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ON MULTIPLE SOLUTIONS OF THE NEUMANN PROBLEM INVOLVING THE CRITICAL SOBOLEV EXPONENT

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Abstract. We consider the Neumann problem involving the critical Sobolev exponent and a nonhomogeneous boundary condition. We establish the existence of two solutions. We use the method of sub- and supersolutions, a local minimization and the mountain-pass principle.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. We consider the Neumann problem

$$(1.1_\lambda)\qquad \begin{cases} -\Delta u + \lambda u = Q(x)u^{2^*-1} & \text{in } \Omega,\\ \partial u/\partial\nu = \phi(x) \quad \text{on } \partial\Omega, & u > 0 \quad \text{on } \Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent. We assume that $Q(x) > 0$ on $\overline{\Omega}$, $\phi(x) \geq 0$ and $\phi(x) \not\equiv 0$ on $\partial\Omega$ and moreover $Q \in C^\alpha(\overline{\Omega})$ and $\phi \in C^\alpha(\partial\Omega)$.

In the case where $Q \equiv 1$ and $\phi \equiv 0$, problem (1.1_λ) has an extensive literature. We refer to papers [2], [3], [8] and [9], where further references can be found. In this case solutions of (1.1_λ) have been obtained as minimizers of the constrained variational problem

$$m_\lambda = \inf_{u \in H^1(\Omega)-\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{2/2^*}}.$$

A suitable multiple of a minimizer u for m_λ is a solution of (1.1_λ) and is called the *least energy solution* of this problem. The main ingredient in the proof of the existence of the least energy solution is the inequality $m_\lambda < S/2^{2/N}$, which is valid for every λ, provided Ω is smooth and bounded. Here S is the best Sobolev constant. This inequality allows us to show that every minimizing sequence for m_λ is relatively compact in $H^1(\Omega)$. These results have been extended to the case $Q \not\equiv \text{const}$ and $\phi \equiv 0$ (see [8] and [9]). In this situation the existence of least energy solutions depends on the relationship between the global maximum $Q_{\mathrm{M}} = \max_{x\in\overline{\Omega}} Q(x)$ and $Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x)$.

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The authors of these papers studied two cases: (i) $Q_{\mathrm{M}} \le 2^{2/(N-2)} Q_{\mathrm{m}}$ and (ii) $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$. In the first case problem (1.1_λ) has the least energy solution for every $\lambda > 0$, provided Q_{m} is achieved at a point $x_\circ \in \partial\Omega$ with a positive mean curvature. In case (ii), the least energy solutions exist only for $\lambda \in (0, \Lambda]$, $\Lambda > 0$. For $\lambda > \Lambda$ problem (1.1_λ) does not have the least energy solutions.

The main purpose of this paper is to establish an existence result for problem (1.1_λ) which involves a nonzero boundary data ϕ. We show that the presence of $\phi \neq 0$ generates the existence of at least two solutions. Results of this nature are known in the cases where a nonhomogeneous term appears in the nonlinear equation ([7], [6] and [13]).

Under an additional assumption on Q we establish the existence of a constant $\lambda_* > 0$ such that for $\lambda > \lambda_*$ problem (1.1_λ) has at least two solutions, at least one solution for $\lambda = \lambda_*$ and no solution for $\lambda < \lambda_*$. In the case where $\lambda > \lambda_*$ the existence of one solution will be established through the method of sub- and supersolutions. A second solution will be obtained via the mountain-pass principle. These existence results are presented in Sections 2, 3 and 4. In these sections we do not impose any restriction on $\|\phi\|_{L^2(\partial\Omega)}$. In Section 5 we show that if $\|\phi\|_{L^2(\partial\Omega)}$ is of order λ (as small as λ), then problem (1.1_λ) has at least two solutions. Section 6 is devoted to the case $\lambda = 0$.

In this paper we use standard notations. In a given Banach space X we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". We recall that a C^1-functional $\Phi : X \to \mathbb{R}$ on a Banach space X satisfies the *Palais–Smale condition at level* c ($(\mathrm{PS})_c$ *condition* for short) if each sequence $\{x_m\}$ such that

$$(*)\ \Phi(x_m) \to c \quad \text{and} \quad (**)\ \Phi'(x_m) \to 0 \text{ in } X^*$$

is relatively compact in X. Finally, any sequence satisfying $(*)$ and $(**)$ is called a *Palais–Smale sequence at level* c (a $(\mathrm{PS})_c$ *sequence* for short).

The norms in the Lebesgue spaces $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$.

2. Sub- and supersolutions. To construct a supersolution to problem (1.1_λ) we need the solution of the problem

$$(2.1_\lambda) \qquad \begin{cases} -\Delta v + \lambda v = 0 & \text{in } \Omega, \\ \partial v/\partial\nu = \phi(x) & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique positive solution $v_\lambda \in C^{1,\alpha}(\overline{\Omega})$. Let v_1 be a solution of (2.1_λ) with $\lambda = 1$. We set

$$\lambda_\circ = \max_{x\in\Omega} Q(x) v_1(x)^{2^*-2} + 1.$$

We then have

$$
\begin{aligned}
-\Delta v_1 + \lambda_\circ v_1 - Q(x)v_1^{2^*-1} \\
&= -\Delta v_1 + (\lambda_\circ - Q(x)v_1^{2^*-2})v_1 \\
&= -\Delta v_1 + (\max_{x\in\overline{\Omega}} Q(x)v_1(x)^{2^*-2} + 1 - Q(x)v_1(x)^{2^*-2})v_1 \\
&\geq -\Delta v_1 + v_1 = 0.
\end{aligned}
$$

Hence $\overline{u} = v_1$ is a supersolution for $(1.1_{\lambda_\circ})$. Since $\underline{u} = 0$ is a subsolution for $(1.1_{\lambda_\circ})$, there exists a minimal solution $u_{\lambda_\circ}$ of $(1.1_{\lambda_\circ})$ satisfying

$$\underline{u} < u_{\lambda_\circ} < \overline{u} \quad \text{on } \Omega.$$

Let

$$\mathcal{S} = \{\lambda;\ (1.1_\lambda) \text{ has a positive solution}\}$$

If $\lambda > \lambda_\circ$, then $u_{\lambda_\circ}$ is a supersolution to (1.1_λ). Indeed, we have

$$
\begin{cases}
-\Delta u_{\lambda_\circ} + \lambda u_{\lambda_\circ} > -\Delta u_{\lambda_\circ} + \lambda_\circ u_{\lambda_\circ} = Q(x)u_{\lambda_\circ}^{2^*-1} & \text{in } \Omega, \\
\partial u_{\lambda_\circ}/\partial\nu = \phi(x) & \text{on } \partial\Omega.
\end{cases}
$$

As before, since $\underline{u} = 0$ is a subsolution, there exists a minimal solution u_λ satisfying

$$\underline{u} < u_\lambda < \overline{u} = u_{\lambda_\circ}.$$

This argument shows that $(\lambda_\circ, \infty) \subset \mathcal{S}$. We set

$$\lambda_* = \inf_{\lambda\in\mathcal{S}} \lambda. \tag{2.1}$$

Repeating the above argument we show that for every $\lambda > \lambda_*$ problem (1.1_λ) has a solution. If $u_\lambda > 0$ is a solution of (1.1_λ), then

$$\int_\Omega \lambda u_\lambda\,dx - \int_{\partial\Omega} \phi(x)\,dS_x = \int_\Omega Q(x)u_\lambda^{2^*-1}\,dx.$$

This yields $\lambda > 0$ and consequently $\lambda_* \geq 0$.

Let $\lambda > \lambda_*$ and let u_λ be a positive solution of (1.1_λ). We now consider the variational problem

$$\mu_\lambda = \inf\Big\{ \int_\Omega (|\nabla v|^2 + \lambda v^2)\,dx;\ v \in H^1(\Omega),\quad (2^*-1)\int_\Omega Q(x)u_\lambda^{2^*-2}v^2\,dx = 1\Big\}. \tag{2.2}$$

PROPOSITION 2.1. *If $\lambda > \lambda_*$, then the constant μ_λ defined by* (2.2) *satisfies $\mu_\lambda > 1$. Moreover, problem* (2.2) *has a minimizer V_λ which is the first eigenfunction of the problem*

$$
\begin{cases}
-\Delta v + \lambda v = \mu_\lambda(2^*-1)Q(x)u_\lambda^{2^*-2}v & \textit{in } \Omega, \\
\partial v/\partial\nu = 0 & \textit{on } \partial\Omega.
\end{cases} \tag{2.3}
$$

Proof. Since the functional $v \mapsto \int_\Omega Q(x)u_\lambda^{2^*-2}v^2\,dx$ is completely continuous on $H^1(\Omega)$, the existence of a minimizer easily follows. We show that $\mu_\lambda > 1$. Let $\overline{\lambda} > \underline{\lambda}$ and let $u_{\overline{\lambda}}$ and $u_{\underline{\lambda}}$ be the corresponding minimal solutions of $(1.1_{\overline{\lambda}})$ and $(1.1_{\underline{\lambda}})$, respectively. It follows from the construction of $\{u_\lambda\}$ that $u_{\underline{\lambda}} > u_{\overline{\lambda}} > 0$. We then have

$$
\begin{aligned}
(2.4)\qquad & -\Delta(u_{\underline{\lambda}} - u_{\overline{\lambda}}) + \overline{\lambda}(u_{\underline{\lambda}} - u_{\overline{\lambda}}) \\
&> -\Delta(u_{\underline{\lambda}} - u_{\overline{\lambda}}) + \underline{\lambda}u_{\underline{\lambda}} - \overline{\lambda}u_{\overline{\lambda}} = Q(x)(u_{\underline{\lambda}}^{2^*-1} - u_{\overline{\lambda}}^{2^*-1}) \\
&= Q(x)[(u_{\overline{\lambda}} + u_{\underline{\lambda}} - u_{\overline{\lambda}})^{2^*-1} - u_{\overline{\lambda}}^{2^*-1}] \\
&= (2^*-1)Q(x)u_{\overline{\lambda}}^{2^*-2}(u_{\underline{\lambda}} - u_{\overline{\lambda}}) \\
&\quad + \tfrac{1}{2}(2^*-1)(2^*-2)Q(x)[u_{\overline{\lambda}} + \theta(u_{\underline{\lambda}} - u_{\overline{\lambda}})]^{2^*-3}(u_{\underline{\lambda}} - u_{\overline{\lambda}})^2 \\
&> (2^*-1)Q(x)u_{\overline{\lambda}}^{2^*-2}(u_{\underline{\lambda}} - u_{\overline{\lambda}})
\end{aligned}
$$

for some $0 < \theta < 1$. Let $V_{\overline{\lambda}}$ be the first eigenfunction of problem (2.3) with $\lambda = \overline{\lambda}$. Since

$$\frac{\partial}{\partial\nu}(u_{\underline{\lambda}} - u_{\overline{\lambda}}) = 0 \quad \text{on } \partial\Omega,$$

testing (2.4) with $V_{\overline{\lambda}}$ and integrating by parts gives

$$\int_\Omega (u_{\underline{\lambda}} - u_{\overline{\lambda}})(-\Delta V_{\overline{\lambda}} + \overline{\lambda}V_{\overline{\lambda}})\,dx > (2^*-1)\int_\Omega Q(x)u_{\overline{\lambda}}^{2^*-2}(u_{\underline{\lambda}} - u_{\overline{\lambda}})V_{\overline{\lambda}}\,dx.$$

Hence

$$\mu_\lambda(2^*-1)\int_\Omega Q(x)(u_{\underline{\lambda}} - u_{\overline{\lambda}})u_{\overline{\lambda}}V_{\overline{\lambda}}\,dx > (2^*-1)\int_\Omega Q(x)u_{\overline{\lambda}}^{2^*-2}(u_{\underline{\lambda}} - u_{\overline{\lambda}})V_{\overline{\lambda}}\,dx$$

and the assertion follows. ■

Let $Q_* = \min_{x\in\overline{\Omega}} Q(x)$.

LEMMA 2.2. *Let u_λ be a solution of problem (1.1_λ) for some $\lambda > 0$. Then*

$$\lambda^{(N+2)/4} \ge Q_*^{(N+2)/4}\frac{\int_\Omega Q(x)u_\lambda^{2^*-1}\,dx + \frac{N+2}{4}\int_{\partial\Omega}\phi(x)\,dS_x}{\int_\Omega Q(x)\,dx}.$$

Proof. Integrating (1.1_λ) we get

$$\lambda\int_\Omega u_\lambda\,dx = \int_\Omega Q(x)u_\lambda^{2^*-1}\,dx + \int_{\partial\Omega}\phi(x)\,dS_x. \tag{2.5}$$

It then follows from the Young inequality that

$$
\begin{aligned}
\lambda\int_\Omega u_\lambda\,dx &\le \lambda Q_*^{-1}\int_\Omega Q(x)u_\lambda\,dx \\
&\le \frac{2^*-2}{2^*-1}\,\lambda^{\frac{2^*-1}{2^*-2}}Q_*^{-\frac{2^*-1}{2^*-2}}\int_\Omega Q(x)\,dx + \frac{1}{2^*-1}\int_\Omega Q(x)u_\lambda^{2^*-1}\,dx.
\end{aligned}
$$

This combined with (2.5) gives

$$\frac{2^*-2}{2^*-1}\int_\Omega Q(x)u_\lambda^{2^*-1}\,dx+\int_{\partial\Omega}\phi(x)\,dS_x\le\frac{2^*-2}{2^*-1}\,\lambda^{\frac{2^*-1}{2^*-2}}Q_*^{-\frac{2^*-1}{2^*-2}}\int_\Omega Q(x)\,dx$$

and the result easily follows. ■

Corollary 2.3. *If*

$$\lambda^{(N+2)/4}\le Q_*^{(N+2)/4}\frac{\int_{\partial\Omega}\phi(x)\,dS_x}{\int_\Omega Q(x)\,dx},$$

then problem (1.1_λ) *has no solution. Consequently*, $\lambda_*>0$, *where* λ_* *is the constant defined by* (2.1).

In Proposition 2.4 below, we derive an estimate for $\|u_\lambda\|_{H^1}$ in terms of the parameter λ and norms of v_1.

Proposition 2.4. *Solutions of* (1.1_λ) *for* $0<\lambda\le 1$ *satisfy the estimate*

$$\|u_\lambda\|_{H^1}^2\le L(\|v_1\|_{H^1}^2+\|v_1\|_{2^*}^{2^*}+(1-\lambda)\|v_1\|_2^2+(1-\lambda)^{N/2})$$

and for $\lambda>1$ *we have*

$$\|u_\lambda\|_{H^1}^2\le L_1$$

for some constants $L>0$ *and* $L_1>0$ *independent of* λ.

Proof. Let u_λ be a solution of (1.1_λ) and v_1 be a solution of (2.1_1). We set $v=u_\lambda-v_1$. Then v satisfies

$$(2.6)\qquad\begin{cases}-\Delta v+v=Q(x)(v+v_1)^{2^*-1}+(1-\lambda)(v+v_1) & \text{in } \Omega,\\ \partial v/\partial\nu=0 \quad \text{on } \partial\Omega.\end{cases}$$

First we consider the case $0<\lambda\le 1$. By the maximum principle $v>0$ on Ω. Testing (2.6) with v we get

$$\int_\Omega(|\nabla v|^2+v^2)\,dx=\int_\Omega Q(x)(v+v_1)^{2^*-1}v\,dx+(1-\lambda)\int_\Omega(v+v_1)v\,dx.$$

Since $0<\lambda\le 1$, it follows from Proposition 2.1 that

$$\begin{aligned}(2^*-1)\int_\Omega Q(x)u_\lambda^{2^*-2}v^2\,dx&\\ &\le\int_\Omega Q(x)(v+v_1)^{2^*-1}v\,dx+(1-\lambda)\int_\Omega(v+v_1)v\,dx\\ &=\int_\Omega Q(x)(v+v_1)^{2^*-2}v^2\,dx+\int_\Omega Q(x)(v+v_1)^{2^*-2}vv_1\,dx\\ &\quad+(1-\lambda)\int_\Omega(v+v_1)v\,dx.\end{aligned}$$

Thus

$$(2.7)\qquad (2^*-2)\int_\Omega Q(x)(v+v_1)^{2^*-2}v^2\,dx \le \int_\Omega Q(x)(v+v_1)^{2^*-2}vv_1\,dx + (1-\lambda)\int_\Omega (v+v_1)v\,dx$$
$$\le \int_\Omega Q(x)(v+v_1)^{2^*-1}v_1\,dx + (1-\lambda)\int_\Omega (v+v_1)v\,dx$$
$$\le 2^{2^*-2}\int_\Omega Q(x)v^{2^*-1}v_1\,dx + 2^{2^*-2}\int_\Omega Q(x)v_1^{2^*}\,dx + (1-\lambda)\int_\Omega v^2\,dx + (1-\lambda)\int_\Omega v_1v\,dx$$
$$\le 2^{2^*-2}\int_\Omega Q(x)v^{2^*-1}v_1\,dx + 2^{2^*-2}\int_\Omega Q(x)v_1^{2^*}\,dx + 2(1-\lambda)\int_\Omega v^2\,dx + (1-\lambda)\int_\Omega v_1^2\,dx.$$

Using the Young inequality we get for $\varepsilon > 0$,

$$(2.8)\qquad \int_\Omega Q(x)v^{2^*-1}v_1\,dx \le \varepsilon\int_\Omega Q(x)v^{2^*}\,dx + C(\varepsilon)\int_\Omega Q(x)v_1^{2^*}\,dx,$$

$$(2.9)\qquad 2(1-\lambda)\int_\Omega v^2\,dx \le \varepsilon\int_\Omega v^{2^*}\,dx + C_1(\varepsilon)(1-\lambda)^{2^*/(2^*-2)}|\Omega|,$$

for some constants $C(\varepsilon) > 0$ and $C_1(\varepsilon) > 0$. Letting $Q_{\mathrm{M}} = \max_{x\in\overline{\Omega}} Q(x)$ we deduce from (2.7)–(2.9) that

$$((2^*-2)Q_* - 2^{2^*-2}Q_{\mathrm{M}}\varepsilon - \varepsilon)\int_\Omega v^{2^*}\,dx \le (2^{2^*-2} + C(\varepsilon))\int_\Omega Q(x)v_1^{2^*}\,dx + C_1(\varepsilon)(1-\lambda)^{2^*/(2^*-2)}|\Omega| + (1-\lambda)\int_\Omega v_1^2\,dx.$$

Choosing $\varepsilon > 0$ small enough we derive from this the estimate

$$(2.10)\qquad \int_\Omega v^{2^*}\,dx \le C\Big[\int_\Omega v_1^{2^*}\,dx + (1-\lambda)^{2^*/(2^*-2)} + (1-\lambda)\int_\Omega v_1^2\,dx\Big].$$

We now use (2.10) to estimate $\|v\|_{H^1}^2$ in terms of λ and v_1. We have

$$\int_\Omega (|\nabla v|^2 + v^2)\,dx = \int_\Omega Q(x)(v+v_1)^{2^*-1}v\,dx + (1-\lambda)\int_\Omega v^2\,dx + (1-\lambda)\int_\Omega v_1v\,dx$$

$$\le 2^{2^*-2}\int_\Omega Q(x)v^{2^*}\,dx + 2^{2^*-2}\int_\Omega Q(x)v_1^{2^*-1}v\,dx + 2(1-\lambda)\int_\Omega v^2\,dx + (1-\lambda)\int_\Omega v_1^2\,dx$$
$$\le 2^{2^*-3}(2^*+1)\int_\Omega Q(x)v^{2^*}\,dx + 2^{2^*-3}(2^*-1)\int_\Omega Q(x)v_1^{2^*}\,dx + 2(1-\lambda)\int_\Omega v^2\,dx + (1-\lambda)\int_\Omega v_1^2\,dx.$$

The last estimate combined with (2.9) and (2.10) gives

$$\int_\Omega (|\nabla v|^2 + v^2)\,dx \le C_1\Big[\int_\Omega v_1^{2^*}\,dx + (1-\lambda)\int_\Omega v_1^2\,dx + (1-\lambda)^{2^*/(2^*-2)}\Big],$$

where $C_1 > 0$ is of the same nature as C in (2.10). Since $\|u_\lambda\|_{H^1} \le \|v\|_{H^1} + \|v_1\|_{H^1}$ the result in the case $0 < \lambda \le 1$ readily follows. If $\lambda > 1$, then $u_\lambda \le u_1$ on Ω, where u_1 is a minimal solution of problem (1.1_1). Thus

$$\int_\Omega (|\nabla u_\lambda|^2 + u_\lambda^2)\,dx \le \int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2)\,dx = \int_\Omega Q(x)u_\lambda^{2^*}\,dx + \int_\Omega \phi(x)u_\lambda\,dS_x \le Q_{\mathrm{M}}\int_\Omega u_1^{2^*}\,dx + \int_\Omega \phi(x)u_1\,dS_x,$$

and the result follows. ∎

Proposition 2.5. *Problem* (1.1_{λ^*}) *has a solution.*

Proof. Let $\lambda_n \to \lambda^*$ and $\lambda_n > \lambda^*$ for each n. By Proposition 2.4 the sequence $\{u_{\lambda_n}\}$ of the corresponding solutions is bounded in $H^1(\Omega)$. It is routine to show that up to a subsequence $u_{\lambda_n} \rightharpoonup u$ in $H^1(\Omega)$ and u is a solution of problem (1.1_{λ^*}). ∎

3. Second solution. Let u_λ be a minimal solution of (1.1_λ). To find the second solution we consider the problem

$$(3.1_\lambda)\qquad \begin{cases} -\Delta v + \lambda v = Q(x)[(v+u_\lambda)^{2^*-1} - u_\lambda^{2^*-1}] & \text{in } \Omega,\\ \partial v/\partial\nu = 0 \quad \text{on } \partial\Omega, \quad v > 0 & \text{on } \Omega, \end{cases}$$

where $\lambda > \lambda_*$. If v is a solution of (3.1_λ), then $U_\lambda = u_\lambda + v$ is a solution of (1.1_λ). A solution of (3.1_λ) will be found as a critical point of the functional

$$J_\lambda(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + \lambda v^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)(u_\lambda + v^+)^{2^*}\,dx + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx + \int_\Omega Q(x)u_\lambda^{2^*-1}v^+\,dx.$$

Proposition 3.1. *Let* $\lambda > \lambda_*$. *There exist constants* $\alpha > 0$ *and* $\varrho > 0$ *such that* $J_\lambda(v) \ge \alpha$ *for* $v \in H^1(\Omega)$ *with* $\|v\|_{H^1} = \varrho$.

Proof. We write J_λ in the form

$$J_\lambda(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + \lambda v^2)\,dx - \frac{2^*-1}{2}\int_\Omega Q(x)u_\lambda^{2^*-2}(v^+)^2\,dx - \int_\Omega \int_0^{v^+} Q(x)[(u_\lambda + s)^{2^*-1} - u_\lambda^{2^*-1} - (2^*-1)u_\lambda^{2^*-2}s]\,ds\,dx. \tag{3.1}$$

Since for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$(u_\lambda + s)^{2^*-1} - u_\lambda^{2^*-1} - (2^*-1)u_\lambda^{2^*-2}s \le \varepsilon u_\lambda^{2^*-2}s + C_\varepsilon s^{2^*-1},$$

we get

$$J_\lambda(v) \ge \frac{1}{2}\int_\Omega [|\nabla v|^2 + \lambda v^2 - (2^*-1)Q(x)u_\lambda^{2^*-2}(v^+)^2]\,dx - \int_\Omega Q(x)\left[\frac{\varepsilon}{2}\,u_\lambda^{2^*-2}(v^+)^2 + C_\varepsilon\,\frac{(v^+)^{2^*}}{2^*}\right]dx.$$

Hence by Proposition 2.1 we have

$$J_\lambda(v) \ge \frac{1}{2}\left(1 - \frac{2^*-1-\varepsilon}{\mu_\lambda(2^*-1)}\right)\int_\Omega (|\nabla v|^2 + \lambda v^2)\,dx - \frac{C_\varepsilon}{2^*}\int_\Omega Q(x)(v^+)^{2^*}\,dx.$$

We choose $0 < \varepsilon < 2^*-1$. An application of the Sobolev inequality completes the proof. ■

LEMMA 3.2. *Let $\{u_m\} \subset H^1(\Omega)$ be a Palais–Smale sequence for J_λ. Then $\{u_m\}$ is bounded in $H^1(\Omega)$.*

Proof. We compute

$$\begin{aligned} J_\lambda(u_m) - \frac{1}{2}\langle J_\lambda'(u_m), u_m\rangle \\ &= -\frac{1}{2^*}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*}\,dx + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx + \int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx \\ &\quad + \frac{1}{2}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1}u_m\,dx - \frac{1}{2}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m\,dx \\ &= \frac{1}{N}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*}\,dx - \frac{1}{2}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1}u_m^-\,dx \\ &\quad - \frac{1}{2}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1}u_\lambda\,dx \\ &\quad + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx + \int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx - \frac{1}{2}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m\,dx \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&= \frac{1}{N}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*}\,dx - \frac{1}{2}\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*-1}u_\lambda\,dx \\
&\quad - \frac{1}{2}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m^-\,dx + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx + \int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx \\
&\quad - \frac{1}{2}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m\,dx \\
&= \frac{1}{N}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*}\,dx - \frac{1}{2}\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*-1}u_\lambda\,dx \\
&\quad + \frac{1}{2}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx.
\end{aligned}
$$

By the Young inequality given $\delta > 0$ we choose $C(\delta) > 0$ so that

$$
\begin{aligned}
&\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*-1}u_\lambda\,dx \\
&\qquad\qquad \le \delta\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*}\,dx + C(\delta)\int_\Omega Q(x)u_\lambda^{2^*}\,dx.
\end{aligned}
\tag{3.3}
$$

Taking $\delta > 0$ small enough and using the fact that $\{u_m\}$ is a $(\mathrm{PS})_c$ sequence we derive from (3.1) and (3.3) that

$$
\int_\Omega Q(x)(u_\lambda+u_m)^{2^*}\,dx \le C_1 + C_2\|u_m\|_{H^1} \tag{3.4}
$$

for every $m \ge 1$. On the other hand, we have

$$
\begin{aligned}
&J_\lambda(u_m) - \frac{1}{2^*}\langle J_\lambda'(u_m), u_m\rangle \\
&\quad = \frac{1}{N}\int_\Omega(|\nabla u_m|^2 + \lambda u_m^2)\,dx + \frac{1}{2^*}\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*-1}(u_m - u_m^+ - u_\lambda)\,dx \\
&\qquad + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx + \int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx - \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m\,dx \\
&\quad = \frac{1}{N}\int_\Omega(|\nabla u_m|^2 + \lambda u_m^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*-1}u_\lambda\,dx \\
&\qquad - \frac{1}{2^*}\int_\Omega Q(x)(u_\lambda+u_m^+)^{2^*-1}u_m^-\,dx + \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx \\
&\qquad + \int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx - \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*-1}u_m\,dx
\end{aligned}
$$

$$= \frac{1}{N}\int_\Omega (|\nabla u_m|^2 + \lambda u_m^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1}u_\lambda\,dx$$
$$+ \frac{1}{2^*}\int_\Omega Q(x)u_\lambda^{2^*}\,dx + \left(1 - \frac{1}{2^*}\right)\int_\Omega Q(x)u_\lambda^{2^*-1}u_m^+\,dx\,dx$$
$$\geq \frac{1}{N}\int_\Omega (|\nabla u_m|^2 + \lambda u_m^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*-1}u_\lambda\,dx.$$

From this we deduce, using the Young inequality, that

$$\|u_m\|_{H^1}^2 \leq C_3 \int_\Omega Q(x)(u_\lambda + u_m^+)^{2^*}\,dx + C_4\|u_m\|_{H^1} + C_5. \tag{3.5}$$

The assertion follows from (3.3) and (3.5). ■

To proceed further we set

$$Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x).$$

We recall that Q_{M} is defined by $Q_{\mathrm{M}} = \max_{x\in\overline{\Omega}} Q(x)$. By S we denote the best Sobolev constant, that is,

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^N} |u|^{2^*}\,dx)^{2/2^*}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space defined by $D^{1,2}(\mathbb{R}^N) = \{u;\ \nabla u \in L^2(\mathbb{R}^N),\ u \in L^{2^*}(\mathbb{R}^N)\}$. The best Sobolev constant is achieved by

$$U(x) = \frac{c_N}{(N(N-2) + |x|^2)^{(N-2)/2}},$$

where $c_N > 0$ is a constant depending on N. The function U, called an *instanton*, satisfies the equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

We have $\int_{\mathbb{R}^N} |\nabla U|^2\,dx = \int_{\mathbb{R}^N} U^{2^*}\,dx = S^{N/2}$. For future use we introduce the notation

$$U_{\varepsilon,y} = \varepsilon^{-(N-2)/2}U\left(\frac{x-y}{\varepsilon}\right), \quad y \in \mathbb{R}^N,\ \varepsilon > 0.$$

We set

$$S_\infty = \min\left(\frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}, \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}\right).$$

PROPOSITION 3.3. *Let $\lambda > \lambda_*$. Suppose that*

$$J_\lambda(u_m) \to c < S_\infty, \tag{3.6}$$
$$J_\lambda'(u_m) \to 0 \quad \text{in } H^{-1}(\Omega). \tag{3.7}$$

Then up to a subsequence $u_m \rightharpoonup v \neq 0$ and v is a solution of problem (3.1_λ).

Proof. By Lemma 3.2, $\{u_m\}$ is bounded in $H^1(\Omega)$. Hence we may assume that $u_m \rightharpoonup v$ in $H^1(\Omega)$, $u_m \to u$ in $L^q(\Omega)$ for each $2 \le q < 2^*$ and $u_m(x) \to v(x)$ a.e. on Ω. Testing $J'_\lambda(u_m) \to 0$ with u_m^- we get

$$\int_\Omega (|\nabla u_m^-|^2 + \lambda (u_m^-)^2)\, dx = o(1).$$

Therefore we may assume that $u_m \ge 0$ on Ω. We now show that $v \ne 0$. Arguing by contradiction assume that $v \equiv 0$. By the P. L. Lions [12] concentration-compactness principle there exist sequences of points $\{x_j\} \subset \mathbb{R}^N$ and numbers $\{\nu_j\}, \{\mu_j\} \subset (0,\infty)$ such that

$$|u_m|^{2^*} \overset{*}{\rightharpoonup} \sum_j \nu_j \delta_{x_j} \quad \text{and} \quad |\nabla u_m|^2 \overset{*}{\rightharpoonup} \sum_j \mu_j \delta_{x_j}$$

in $\mathcal{M}$, where $\mathcal{M}$ is a space of measures. Moreover,

$$S\nu_j^{2/2^*} \le \mu_j \quad \text{if } x_j \in \Omega \quad \text{and} \quad S\,\frac{\nu_j^{2/2^*}}{2^{2/N}} \le \mu_j \quad \text{if } x_j \in \partial\Omega.$$

Testing (3.7) with $u_m\phi_\delta$, where ϕ_δ, $\delta > 0$, is a family of functions concentrating at x_j as $\delta \to 0$, we deduce that $\mu_j \le Q(x_j)\nu_j$ for every j. If $\nu_j > 0$ and $x_j \in \Omega$, then $\nu_j \ge S^{N/2}/Q(x_j)^{N/2}$, and if $x_j \in \partial\Omega$, then $\nu_j \ge S^{N/2}/2Q(x_j)^{N/2}$. By the Brézis–Lieb lemma we have

$$\begin{aligned}
J_\lambda(u_m) &- \frac{1}{2}\langle J'_\lambda(u_m), u_m\rangle \\
&= \frac{1}{N}\int_\Omega Q(x)(u_\lambda + u_m)^{2^*}\, dx - \frac{1}{2}\int_\Omega Q(x)(u_\lambda + u_m)^{2^*-1} u_\lambda\, dx \\
&\quad + \frac{1}{2^*}\int_\Omega Q(x) u_\lambda^{2^*}\, dx + \frac{1}{2}\int_\Omega Q(x) u_\lambda^{2^*-1} u_m\, dx + o(1) \\
&= \frac{1}{N}\sum_{x_j \in \partial\Omega} Q(x_j)\nu_j + \frac{1}{N}\sum_{x_j \in \Omega} Q(x_j)\nu_j + o(1) \\
&\ge \frac{1}{2N}\sum_{x_j \in \partial\Omega} \frac{S^{N/2}}{Q(x_j)^{(N-2)/2}} + \frac{1}{N}\sum_{x_j \in \Omega} \frac{S^{N/2}}{Q(x_j)^{(N-2)/2}} + o(1).
\end{aligned}$$

If $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$, then letting $m \to \infty$ we derive that $c \ge S^{N/2}/NQ_{\mathrm{M}}^{N-2/2}$, and if $Q_{\mathrm{M}} \le 2^{2/(N-2)} Q_{\mathrm{m}}$, then $c \ge S^{N/2}/2NQ_{\mathrm{m}}^{(N-2)/2}$. In both cases we obtain a contradiction. ■

4. Main result. In order to apply the mountain-pass theorem we set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\Omega));\ \gamma(0), \gamma(1) = t_\circ\}$$

and the constant $t_\circ$ is chosen so large that $J_\lambda(tt_\circ) \le 0$ for $t \ge 1$. It follows from Proposition 3.1 that $c > 0$.

We need the following relations for $U_{\varepsilon,y}$ with $y \in \partial\Omega$ (see [1] or [14]):

$$\frac{\int_\Omega (|\nabla U_{\varepsilon,y}|^2 + \lambda U_{\varepsilon,y}^2)\,dx}{(\int_\Omega U_{\varepsilon,y}^{2^*}\,dx)^{2/2^*}} \le \begin{cases} S/2^{2/N} - A_N H(y)\varepsilon \log\frac{1}{\varepsilon} + a_N\lambda\varepsilon + O(\varepsilon) + o(\lambda\varepsilon) & \text{if } N = 3, \\ S/2^{2/N} - A_N H(y)\varepsilon + a_N\lambda\varepsilon^2\log\frac{1}{\varepsilon} + O(\varepsilon^2\log\frac{1}{\varepsilon}) & \\ \quad + o(\lambda\varepsilon^2\log\frac{1}{\varepsilon}) & \text{if } N = 4, \\ S/2^{2/N} - A_N H(y)\varepsilon + a_N\lambda\varepsilon^2 + O(\varepsilon^2) + o(\lambda\varepsilon^2) & \text{if } N \ge 5, \end{cases} \tag{4.1}$$

where $H(y)$ denotes the mean curvature of $\partial\Omega$ at y.

It is known that

$$c \le c^* = \inf_{u\in H^1(\Omega),\, u\ne 0} \sup_{t\ge 0} J_\lambda(tu). \tag{4.2}$$

THEOREM 4.1. *Suppose that $Q_{\rm M} \ge 2^{2/(N-2)} Q_{\rm m}$ and that at some point $y \in \partial\Omega$ with $H(y) > 0$ we have*

$$|Q(x) - Q(y)| = o(|x-y|) \quad \textit{for } x \textit{ close to } y. \tag{4.3}$$

Then problem (3.1_λ) *has a solution for every* $\lambda > 0$.

Proof. It follows from (4.2) that

$$c \le c^* \le \frac{1}{N}\,\frac{(\int_\Omega (|\nabla U_{\varepsilon,y}|^2 + \lambda U_{\varepsilon,y}^2)\,dx)^{N/2}}{(\int_\Omega U_{\varepsilon,y}^{2^*}\,dx)^{(N-2)/2}}.$$

Thus (4.1) and (4.3) yield

$$c < \frac{S^{N/2}}{2NQ_{\rm m}^{(N-2)/2}}$$

for $\varepsilon > 0$ sufficiently small. By Proposition 3.3 problem (3.1_λ) has a solution. ∎

COROLLARY 4.2. *Under the assumptions of Theorem* 4.1 *there exists* $\lambda^* > 0$ *such that problem* (1.1_λ) *has at least two solutions for* $\lambda > \lambda^*$.

5. Existence of solutions for small boundary data. Lemma 2.2 (see also Corollary 2.3) provides the estimate of λ^* in terms of $\int_{\partial\Omega}\phi\,dS_x$. For $\lambda < \lambda^*$ problem (1.1_λ) does not have a solution for a given ϕ. In this section we establish the existence of a solution of problem (1.1_λ) for every $\lambda > 0$ if $\int_{\partial\Omega}\phi\,dS_x$ is small. Obviously, the size of $\int_{\partial\Omega}\phi\,dS_x$ will depend on λ.

Let

$$I_\lambda(u)=\frac{1}{2}\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx-\frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx-\int_{\partial\Omega}u\phi(x)\,dS_x$$

for $u\in H^1(\Omega)$ be a variational functional corresponding to problem (1.1_λ). In what follows we shall use the Sobolev inequality

$$\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{2/2^*}\le C_s\int_\Omega(|\nabla u|^2+u^2)\,dx$$

for $u\in H^1(\Omega)$, where $C_s>0$ is a constant. Letting $C_s(\lambda)=C_s$ for $\lambda\ge 1$ and $C_s(\lambda)=C_s/\lambda$ for $0<\lambda<1$, we can write this inequality in the form

$$\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{2/2^*}\le C_s(\lambda)\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx.$$

PROPOSITION 5.1. *Given $0<\lambda<\infty$ there exists a constant $\varrho_1=\varrho_1(\lambda)$ such that for a boundary data ϕ satisfying $\|\phi\|_{L^2(\partial\Omega)}\le\varrho_1$ problem (1.1_λ) has a solution. (If $\lambda\ge 1$ the choice of ϱ_1 can be made independent of λ.)*

Proof. A solution will be found as a local minimizer of I_λ. We commence by estimating I_λ from below:

$$\begin{aligned}I_\lambda(u)&\ge\frac{1}{2}\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx-\frac{C_s(\lambda)^{2^*/2}}{2^*}\,Q_{\mathrm{M}}\Big(\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx\Big)^{2^*/2}\\&\quad-K\Big(\int_{\partial\Omega}\phi^2\,dS_x\Big)^{1/2}\Big(\int_\Omega(|\nabla u|^2+u^2)\,dx\Big)^{1/2},\end{aligned}$$

where $K>0$ is the best constant for the embedding of $H^1(\Omega)$ into $L^2(\partial\Omega)$, that is,

$$K=\inf\Big\{\int_\Omega(|\nabla u|^2+u^2)\,dx;\ u\in H^1(\Omega),\ \int_{\partial\Omega}u^2\,dS_x=1\Big\}.$$

Letting $\|u\|_\lambda^2=\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx$ we can write this estimate as

$$I_\lambda(u)\ge\|u\|_\lambda\Big(\|u\|_\lambda-\frac{C_s(\lambda)^{2^*/2}}{2^*}\,Q_{\mathrm{M}}\|u\|_\lambda^{2^*-1}-K(\lambda)\|\phi\|_{L^2(\partial\Omega)}\Big),$$

where $K(\lambda)=K$ for $\lambda\ge1$ and $K(\lambda)=K/\lambda$ for $0<\lambda<1$. First we choose $\varrho>0$ such that

$$\varrho-\frac{C_s(\lambda)^{2^*/2}}{2^*}\,Q_{\mathrm{M}}\varrho^{2^*-1}\ge\frac{3}{4}\,\varrho.$$

If $\|\phi\|_{L^2(\partial\Omega)}\le\varrho/K(\lambda)=\varrho_1$, then

$$I_\lambda(u)\ge\varrho^2/4\quad\text{ for }\|u\|_\lambda=\varrho.\tag{5.1}$$

Testing $I_\lambda(u)$ with a constant function $u = t$ we get

$$I_\lambda(t) = \frac{|\Omega|}{2} t^2 - \frac{t^{2^*}}{2^*} \int_\Omega Q(x)\,dx - t \int_{\partial\Omega} \phi(x)\,dS_x < 0$$

for sufficiently small t. Therefore

$$c_1 = \inf_{\|u\|_\lambda = \varrho} I_\lambda(u) < 0. \tag{5.2}$$

It follows from (5.1), (5.2) and the Ekeland variational principle that there exists a minimizing sequence $\{u_m\}$ satisfying

$$I_\lambda(u_m) \to c_1 \quad \text{and} \quad I'_\lambda(u_m) \to 0 \quad \text{in } H^{-1}(\Omega).$$

It is clear that $\{u_m\}$ is bounded in $H^1(\Omega)$. Thus we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, $u_m \to u$ in $L^p(\Omega)$ for $2 \le p < 2^*$ and $u_m \to a$ a.e. on Ω. Moreover, u is a solution of (1.1_λ). We now observe that $\|u\|_\lambda \le \varrho$ and $I_\lambda(u) \ge c_1$. Since $\langle I'_\lambda(u), u\rangle = 0$, we see that

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \left(1 - \frac{1}{2^*}\right) \int_\Omega \phi u\,dS_x \ge c_1.$$

The weak lower semicontinuity of $\int_\Omega |\nabla u|^2\,dx$ yields

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \left(1 - \frac{1}{2^*}\right) \int_\Omega \phi u\,dS_x \le c_1$$

and consequently $I_\lambda(u) = c_1$. ■

To prove the existence of a second solution we use the method of Section 3. In what follows we assume that the boundary data ϕ satisfies

$$\|\phi\|_{L^2(\partial\Omega)} < \varrho_1 = \frac{\varrho}{4K(\lambda)}. \tag{5.3}$$

This condition on ϕ guarantees the existence of a local minimizer v_λ of the functional I_λ. As in Section 3 we consider the problem

$$\begin{cases} -\Delta v + \lambda v = Q(x)[(v + v_\lambda)^{2^*-1} - v_\lambda^{2^*-1}] & \text{in } \Omega, \\ \partial v/\partial\nu = 0. \end{cases} \tag{5.3_λ}$$

If problem (5.3_λ) has a solution w, then $w + v_\lambda$ is a solution of problem (1.1_λ). Let $\widetilde{I}_\lambda(v)$ be a variational functional corresponding to problem (5.3_λ). We now consider the variational problem

$$\widetilde{\mu}_\lambda = \inf\Big\{ \int_\Omega (|\nabla v|^2 + \lambda v^2)\,dx;\ v \in H^1(\Omega),\\ (2^* - 2) \int_\Omega Q(x) v_\lambda^{2^*-1} v^2\,dx = 1\Big\}. \tag{5.4}$$

Proposition 5.2. *Problem* (5.4) *has a minimizer w_λ which is the first eigenfunction of the eigenvalue problem*

$$\begin{cases} -\Delta v + \lambda v = \widetilde{\mu}_\lambda (2^* - 1) Q(x) v_\lambda^{2^*-2} v & \text{in } \Omega, \\ \partial v/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. As in the proof of Proposition 2.1 we obtain the existence of a minimizer w_λ. To show that $\widetilde{\mu}_\lambda > 1$ we take $\underline{\lambda} < \overline{\lambda} = \lambda$, where $\underline{\lambda}$ is chosen so that

$$\|\phi\|_{L^2(\partial\Omega)} < \frac{\varrho}{4K(\underline{\lambda})} < \frac{\varrho}{4K(\lambda)}.$$

Hence $I_{\underline{\lambda}}$ and $I_{\overline{\lambda}}$ have local minimizers $v_{\underline{\lambda}}$ and $v_{\overline{\lambda}}$, respectively. Let $z_{\underline{\lambda}}$ and $z_{\overline{\lambda}}$ be the minimal solutions of (1.1_λ) satisfying $0 \le z_{\underline{\lambda}} \le v_{\underline{\lambda}}$ and $z_{\overline{\lambda}} \le v_{\overline{\lambda}}$. Repeating estimates (2.2) with $u_{\underline{\lambda}}$ and $u_{\overline{\lambda}}$ replaced by $z_{\underline{\lambda}}$ and $z_{\overline{\lambda}}$ we derive that $\widetilde{\mu}_\lambda > 1$. ∎

This allows us to show that the variational functional $\widetilde{J}_\lambda$ for problem (5.3_λ) has a mountain-pass geometry. It is also easy to see that Proposition 3.3 continues to hold for the functional $\widetilde{J}_\lambda$ for every $\lambda > 0$.

We are now in a position to formulate the following existence result:

Theorem 5.3. *Suppose that ϕ satisfies* (5.3).

(i) *If $Q_{\mathrm{M}} \le 2^{2/(N-2)} Q_{\mathrm{m}}$ and at some point $y \in \partial\Omega$ the function Q satisfies condition* (4.3) *of Theorem* 4.1, *then problem* (5.3_λ) *has a solution.*

(ii) *If $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$, then there exists a $\widetilde{\lambda} > 0$ such that problem* (5.3_λ) *has a solution for $\lambda < \widetilde{\lambda}$.*

The proof of part (i) is identical to that of Theorem 4.1. To establish part (ii) we observe that $S_\infty = S^{N/2}/NQ_{\mathrm{M}}^{(N-2)/2}$. Testing $\widetilde{I}_\lambda$ with a constant function $u = 1$, we see that the mountain-pass level is below S_∞ if λ is small.

6. Case $\lambda = 0$. If $\lambda = 0$, then problem (1.1_0) cannot have a solution. Indeed, integrating equation (1.1_0) we get

$$-\int_{\partial\Omega} \phi(x)\, dS_x = \int_{\Omega} Q(x) u^{2^*-1}\, dx,$$

which is impossible. Therefore we assume throughout this section that

(**Q**) Q changes sign on Ω and $\int_\Omega Q(x)\, dx < 0$.

Since 0 is the first eigenvalue of the linear part of equation (1.1_λ) with the Neumann boundary conditions, it is convenient to decompose $H^1(\Omega) = \mathbb{R} \oplus V$, where the space V consists of functions v satisfying $\int_\Omega v(x)\, dx = 0$.

Having this decomposition we define an equivalent norm in $H^1(\Omega)$ by

$$\|u\|_V^2 = t^2 + \int_\Omega |\nabla v|^2\,dx.$$

LEMMA 6.1. *Suppose that $Q(x)$ satisfies* (**Q**). *Then there exists a constant $\eta > 0$ such that for each $t \in \mathbb{R}$ and $v \in V$ the inequality*

$$\Big(\int_\Omega |\nabla v|^2\,dx\Big)^{1/2} \le \eta|t|$$

implies

$$\int_\Omega Q(x)|t+v(x)|^{2^*}\,dx \le \frac{|t|^{2^*}}{2}\int_\Omega Q(x)\,dx.$$

For the proof we refer to [4].

PROPOSITION 6.2. *Suppose that $Q(x)$ satisfies* (**Q**). *Then there exist constants $\beta > 0$, $\beta_\circ > 0$ and $\varrho > 0$ such that*

$$I_\circ(u) \ge \beta \quad \text{for } \|u\|_V = \varrho \quad \text{and} \quad \|\phi\|_{L^2(\partial\Omega)} \le \beta_\circ. \tag{6.1}$$

Proof. We write

$$\begin{aligned} I_\circ(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx - \int_{\partial\Omega} u\phi(x)\,dS_x \\ &= \bar{I}_\circ(u) - \int_{\partial\Omega} u\phi(x)\,dS_x. \end{aligned}$$

We now consider two cases: (i) $\|\nabla v\|_2 \le \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$. If $\|\nabla v\|_2 \le \eta|t|$ and $\|\nabla v\|_2^2 + t^2 = \varrho^2$, then $t^2 \ge \varrho^2/(1+\eta^2)$. It then follows from Lemma 6.1 that

$$\int_\Omega Q(x)|t+v(x)|^{2^*}\,dx \le \frac{|t|^{2^*}}{2}\int_\Omega Q(x)\,dx = -|t|^{2^*}\alpha,$$

where $\alpha = -\frac{1}{2}\int_\Omega Q(x)\,dx > 0$. Hence we have

$$\bar{I}_\circ(u) \ge \frac{|t|^{2^*}}{2^*}\,\alpha \ge \frac{\alpha\varrho^{2^*}}{2^*(1+\eta^2)^{2^*/2}}.$$

In case (ii) we have $\|\nabla u\|_V \le \|\nabla v\|_2(1+1/\eta^2)^{1/2}$. Thus applying the Sobolev inequality we get

$$\int_\Omega Q(x)|u|^{2^*}\,dx \le C_1\|u\|_V^{2^*} \le C_1(1+1/\eta^2)^{2^*/2}\|\nabla v\|_2^{2^*}$$

for some constant $C_1 > 0$. Hence

$$\bar{I}_\circ(u) \ge \frac{1}{2}\|\nabla v\|_2^2 - C_1(1+1/\eta^2)^{2^*/2}\|\nabla v\|_2^{2^*}.$$

Taking $\|\nabla v\|_2 \leq \varrho$ small enough we deduce from the above inequality the estimate

$$\bar{I}_\circ(u) \geq \frac{1}{4}\|\nabla v\|_2^2.$$

On the other hand, if $\|u\|_V = \varrho$, then $\varrho \leq (\|\nabla v\|_2/\eta)(1+\eta^2)^{1/2}$. Hence

$$\bar{I}_\circ(u) \geq \frac{\eta^2\varrho^2}{4(1+\eta^2)}.$$

Taking

$$\beta_1 = \min\left(\frac{\eta^2\varrho^2}{4(1+\eta^2)}, \frac{\alpha\varrho^{2^*}}{2^*(1+\eta^2)^{2^*/2}}\right),$$

we obtain the following estimate for $\|u\|_V = \varrho$:

$$I_\circ(u) \geq \beta_1 - C_2\|\phi\|_{L^2(\partial\Omega)}\|u\|_V = \beta_1 - C_2\varrho\|\phi\|_{L^2(\partial\Omega)}$$

for some constant $C_2 > 0$. We now choose $\|\phi\|_{L^2(\partial\Omega)}$ so that

$$\|\phi\|_{L^2(\partial\Omega)} \leq \frac{\beta_1}{2C_2\varrho}.$$

This gives the desired estimate for $I_\circ(u)$ with $\beta = \beta_1/2$ and $\beta_\circ = \beta_1/2C_2\varrho$. ∎

Testing $I_\circ(u)$ with a constant function $u = t$, with t sufficiently small, we get $I_\circ(t) < 0$. Hence

$$c_2 = \inf_{\|u\|_V \leq \varrho} I_\circ(u) < 0.$$

Repeating the argument used in the proof of Proposition 5.1 we obtain

PROPOSITION 6.3. *Suppose that* (**Q**) *holds. Then there exists a constant* $\beta_\circ > 0$ *such that for* ϕ *satisfying* $\|\phi\|_{L^2(\partial\Omega)} \leq \beta_\circ$ *problem* (1.1$_0$) *admits a solution which is a local minimizer of* $I_\circ(u)$.

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