\lambda^{2^*_v-1} \, dx + \frac{N+2}{4} \int _{\partial \Omega} \phi(x) \, dS_x.
\]

**Proof.** Integrating (1.1_\lambda) we get

\[
(2.5) \quad \lambda \int _\Omega u_\lambda \, dx = \int _\Omega Q(x) u_\lambda^{2^*_v-1} \, dx + \int _{\partial \Omega} \phi(x) \, dS_x .
\]

It then follows from the Young inequality that

\[
\lambda \int _\Omega u_\lambda \, dx \leq \lambda Q_*^{-1} \int _\Omega Q(x) u_\lambda \, dx \leq \frac{2^* - 2}{2^* - 1} \lambda^{\frac{2^*_v-1}{2^* - 2}} Q_*^{-\frac{2^*_v-1}{2^* - 2}} \int _\Omega Q(x) \, dx + \frac{1}{2^* - 1} \int _\Omega Q(x) u_\lambda^{2^*_v-1} \, dx.
\]
This combined with (2.5) gives
\[
\frac{2^* - 2}{2^* - 1} \int_\Omega Q(x) u_\lambda^{2^*-1} dx + \int_{\partial \Omega} \phi(x) dS_x \leq \frac{2^* - 2}{2^* - 1} \lambda^{\frac{2^*-1}{2^* - 2}} Q_\star^{\frac{2^*-1}{2^* - 2}} \int_\Omega Q(x) dx
\]
and the result easily follows. 

**Corollary 2.3.** If
\[
\lambda^{(N+2)/4} \leq Q_\star^{(N+2)/4} \frac{\int_{\partial \Omega} \phi(x) dS_x}{\int_\Omega Q(x) dx},
\]
then problem \((1.1_\lambda)\) has no solution. Consequently, \(\lambda_\ast > 0\), where \(\lambda_\ast\) is the constant defined by (2.1).

In Proposition 2.4 below, we derive an estimate for \(\|u_\lambda\|_{H^1}\) in terms of the parameter \(\lambda\) and norms of \(v_1\).

**Proposition 2.4.** Solutions of \((1.1_\lambda)\) for \(0 < \lambda \leq 1\) satisfy the estimate
\[
\|u_\lambda\|_{H^1}^2 \leq L(\|v_1\|_{H^1}^2 + \|v_1\|_2^2 + (1 - \lambda)\|v_1\|_2^2 + (1 - \lambda)^{N/2})
\]
and for \(\lambda > 1\) we have
\[
\|u_\lambda\|_{H^1}^2 \leq L_1
\]
for some constants \(L > 0\) and \(L_1 > 0\) independent of \(\lambda\).

*Proof.* Let \(u_\lambda\) be a solution of \((1.1_\lambda)\) and \(v_1\) be a solution of \((2.1)\). We set \(v = u_\lambda - v_1\). Then \(v\) satisfies
\[
\begin{cases}
-\Delta v + v = Q(x)(v + v_1)^{2^*-1} + (1 - \lambda)(v + v_1) & \text{in } \Omega, \\
\partial v / \partial n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(2.6) First we consider the case \(0 < \lambda \leq 1\). By the maximum principle \(v > 0\) on \(\Omega\). Testing (2.6) with \(v\) we get
\[
\int_\Omega \left( |\nabla v|^2 + v^2 \right) dx = \int_\Omega Q(x)(v + v_1)^{2^*-1} v dx + (1 - \lambda) \int_\Omega (v + v_1)v dx.
\]
Since \(0 < \lambda \leq 1\), it follows from Proposition 2.1 that
\[
(2^* - 1) \int_\Omega Q(x) u_\lambda^{2^*-2} v^2 dx 
\]
\[
\leq \int_\Omega Q(x)(v + v_1)^{2^*-1} v dx + (1 - \lambda) \int_\Omega (v + v_1)v dx 
\]
\[
= \int_\Omega Q(x)(v + v_1)^{2^*-2} v^2 dx + \int_\Omega Q(x)(v + v_1)^{2^*-2} v v_1 dx 
\]
\[
+ (1 - \lambda) \int_\Omega (v + v_1)v dx.
\]
Thus

\[(2.7) \quad (2^* - 2) \int_\Omega Q(x)(v + v_1)^{2^*-2}v^2 \, dx \]

\[
\leq \int_\Omega Q(x)(v + v_1)^{2^*-2}vv_1 \, dx + (1 - \lambda) \int_\Omega (v + v_1)v \, dx \\
\leq \int_\Omega Q(x)(v + v_1)^{2^*-1}v_1 \, dx + (1 - \lambda) \int_\Omega (v + v_1)v \, dx \\
\leq 2^{2^*-2} \int_\Omega Q(x)v_1^{2^*-1}v_1 \, dx + 2^{2^*-2} \int_\Omega Q(x)v_1^{2^*} \, dx \\
+ (1 - \lambda) \int_\Omega v^2 \, dx + (1 - \lambda) \int_\Omega v_1v \, dx \\
\leq 2^{2^*-2} \int_\Omega Q(x)v_1^{2^*-1}v_1 \, dx + 2^{2^*-2} \int_\Omega Q(x)v_1^{2^*} \, dx \\
+ 2(1 - \lambda) \int_\Omega v^2 \, dx + (1 - \lambda) \int_\Omega v_1^2 \, dx.
\]

Using the Young inequality we get for \( \varepsilon > 0, \)

\[(2.8) \quad \int_\Omega Q(x)v_1^{2^*-1}v_1 \, dx \leq \varepsilon \int_\Omega Q(x)v_1^{2^*} \, dx + C(\varepsilon) \int_\Omega Q(x)v_1^{2^*} \, dx,\]

\[(2.9) \quad 2(1 - \lambda) \int_\Omega v^2 \, dx \leq \varepsilon \int_\Omega v_1^{2^*} \, dx + C_1(\varepsilon)(1 - \lambda)^{2^*/(2^* - 2)}|\Omega|,\]

for some constants \( C(\varepsilon) > 0 \) and \( C_1(\varepsilon) > 0. \) Letting \( Q_M = \max_{x \in \Omega_1} Q(x) \) we deduce from (2.7)–(2.9) that

\[
((2^* - 2)Q_M - 2^{2^*-2}Q_M \varepsilon - \varepsilon) \int_\Omega v_1^{2^*} \, dx \leq (2^{2^*-2} + C(\varepsilon)) \int_\Omega Q(x)v_1^{2^*} \, dx \\
+ C_1(\varepsilon)(1 - \lambda)^{2^*/(2^* - 2)}|\Omega| + (1 - \lambda) \int_\Omega v_1^2 \, dx.
\]

Choosing \( \varepsilon > 0 \) small enough we derive from this the estimate

\[(2.10) \quad \int_\Omega v^{2^*} \, dx \leq C \left[ \int_\Omega v_1^{2^*} \, dx + (1 - \lambda)^{2^*/(2^* - 2)} + (1 - \lambda) \int_\Omega v_1^2 \, dx \right].\]

We now use (2.10) to estimate \( \|v\|_{H^1}^2 \) in terms of \( \lambda \) and \( v_1. \) We have

\[
\int_\Omega (|\nabla v|^2 + v^2) \, dx = \int_\Omega Q(x)(v + v_1)^{2^*-1}v \, dx + (1 - \lambda) \int_\Omega v^2 \, dx \\
+ (1 - \lambda) \int_\Omega v_1v \, dx
\]
\[
\begin{align*}
&\leq 2^{2r-2} \int_{\Omega} Q(x)v^{2r} \, dx + 2^{2r-2} \int_{\Omega} Q(x)v_1^{2r-1} v \, dx \\
&\quad + 2(1 - \lambda) \int_{\Omega} v^2 \, dx + (1 - \lambda) \int_{\Omega} v_1^2 \, dx \\
&\leq 2^{2r-3}(2^r + 1) \int_{\Omega} Q(x)v^{2r} \, dx + 2^{2r-3}(2^r - 1) \int_{\Omega} Q(x)v_1^{2r} \, dx \\
&\quad + 2(1 - \lambda) \int_{\Omega} v^2 \, dx + (1 - \lambda) \int_{\Omega} v_1^2 \, dx.
\end{align*}
\]

The last estimate combined with (2.9) and (2.10) gives
\[
\int_{\Omega} (|\nabla v|^2 + v^2) \, dx \leq C_1 \left[ \int_{\Omega} v_1^{2r} \, dx + (1 - \lambda) \int_{\Omega} v_1^2 \, dx + (1 - \lambda)^{2r/(2^r - 2)} \right],
\]
where \( C_1 > 0 \) is of the same nature as \( C \) in (2.10). Since \( \|u_\lambda\|_{H^1} \leq \|v\|_{H^1} + \|v_1\|_{H^1} \) the result in the case \( 0 < \lambda \leq 1 \) readily follows. If \( \lambda > 1 \), then \( u_\lambda \leq u_1 \) on \( \Omega \), where \( u_1 \) is a minimal solution of problem (1.1). Thus
\[
\int_{\Omega} (|\nabla u_\lambda|^2 + u_\lambda^2) \, dx \leq \int_{\Omega} (|\nabla u_\lambda|^2 + \lambda u_\lambda^2) \, dx = \int_{\Omega} Q(x)u_\lambda^2 \, dx + \int_{\Omega} \phi(x)u_\lambda \, dS_x
\]
\[
\leq Q_M \int_{\Omega} u_\lambda^2 \, dx + \int_{\Omega} \phi(x)u_1 \, dS_x,
\]
and the result follows.

**Proposition 2.5.** Problem (1.1) has a solution.

**Proof.** Let \( \lambda_n \to \lambda^* \) and \( \lambda_n > \lambda^* \) for each \( n \). By Proposition 2.4 the sequence \( \{u_{\lambda_n}\} \) of the corresponding solutions is bounded in \( H^1(\Omega) \). It is routine to show that up to a subsequence \( u_{\lambda_n} \rightharpoonup u \) in \( H^1(\Omega) \) and \( u \) is a solution of problem (1.1). Thus

**Proposition 3.1.** Let \( \lambda > \lambda_+ \). There exist constants \( \alpha > 0 \) and \( \varrho > 0 \) such that \( J_\lambda(v) \geq \alpha \) for \( v \in H^1(\Omega) \) with \( \|v\|_{H^1} = \varrho \).
Proof. We write \( J_\lambda \) in the form
\[
J_\lambda(v) = \frac{1}{2} \int_\Omega \left( |\nabla v|^2 + \lambda v^2 \right) \, dx - \frac{2^* - 1}{2} \int_\Omega Q(x) u_\lambda^{2^*-2} (v^+)^2 \, dx
\]
\[
- \int_0^{v^+} \int_\Omega Q(x)[(u_\lambda + s)^{2^*-1} - u_\lambda^{2^*-1} - (2^* - 1)u_\lambda^{2^*-2}s] \, ds \, dx.
\]
Since for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
(u_\lambda + s)^{2^*-1} - u_\lambda^{2^*-1} - (2^* - 1)u_\lambda^{2^*-2}s \leq \varepsilon u_\lambda^{2^*-2}s + C_\varepsilon s^{2^*-1},
\]
we get
\[
J_\lambda(v) \geq \frac{1}{2} \int_\Omega \left( |\nabla v|^2 + \lambda v^2 - (2^* - 1)Q(x) u_\lambda^{2^*-2}(v^+)^2 \right) \, dx
\]
\[- \int_\Omega Q(x) \left[ \frac{\varepsilon}{2} u_\lambda^{2^*-2}(v^+)^2 + C_\varepsilon \frac{(v^+)^2}{2^*} \right] \, dx.
\]
Hence by Proposition 2.1 we have
\[
J_\lambda(v) \geq \frac{1}{2} \left( 1 - \frac{2^* - 1 - \varepsilon}{\mu_\lambda(2^* - 1)} \right) \int_\Omega (|\nabla v|^2 + \lambda v^2) \, dx - \frac{C_\varepsilon}{2^*} \int_\Omega Q(x)(v^+)^2 \, dx.
\]
We choose \( 0 < \varepsilon < 2^*-1 \). An application of the Sobolev inequality completes the proof. \( \blacksquare \)

Lemma 3.2. Let \( \{u_m\} \subset H^1(\Omega) \) be a Palais–Smale sequence for \( J_\lambda \). Then \( \{u_m\} \) is bounded in \( H^1(\Omega) \).

Proof. We compute
\[
J_\lambda(u_m) - \frac{1}{2} \langle J_\lambda'(u_m), u_m \rangle = -\frac{1}{2^*} \int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*} \, dx + \frac{1}{2^*} \int_\Omega Q(x) u_\lambda^{2^*} \, dx + \int_\Omega Q(x) u_\lambda^{2^*-1} u_m^+ \, dx
\]
\[
+ \frac{1}{2} \int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1} u_m \, dx - \frac{1}{2} \int_\Omega Q(x) u_\lambda^{2^*-1} u_m \, dx
\]
\[
= \frac{1}{N} \int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*} \, dx - \frac{1}{2} \int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1} u_m \, dx
\]
\[- \frac{1}{2} \int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1} u_\lambda \, dx
\]
\[
+ \frac{1}{2^*} \int_\Omega Q(x) u_\lambda^{2^*} \, dx + \int_\Omega Q(x) u_\lambda^{2^*-1} u_m^+ \, dx - \frac{1}{2} \int_\Omega Q(x) u_\lambda^{2^*-1} u_m \, dx
\]
\[ = \frac{1}{N} \int_\Omega Q(x)(u_\lambda + u^+_m)^2 \, dx - \frac{1}{2} \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*-1} u_\lambda \, dx \]
\[ - \frac{1}{2} \int_\Omega Q(x)u^{2^*-1}_\lambda u_m \, dx + \frac{1}{2^*} \int_\Omega Q(x)u^2_\lambda \, dx + \int_\Omega Q(x)u^{2^*-1}_\lambda u_m^+ \, dx \]
\[ - \frac{1}{2} \int_\Omega Q(x)u^{2^*-1}_\lambda u_m \, dx \]
\[ = \frac{1}{N} \int_\Omega Q(x)(u_\lambda + u^+_m)^2 \, dx - \frac{1}{2} \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*-1} u_\lambda \, dx \]
\[ + \frac{1}{2} \int_\Omega Q(x)u^{2^*-1}_\lambda u_m^+ \, dx + \frac{1}{2^*} \int_\Omega Q(x)u^2_\lambda \, dx. \]

By the Young inequality given \( \delta > 0 \) we choose \( C(\delta) > 0 \) so that
\[
(3.3) \quad \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*-1} u_\lambda \, dx 
\leq \delta \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*} \, dx + C(\delta) \int_\Omega Q(x)u^2_\lambda \, dx.
\]

Taking \( \delta > 0 \) small enough and using the fact that \( \{u_m\} \) is a \((PS)_c\) sequence we derive from (3.1) and (3.3) that
\[
(3.4) \quad \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*} \, dx \leq C_1 + C_2 \|u_m\|_{H^1}
\]
for every \( m \geq 1 \). On the other hand, we have
\[
J_\lambda(u_m) - \frac{1}{2^*}(J'_\lambda(u_m), u_m)
\]
\[
= \frac{1}{N} \int_\Omega (|\nabla u_m|^2 + \lambda u_m^2) \, dx + \frac{1}{2^*} \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*-1}(u_m - u^+_m - u_\lambda) \, dx \]
\[ + \frac{1}{2^*} \int_\Omega Q(x)u^2_\lambda \, dx + \int_\Omega Q(x)u^{2^*-1}_\lambda u_m^+ \, dx - \frac{1}{2^*} \int_\Omega Q(x)u^{2^*-1}_\lambda u_m \, dx \]
\[
= \frac{1}{N} \int_\Omega (|\nabla u_m|^2 + \lambda u_m^2) \, dx - \frac{1}{2^*} \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*-1} u_\lambda \, dx \]
\[ - \frac{1}{2^*} \int_\Omega Q(x)(u_\lambda + u^+_m)^{2^*-1} u_m^- \, dx + \frac{1}{2^*} \int_\Omega Q(x)u^2_\lambda \, dx \]
\[ + \int_\Omega Q(x)u^{2^*-1}_\lambda u_m^+ \, dx - \frac{1}{2^*} \int_\Omega Q(x)u^{2^*-1}_\lambda u_m \, dx \]
\[
\frac{1}{N} \int_{\Omega} (|\nabla u_m|^2 + \lambda u_m^2) \, dx - \frac{1}{2^*} \int_{\Omega} Q(x)(u_\lambda + u_m^{+})^{2^{*}-1} u_\lambda \, dx \\
+ \frac{1}{2^*} \int_{\Omega} Q(x) u_\lambda^{2^*} \, dx + \left( 1 - \frac{1}{2^*} \right) \int_{\Omega} Q(x) u_m^{2^{*}-1} u_m^{+} \, dx \\
\geq \frac{1}{N} \int_{\Omega} (|\nabla u_m|^2 + \lambda u_m^2) \, dx - \frac{1}{2^*} \int_{\Omega} Q(x)(u_\lambda + u_m^{+})^{2^{*}-1} u_\lambda \, dx.
\]

From this we deduce, using the Young inequality, that

\[(3.5) \quad \|u_m\|_{H^1}^2 \leq C_3 \int_{\Omega} Q(x)(u_\lambda + u_m^{+})^{2^*} \, dx + C_4 \|u_m\|_{H^1} + C_5.\]

The assertion follows from (3.3) and (3.5).

To proceed further we set

\[Q_m = \max_{x \in \partial \Omega} Q(x).\]

We recall that \(Q_M\) is defined by \(Q_M = \max_{x \in \Omega} Q(x)\). By \(S\) we denote the best Sobolev constant, that is,

\[S = \inf_{u \in D^{1,2}(\mathbb{R}^N) - \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} |u|^{2^*} \, dx)^{2/2^*}},\]

where \(D^{1,2}(\mathbb{R}^N)\) is the Sobolev space defined by \(D^{1,2}(\mathbb{R}^N) = \{u; \nabla u \in L^2(\mathbb{R}^N), u \in L^{2^*}(\mathbb{R}^N)\}\). The best Sobolev constant is achieved by

\[U(x) = \frac{c_N}{(N(N-2) + |x|^2)^{(N-2)/2}},\]

where \(c_N > 0\) is a constant depending on \(N\). The function \(U\), called an instanton, satisfies the equation

\[-\Delta U = U^{2^*-1} \quad \text{in} \quad \mathbb{R}^N.\]

We have \(\int_{\mathbb{R}^N} |\nabla U|^2 \, dx = \int_{\mathbb{R}^N} U^{2^*} \, dx = S^{N/2}\). For future use we introduce the notation

\[U_{\varepsilon,y} = \varepsilon^{-(N-2)/2} U\left( \frac{x - y}{\varepsilon} \right), \quad y \in \mathbb{R}^N, \ \varepsilon > 0.\]

We set

\[S_\infty = \min\left( \frac{S^{N/2}}{2 N Q_m^{(N-2)/2}}, \frac{S^{N/2}}{N Q_M^{(N-2)/2}} \right).\]

**Proposition 3.3.** Let \(\lambda > \lambda_\ast\). Suppose that

\[(3.6) \quad J_\lambda(u_m) \to c < S_\infty,\]

\[(3.7) \quad J'_\lambda(u_m) \to 0 \quad \text{in} \quad H^{-1}(\Omega).\]

Then up to a subsequence \(u_m \to v \neq 0\) and \(v\) is a solution of problem (3.1\(\lambda\)).
Proof. By Lemma 3.2, \( \{u_m\} \) is bounded in \( H^1(\Omega) \). Hence we may assume that \( u_m \to v \) in \( H^1(\Omega) \), \( u_m \to u \) in \( L^q(\Omega) \) for each \( 2 \leq q < 2^* \) and \( u_m(x) \to v(x) \) a.e. on \( \Omega \). Testing \( J_\lambda'(u_m) \to 0 \) with \( u_m^- \) we get
\[
\int_{\Omega} (|\nabla u_m^-|^2 + \lambda(u_m^-)^2) \, dx = o(1).
\]
Therefore we may assume that \( u_m \geq 0 \) on \( \Omega \). We now show that \( v \neq 0 \). Arguing by contradiction assume that \( v \equiv 0 \). By the P. L. Lions [12] concentration-compactness principle there exist sequences of points \( \{x_j\} \subset \mathbb{R}^N \) and numbers \( \{\nu_j\}, \{\mu_j\} \subset (0, \infty) \) such that
\[
|u_m|^{2^*} \overset{*}{\rightharpoonup} \sum_j \nu_j \delta_{x_j} \quad \text{and} \quad |\nabla u_m|^2 \overset{*}{\rightharpoonup} \sum_j \mu_j \delta_{x_j}
\]
in \( \mathcal{M} \), where \( \mathcal{M} \) is a space of measures. Moreover,
\[
S\nu_j^{2/2^*} \leq \mu_j \quad \text{if} \quad x_j \in \Omega \quad \text{and} \quad S\nu_j^{2/2^*} \leq \mu_j \quad \text{if} \quad x_j \in \partial \Omega.
\]
Testing (3.7) with \( u_m \phi_\delta \), where \( \phi_\delta, \delta > 0 \), is a family of functions concentrating at \( x_j \) as \( \delta \to 0 \), we deduce that \( \mu_j \leq Q(x_j)\nu_j \) for every \( j \). If \( \nu_j > 0 \) and \( x_j \in \Omega \), then \( \nu_j \geq S^{N/2}/Q(x_j)^{N/2} \), and if \( x_j \in \partial \Omega \), then \( \nu_j \geq S^{N/2}/2Q(x_j)^{N/2} \). By the Brézis–Lieb lemma we have
\[
J_\lambda(u_m) - \frac{1}{2} \langle J_\lambda'(u_m), u_m \rangle = \frac{1}{N} \int_{\Omega} Q(x)(u_\lambda + u_m)^{2^*} \, dx - \frac{1}{2} \int_{\Omega} Q(x)(u_\lambda + u_m)^{2^*} - 1 u_\lambda \, dx
\]
\[
+ \frac{1}{2^*} \int_{\Omega} Q(x)u_\lambda^{2^*} \, dx + \frac{1}{2} \int_{\Omega} Q(x)u_\lambda^{2^*} - 1 u_m \, dx + o(1)
\]
\[
= \frac{1}{N} \sum_{x_j \in \partial \Omega} Q(x_j)\nu_j + \frac{1}{N} \sum_{x_j \in \Omega} Q(x_j)\nu_j + o(1)
\]
\[
\geq \frac{1}{2N} \sum_{x_j \in \partial \Omega} \frac{S^{N/2}}{Q(x_j)(N-2)/2} + \frac{1}{N} \sum_{x_j \in \Omega} \frac{S^{N/2}}{Q(x_j)(N-2)/2} + o(1).
\]
If \( Q_M > 2^{2/(N-2)} Q_m \), then letting \( m \to \infty \) we derive that \( c \geq S^{N/2}/NQ_M^{N-2}/2 \), and if \( Q_M \leq 2^{2/(N-2)} Q_m \), then \( c \geq S^{N/2}/2NQ_m^{N-2}/2 \). In both cases we obtain a contradiction. \( \blacksquare \)

4. Main result. In order to apply the mountain-pass theorem we set
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)) ,
\]
where
\[ \Gamma = \{ \gamma \in C([0, 1], H^1(\Omega)) : \gamma(0), \gamma(1) = t_0 \} \]
and the constant \( t_0 \) is chosen so large that \( J_\lambda(t_0 t) \leq 0 \) for \( t \geq 1 \). It follows from Proposition 3.1 that \( c > 0 \).

We need the following relations for \( U_{\varepsilon,y} \) with \( y \in \partial \Omega \) (see [1] or [14]):
\[
\text{(4.1)} \quad \frac{\int_\Omega (|\nabla U_{\varepsilon,y}|^2 + \lambda U_{\varepsilon,y}^2) \, dx}{(\int_\Omega U_{\varepsilon,y}^2 \, dx)^{2/2}} \leq \begin{cases}
S/2^{2/N} - A_N H(y) \varepsilon \log \frac{1}{\varepsilon} + a_N \lambda \varepsilon + O(\varepsilon) + o(\lambda \varepsilon) & \text{if } N = 3, \\
S/2^{2/N} - A_N H(y) \varepsilon + a_N \lambda \varepsilon^2 \log \frac{1}{\varepsilon} + O(\varepsilon^2 \log \frac{1}{\varepsilon}) & \text{if } N = 4,
\end{cases}
\]
where \( H(y) \) denotes the mean curvature of \( \partial \Omega \) at \( y \).

It is known that
\[
\text{(4.2)} \quad c \leq c^* = \inf_{u \in H^1(\Omega), u \neq 0} \sup_{t \geq 0} J_\lambda(tu).
\]

**Theorem 4.1.** Suppose that \( Q_M \geq 2^{2/(N-2)} Q_m \) and that at some point \( y \in \partial \Omega \) with \( H(y) > 0 \) we have
\[
\text{(4.3)} \quad |Q(x) - Q(y)| = o(|x - y|) \quad \text{for } x \text{ close to } y.
\]
Then problem (3.1) has a solution for every \( \lambda > 0 \).

**Proof.** It follows from (4.2) that
\[
c \leq c^* \leq \frac{1}{N} \frac{\int_\Omega (|\nabla U_{\varepsilon,y}|^2 + \lambda U_{\varepsilon,y}^2) \, dx)^{N/2}}{(\int_\Omega U_{\varepsilon,y}^2 \, dx)^{(N-2)/2}}.
\]
Thus (4.1) and (4.3) yield
\[
c \leq \frac{S^{N/2}}{2 N Q_m^{(N-2)/2}}
\]
for \( \varepsilon > 0 \) sufficiently small. By Proposition 3.3 problem (3.1) has a solution.

**Corollary 4.2.** Under the assumptions of Theorem 4.1 there exists \( \lambda^* > 0 \) such that problem (1.1) has at least two solutions for \( \lambda > \lambda^* \).

**5. Existence of solutions for small boundary data.** Lemma 2.2 (see also Corollary 2.3) provides the estimate of \( \lambda^* \) in terms of \( \int_{\partial \Omega} \phi dS_x \).

For \( \lambda < \lambda^* \) problem (1.1) does not have a solution for a given \( \phi \). In this section we establish the existence of a solution of problem (1.1) for every \( \lambda > 0 \) if \( \int_{\partial \Omega} \phi dS_x \) is small. Obviously, the size of \( \int_{\partial \Omega} \phi dS_x \) will depend on \( \lambda \).
Let 
\[ I_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx - \frac{1}{2^*} \int_\Omega Q(x)|u|^{2^*} \, dx - \int_{\partial \Omega} u\phi(x) \, dS_x \]
for \( u \in H^1(\Omega) \) be a variational functional corresponding to problem (1.1). In what follows we shall use the Sobolev inequality
\[ \left( \int_\Omega |u|^{2^*} \, dx \right)^{2/2^*} \leq C_s \left( \int_\Omega (|\nabla u|^2 + u^2) \, dx \right)^{2/2^*} \]
for \( u \in H^1(\Omega) \), where \( C_s > 0 \) is a constant. Letting \( C_s(\lambda) = C_s \) for \( \lambda \geq 1 \) and \( C_s(\lambda) = C_s/\lambda \) for \( 0 < \lambda < 1 \), we can write this inequality in the form
\[ \left( \int_\Omega |u|^{2^*} \, dx \right)^{2/2^*} \leq C_s(\lambda) \left( \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx \right)^{2/2^*}. \]

**Proposition 5.1.** Given \( 0 < \lambda < \infty \) there exists a constant \( \varrho_1 = \varrho_1(\lambda) \) such that for a boundary data \( \phi \) satisfying \( \|\phi\|_{L^2(\partial \Omega)} \leq \varrho_1 \) problem (1.1) has a solution. (If \( \lambda \geq 1 \) the choice of \( \varrho_1 \) can be made independent of \( \lambda \).)

**Proof.** A solution will be found as a local minimizer of \( I_\lambda \). We commence by estimating \( I_\lambda \) from below:
\[ I_\lambda(u) \geq \frac{1}{2} \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx - \frac{C_s(\lambda)^{2^{*}/2}}{2^{*}} Q_M \left( \int_\Omega (|\nabla u|^2 + u^2) \, dx \right)^{2^{*}/2} \]
\[ - K \left( \int_{\partial \Omega} \phi^2 \, dS_x \right)^{1/2} \left( \int_\Omega (|\nabla u|^2 + u^2) \, dx \right)^{1/2}, \]
where \( K > 0 \) is the best constant for the embedding of \( H^1(\Omega) \) into \( L^2(\partial \Omega) \), that is,
\[ K = \inf \left\{ \int_\Omega (|\nabla u|^2 + u^2) \, dx; \ u \in H^1(\Omega), \ \int_{\partial \Omega} u^2 \, dS_x = 1 \right\}. \]
Letting \( \|u\|_\lambda^2 = \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx \) we can write this estimate as
\[ I_\lambda(u) \geq \|u\|_\lambda \left( \|u\|_\lambda - \frac{C_s(\lambda)^{2^{*}/2}}{2^{*}} Q_M \|u\|_\lambda^{2^{*}-1} - K(\lambda) \|\phi\|_{L^2(\partial \Omega)} \right), \]
where \( K(\lambda) = K \) for \( \lambda \geq 1 \) and \( K(\lambda) = K/\lambda \) for \( 0 < \lambda < 1 \). First we choose \( \varrho > 0 \) such that
\[ \varrho - \frac{C_s(\lambda)^{2^{*}/2}}{2^{*}} Q_M \varrho^{2^{*}-1} \geq \frac{3}{4} \varrho. \]
If \( \|\phi\|_{L^2(\partial \Omega)} \leq \varrho/K(\lambda) = \varrho_1 \), then
\[ I_\lambda(u) \geq \frac{\varrho^2}{4} \quad \text{for} \ |\|u\|_\lambda = \varrho. \]
Testing $I_{\lambda}(u)$ with a constant function $u = t$ we get

$$I_{\lambda}(t) = \frac{|\Omega|}{2} t^2 - \frac{t^{2^*}}{2^*} \int_{\Omega} Q(x) \, dx - t \int_{\partial \Omega} \phi(x) \, dS_x < 0$$

for sufficiently small $t$. Therefore

$$c_1 = \inf_{\|u\|_{\lambda} = \rho} I_{\lambda}(u) < 0. \quad (5.2)$$

It follows from (5.1), (5.2) and the Ekeland variational principle that there exists a minimizing sequence $\{u_m\}$ satisfying

$$I_{\lambda}(u_m) \to c_1 \quad \text{and} \quad I'_{\lambda}(u_m) \to 0 \quad \text{in } H^{-1}(\Omega).$$

It is clear that $\{u_m\}$ is bounded in $H^1(\Omega)$. Thus we may assume that $u_m \to u$ in $H^1(\Omega)$, $u_m \to u$ in $L^p(\Omega)$ for $2 \leq p < 2^*$ and $u_m \to a$ a.e. on $\Omega$. Moreover, $u$ is a solution of (1.1$\lambda$). We now observe that $\|u\|_{\lambda} \leq \rho$ and $I_{\lambda}(u) \geq c_1$. Since $\langle I'_{\lambda}(u), u \rangle = 0$, we see that

$$\left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\Omega} (|\nabla u|^2 + \lambda u^2) \, dx - \left( 1 - \frac{1}{2^*} \right) \int_{\Omega} \phi u \, dS_x \geq c_1.$$

The weak lower semicontinuity of $\int_{\Omega} |\nabla u|^2 \, dx$ yields

$$\left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\Omega} (|\nabla u|^2 + \lambda u^2) \, dx - \left( 1 - \frac{1}{2^*} \right) \int_{\Omega} \phi u \, dS_x \leq c_1$$

and consequently $I_{\lambda}(u) = c_1$. ■

To prove the existence of a second solution we use the method of Section 3. In what follows we assume that the boundary data $\phi$ satisfies

$$\|\phi\|_{L^2(\partial \Omega)} < \varrho_1 = \frac{\varrho}{4K(\lambda)}. \quad (5.3)$$

This condition on $\phi$ guarantees the existence of a local minimizer $v_{\lambda}$ of the functional $I_{\lambda}$. As in Section 3 we consider the problem

$$(5.3_{\lambda}) \quad \begin{cases} -\Delta v + \lambda v = Q(x)[(v + v_{\lambda})^{2^*-1} - v_{\lambda}^{2^*-1}] \quad \text{in } \Omega, \\ \partial v / \partial \nu = 0. \end{cases}$$

If problem $(5.3_{\lambda})$ has a solution $w$, then $w + v_{\lambda}$ is a solution of problem (1.1$\lambda$). Let $\tilde{I}_{\lambda}(v)$ be a variational functional corresponding to problem $(5.3_{\lambda})$. We now consider the variational problem

$$(5.4) \quad \tilde{\mu}_{\lambda} = \inf \left\{ \int_{\Omega} (|\nabla v|^2 + \lambda v^2) \, dx; \ v \in H^1(\Omega), \right.$$}

$$\left. (2^* - 2) \int_{\Omega} Q(x)v_{\lambda}^{2^*-1}v^2 \, dx = 1 \right\}.$$
Proposition 5.2. Problem (5.4) has a minimizer \( w_\lambda \) which is the first eigenfunction of the eigenvalue problem

\[
\begin{align*}
-\Delta v + \lambda v &= \tilde{\mu}_\lambda (2^*-1)Q(x)v^{2^*-2} \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Proof. As in the proof of Proposition 2.1 we obtain the existence of a minimizer \( w_\lambda \). To show that \( \tilde{\mu}_\lambda > 1 \) we take \( \mu = \mu_\lambda \), where \( \mu \) is chosen so that

\[
\|\phi\|_{L^2(\partial \Omega)} < \frac{\theta}{4K(\lambda)} < \frac{\theta}{4K(\lambda)}.
\]

Hence \( I_\lambda \) and \( I_{\tilde{\lambda}} \) have local minimizers \( v_\lambda \) and \( v_{\tilde{\lambda}} \), respectively. Let \( z_\lambda \) and \( z_{\tilde{\lambda}} \) be the minimal solutions of (1.1\( \lambda \)) satisfying \( 0 \leq z_\lambda \leq v_\lambda \) and \( z_{\tilde{\lambda}} \leq v_{\tilde{\lambda}} \). Repeating estimates (2.2) with \( u_\lambda \) and \( u_{\tilde{\lambda}} \) replaced by \( z_\lambda \) and \( z_{\tilde{\lambda}} \) we derive that \( \tilde{\mu}_\lambda > 1 \). \( \blacksquare \)

This allows us to show that the variational functional \( \tilde{J}_\lambda \) for problem (5.3\( \lambda \)) has a mountain-pass geometry. It is also easy to see that Proposition 3.3 continues to hold for the functional \( \tilde{J}_\lambda \) for every \( \lambda > 0 \).

We are now in a position to formulate the following existence result:

**Theorem 5.3.** Suppose that \( \phi \) satisfies (5.3).

(i) If \( Q_M \leq 2^{2/(N-2)}Q_m \) and at some point \( y \in \partial \Omega \) the function \( Q \) satisfies condition (4.3) of Theorem 4.1, then problem (5.3\( \lambda \)) has a solution.

(ii) If \( Q_M > 2^{2/(N-2)}Q_m \), then there exists a \( \tilde{\lambda} > 0 \) such that problem (5.3\( \lambda \)) has a solution for \( \lambda < \tilde{\lambda} \).

The proof of part (i) is identical to that of Theorem 4.1. To establish part (ii) we observe that \( S_\infty = S_{N/2}/NQ_M^{(N-2)/2} \). Testing \( \tilde{J}_\lambda \) with a constant function \( u = 1 \), we see that the mountain-pass level is below \( S_\infty \) if \( \lambda \) is small.

6. Case \( \lambda = 0 \). If \( \lambda = 0 \), then problem (1.1\( \lambda \)) cannot have a solution. Indeed, integrating equation (1.1\( \lambda \)) we get

\[
-\int_{\partial \Omega} \phi(x) \, dS_x = \int_{\Omega} Q(x)u^{2^*-1} \, dx,
\]

which is impossible. Therefore we assume throughout this section that

(Q) \( Q \) changes sign on \( \Omega \) and \( \int_{\Omega} Q(x) \, dx < 0 \).

Since 0 is the first eigenvalue of the linear part of equation (1.1\( \lambda \)) with the Neumann boundary conditions, it is convenient to decompose \( H^1(\Omega) = \mathbb{R} \oplus V \), where the space \( V \) consists of functions \( v \) satisfying \( \int_{\Omega} v(x) \, dx = 0 \).
Having this decomposition we define an equivalent norm in $H^1(\Omega)$ by

$$\|u\|^2_V = t^2 + \int_\Omega |\nabla v|^2 \, dx.$$  

**Lemma 6.1.** Suppose that $Q(x)$ satisfies (Q). Then there exists a constant $\eta > 0$ such that for each $t \in \mathbb{R}$ and $v \in V$ the inequality

$$\left( \int_\Omega |\nabla v|^2 \, dx \right)^{1/2} \leq \eta |t|$$  

implies

$$\int_\Omega Q(x)|t + v(x)|^{2^*} \, dx \leq \frac{|t|^{2^*}}{2} \int_\Omega Q(x) \, dx.$$  

For the proof we refer to [4].

**Proposition 6.2.** Suppose that $Q(x)$ satisfies (Q). Then there exist constants $\beta > 0$, $\beta_0 > 0$ and $\varrho > 0$ such that

(6.1) \quad $I_0(u) \geq \beta$ \quad for $\|u\|_V = \varrho$ \quad and \quad $\|\phi\|_{L^2(\partial\Omega)} \leq \beta_0$.  

Proof. We write

$$I_0(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*} \int_\Omega Q(x)|u|^{2^*} \, dx - \int_{\partial\Omega} u\phi(x) \, dS_x$$  

$$= I_o(u) - \int_{\partial\Omega} u\phi(x) \, dS_x.$$  

We now consider two cases: (i) $\|\nabla v\|_2 \leq \eta |t|$ and (ii) $\|\nabla v\|_2 > \eta |t|$. If $\|\nabla v\|_2 \leq \eta |t|$ and $\|\nabla v\|_2^2 + t^2 = \varrho^2$, then $t^2 \geq \varrho^2/(1 + \eta^2)$. It then follows from Lemma 6.1 that

$$\int_\Omega Q(x)|t + v(x)|^{2^*} \, dx \leq \frac{|t|^{2^*}}{2} \int_\Omega Q(x) \, dx = -|t|^{2^*} \alpha,$$

where $\alpha = -\frac{1}{2} \int_\Omega Q(x) \, dx > 0$. Hence we have

$$I_o(u) \geq \frac{|t|^{2^*}}{2^*} \alpha \geq \frac{\alpha \varrho^{2^*}}{2^*(1 + \eta^2)^{2^*/2}}.$$  

In case (ii) we have $\|\nabla u\|_V \leq \|\nabla v\|_2(1 + 1/\eta^2)^{1/2}$. Thus applying the Sobolev inequality we get

$$\int_\Omega Q(x)|u|^{2^*} \, dx \leq C_1 \|u\|_V^{2^*} \leq C_1 (1 + 1/\eta^2)^{2^*/2} \|\nabla v\|_2^{2^*}$$  

for some constant $C_1 > 0$. Hence

$$I_o(u) \geq \frac{1}{2} \|\nabla v\|_2^{2^*} - C_1 (1 + 1/\eta^2)^{2^*/2} \|\nabla v\|_2^{2^*}.$$
Taking \( \| \nabla v \|_2 \leq \varrho \) small enough we deduce from the above inequality the estimate
\[
\tilde{I}_\circ(u) \geq \frac{1}{4} \| \nabla v \|_2^2.
\]
On the other hand, if \( \| u \|_V = \varrho \), then \( \varrho \leq (\| \nabla v \|_2 / \eta)(1 + \eta^2)^{1/2} \). Hence
\[
\tilde{I}_\circ(u) \geq \frac{\eta^2 \varrho^2}{4(1 + \eta^2)}.
\]
Taking
\[
\beta_1 = \min \left( \frac{\eta^2 \varrho^2}{4(1 + \eta^2)}, \frac{\alpha \varrho^{2*}}{2^*(1 + \eta^2)^{2*}/2} \right),
\]
we obtain the following estimate for \( \| u \|_V = \varrho \):
\[
I_\circ(u) \geq \beta_1 - C_2 \| \phi \|_{L^2(\partial \Omega)} \| u \|_V = \beta_1 - C_2 \varrho \| \phi \|_{L^2(\partial \Omega)}
\]
for some constant \( C_2 > 0 \). We now choose \( \| \phi \|_{L^2(\partial \Omega)} \) so that
\[
\| \phi \|_{L^2(\partial \Omega)} \leq \frac{\beta_1}{2C_2 \varrho}.
\]
This gives the desired estimate for \( I_\circ(u) \) with \( \beta = \beta_1/2 \) and \( \beta_\circ = \beta_1/2C_2 \varrho \).

Testing \( I_\circ(u) \) with a constant function \( u = t \), with \( t \) sufficiently small, we get \( I_\circ(t) < 0 \). Hence
\[
c_2 = \inf_{\| u \|_V \leq \varrho} I_\circ(u) < 0.
\]

Repeating the argument used in the proof of Proposition 5.1 we obtain

**Proposition 6.3.** Suppose that (Q) holds. Then there exists a constant \( \beta_\circ > 0 \) such that for \( \phi \) satisfying \( \| \phi \|_{L^2(\partial \Omega)} \leq \beta_\circ \) problem (1.10) admits a solution which is a local minimizer of \( I_\circ(u) \).

**References**


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