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GALOIS COVERINGS AND SPLITTING PROPERTIES OF THE IDEAL GENERATED BY HALFLINES

BY

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Abstract. Given a locally bounded k-category R and a group $G \subseteq \mathrm{Aut}_k(R)$ acting freely on R we study the properties of the ideal generated by a class of indecomposable locally finite-dimensional modules called halflines (Theorem 3.3). They are applied to prove that under certain circumstances the Galois covering reduction to stabilizers, for the Galois covering $F : R \to R/G$, is strictly full (Theorems 1.5 and 4.2).

Introduction. The Galois coverings technique is one of the most efficient tools of the modern representation theory, used successfully in solution of many important theoretical and classification problems. Originally invented for studying algebras of finite representation type [23, 17, 3, 19], the covering method was adopted for the representation infinite case [13, 12, 14] (see e.g. [30, 31, 32, 18] for applications), and also for matrix problems [25, 26, 28, 16, 11].

The fundamental and still unsolved problem concerning a Galois covering $\widetilde{\Lambda} \to \Lambda$ of an algebra Λ is if the representation type of the algebra Λ is determined by the representation type of its cover $\widetilde{\Lambda}$; in particular we do not know if Λ is tame provided so is $\widetilde{\Lambda}$.

There are two different approaches to this problem. The first, stated in [13, 12, 14] and developed in [4, 5, 8] (where the concept of direct Galois covering reduction to stabilizers was introduced) relies on a description of the structure of the category of all indecomposable Λ-modules that cannot be obtained "directly" from indecomposables over $\widetilde{\Lambda}$. This is done in terms of some group representation categories (the groups involved are usually infinite). The second approach bases on a reduction of the original problem to the analogous one for special classes of BOCS's and using methods specific to matrix problems (see [15]).

The first approach depends strongly on a splitting property of the Jacobson radical (viewed as a representation of certain group) between $\widetilde{\Lambda}$-modules

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of some special form (see 1.4, [8, Theorem A]). This property is usually a consequence of the description of nonisomorphisms between G-atoms (see 1.3), where $\widetilde{\Lambda}/G$ is a locally bounded category corresponding to Λ. Sometimes it can be expressed in terms of factorization through modules from some fixed class (see the example of the ideal $\mathcal{P}u$ of $\operatorname{Mod} R$ generated by finite-dimensional modules in [6, Theorem A] and [8, Theorem B]).

One of the main results of this paper, the Main Theorem of 1.5, is a development of [8, Theorem B]. It is obtained by a detailed analysis of splitting properties (with respect to injectivity) of the ideal generated by the class of locally finite-dimensional modules called halflines (see Definition 3.1 and Theorem 3.3). It seems to us that the halflines form a class much more suitable and natural with respect to that kind of properties than the finite-dimensional modules; this is especially visible when one compares the proofs of the corresponding facts (in the case of halflines they are much simpler and better reflect the general intuitions concerning the injectivity concept).

The paper is organized as follows. In Section 1 we recall basic notions used in the paper and the general context of the problems considered. We also formulate the main result of the paper, Main Theorem 1.5. In Section 2, properties of special homomorphism spaces consisting of all morphisms whose supports are halflines are studied. Section 3 is devoted to the properties of the ideal generated by the class of indecomposable locally finite-dimensional modules consisting of all halflines; our main results there are Theorem 3.3 and Proposition 3.4. In Section 4, by applying results of the previous sections, we prove Theorem 1.5 and we state its specialization (Theorem 4.2).

1. Preliminaries and the result. Before we formulate our main results, we recall from [5, 8], for the convenience of the reader, the basic notions and we sketch more precisely the situation we are dealing with in this paper. For basic information concerning representation theory of algebras (resp. rings and modules, and notions of the theory of categories) we refer to [2, 24, 27] (resp. [1], [21]).

1.1. Let k be a field and R be a *locally bounded k-category*, i.e. all objects of R have local endomorphism rings, different objects are nonisomorphic, and both sums $\sum_{y\in R}\dim_k R(x,y)$ and $\sum_{y\in R}\dim_k R(y,x)$ are finite for each $x\in R$. By an *R-module* we mean a contravariant k-linear functor from R to the category of k-vector spaces. An R-module M is *locally finite-dimensional* (resp. *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x\in R$ (resp. the *dimension* $\dim_k M=\sum_{x\in R}\dim_k M(x)$ of M is finite). We denote by $\operatorname{MOD} R$ the category of all R-modules, and by $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) the full subcategory formed by all locally finite-dimensional (resp. finite-dimensional) R-modules. By the *support* of an object M in $\mathrm{MOD}\,R$ we mean the full subcategory $\operatorname{supp} M$ of R formed by the set $\{x \in R : M(x) \neq 0\}$. If $f : M \to N$ is a homomorphism of R-modules then $\operatorname{supp}(\operatorname{Im} f)$ is called simply the support of f and briefly denoted by $\operatorname{supp} f$. We denote by $\mathcal{J}_R$ the Jacobson radical of the category $\mathrm{Mod}\,R$ (see [22]).

For any k-algebra A we denote by $\mathrm{MOD}\,A$ (resp. $\mathrm{mod}\,A$) the category of all (resp. all finite-dimensional) right A-modules.

1.2. Let G be a group of k-linear automorphisms of R acting freely on the set $\mathrm{ob}\,R$ of all objects of R. Then G acts on the category $\mathrm{MOD}\,R$ by translations ${}^{g}(-)$, which assign to each M in $\mathrm{MOD}\,R$ the R-module ${}^{g}M = M \circ g^{-1}$ and to each $f : M \to N$ in $\mathrm{MOD}\,R$ the R-homomorphism ${}^{g}f : {}^{g}M \to {}^{g}N$ given by the family $\big(f(g^{-1}(x))\big)_{x\in R}$ of k-linear maps. Given M in $\mathrm{MOD}\,R$ the subgroup

$$G_M = \{g \in G : {}^{g}M \simeq M\}$$

of G is called the *stabilizer* of M. We do not assume here that G acts freely on the set of isoclasses of indecomposable finite-dimensional R-modules (briefly $(\operatorname{ind} R)/\!\simeq$), i.e. that $G_M = \{\mathrm{id}_R\}$ for every indecomposable M in $\mathrm{mod}\,R$.

We consider the orbit category $\overline{R} = R/G$, which is again a locally bounded k-category (see [17]), and we study the module category $\mathrm{mod}\,\overline{R}$ in terms of the category $\mathrm{Mod}\,R$. The tool we have at our disposal is a pair of functors

$$\mathrm{MOD}\,R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \mathrm{MOD}\,\overline{R},$$

where $F_\bullet : \mathrm{MOD}\,\overline{R} \to \mathrm{MOD}\,R$ is the "pull-up" functor associated with the canonical Galois covering functor $F : R \to \overline{R}$, assigning to each X in $\mathrm{MOD}\,\overline{R}$ the R-module $X \circ F$, and the "push-down" functor $F_\lambda : \mathrm{MOD}\,R \to \mathrm{MOD}\,\overline{R}$ is the left adjoint to $F_\bullet$.

The classical results from [17] state that if G acts freely on $(\operatorname{ind} R)/\!\simeq$ then F_λ induces an embedding of the set $((\operatorname{ind} R)/\!\simeq)/G$ of G-orbits into $(\operatorname{ind} \overline{R})/\!\simeq$.

Let H be a subgroup of the stabilizer G_M of a given M in $\mathrm{MOD}\,R$. By an *R-action* of H on M we mean a family

$$\mu = (\mu_g : M \to {}^{g^{-1}}M)_{g\in H}$$

of R-homomorphisms such that $\mu_e = \mathrm{id}_M$, where $e = \mathrm{id}_R$ is the unit of H, and ${}^{g_1^{-1}}\mu_{g_2} \cdot \mu_{g_1} = \mu_{g_2 g_1}$ for all $g_1, g_2 \in H$ (see [17]). Observe that if H is a free group then M admits an R-action of H (see [4, Lemma 4.1]). We denote by $\mathrm{Mod}^H R$ the category of pairs (M, μ), where M is a locally finite-dimensional R-module and μ an R-action of H on M. For any $M = (M, \mu)$ and $N = (N, \nu)$ in $\mathrm{Mod}^H R$ the space $\mathrm{Hom}^H_R(M, N)$ of morphisms from M

to N in $\operatorname{Mod}^H R$ consists of all $f \in \operatorname{Hom}_R(M, N)$ such that ${}^{g^{-1}}f \cdot \mu_g = \nu_g \cdot f$ for every $g \in H$. Note that $\operatorname{Hom}_R^H(M, N)$ is the set of H-invariant elements in $\operatorname{Hom}_R(M, N)$ with respect to the action $\operatorname{Hom}_R(\mu, \nu) : H \times \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N)$ given by $(g, f) \mapsto {}^h\nu_g\, {}^g f \mu_{g^{-1}}$ for $g \in H$, $f \in \operatorname{Hom}_R(M, N)$, and that $\mathcal{J}(M, N)$ forms an H-invariant subspace in $\operatorname{Hom}_R(M, N)$. We denote by $\mathcal{J}_R^H$ the ideal $\operatorname{Hom}_R^H \cap \mathcal{J}_R$ of the category $\operatorname{Mod}^H R$.

A useful interpretation of $\operatorname{mod} \overline{R}$ is the category $\operatorname{Mod}_{\mathrm{f}}^G R$ consisting of pairs (M, μ) in $\operatorname{Mod}^G R$ such that $\operatorname{supp} M$ is contained in the union of a finite number of G-orbits in R (see [4, 17]). The functor $F_\bullet$ associating with any X in $\operatorname{mod} \overline{R}$ the R-module $F_\bullet X$ endowed with the trivial R-action of G yields an equivalence

$$\operatorname{mod}(\overline{R}) \simeq \operatorname{Mod}_{\mathrm{f}}^G R.$$

We denote by $\mathcal{I}_{\overline{R}}$ the ideal $F_\bullet^{-1}(\mathcal{J}_R^G)$ which constitutes an essential class of morphisms in $\operatorname{mod} \overline{R}$. It is clear that $\mathcal{I}_{\overline{R}}$ is contained in the Jacobson radical $\mathcal{J}_{\overline{R}}$, but usually we have $\mathcal{I}_{\overline{R}} \neq \mathcal{J}_{\overline{R}}$.

1.3. An important role in understanding the nature of objects from $\operatorname{Mod}_{\mathrm{f}}^G R$, or equivalently from $\operatorname{mod} \overline{R}$, is played by a class of indecomposable locally finite-dimensional R-modules called *G-atoms*. Following [4], an indecomposable B in $\operatorname{Mod} R$ (with local endomorphism ring) is called a *G-atom* if $\operatorname{supp} B$ is contained in the union of a finite number of G_B-orbits in R.

Denote by $\mathcal{A}$ a fixed set of representatives of isoclasses of all G-atoms, by $\mathcal{A}_{\mathrm{o}}$ a fixed set of representatives of G-orbits of the induced action of G on $\mathcal{A}$, and by $\overline{\mathcal{A}}$ the set of all $B \in \mathcal{A}$ such that $\operatorname{End}_R(B)/J(\operatorname{End}_R(B)) \simeq k$. Given a subset $\mathcal{U} \subset \mathcal{A}$ we set $\mathcal{U}_{\mathrm{o}} = {}^G\mathcal{U} \cap \mathcal{A}_{\mathrm{o}}$ (resp. $\overline{\mathcal{U}} = {}^G\mathcal{U} \cap \overline{\mathcal{A}}$), where ${}^G\mathcal{U}$ is the union of all orbits of elements from $\mathcal{U}$ in $\mathcal{A}$. For any $B \in \mathcal{A}$, denote by S_B a fixed set of representatives of left cosets of G_B in G, containing the unit e of G.

It is well known that the set of isoclasses of R-modules M in $\operatorname{Mod} R$ such that $G_M = G$ and $\operatorname{supp} M/G$ is finite, is in bijective correspondence with the set $(\mathbb{N}^{\mathcal{A}_{\mathrm{o}}})_0$ of all sequences $n = (n_B)_{B \in \mathcal{A}_{\mathrm{o}}}$ of natural numbers such that almost all n_B are zero. This correspondence is given by $n \mapsto M_n$, where

$$M_n = \bigoplus_{B \in \mathcal{A}_{\mathrm{o}}} \Big(\bigoplus_{g \in S_B} {}^g(B^{n_B}) \Big)$$

(see [8, Corollary 2.4]). In consequence, $\operatorname{mod} \overline{R}$ is equivalent via $F_\bullet$ to the full subcategory of $\operatorname{Mod}_{\mathrm{f}}^G R$ formed by all pairs (M_n, μ), where $n \in (\mathbb{N}^{\mathcal{A}_{\mathrm{o}}})_0$ and μ is an arbitrary R-action of G on M_n. Therefore to any X in $\operatorname{mod} \overline{R}$ one can attach the finite set $\operatorname{dss}(X)$, called the *direct summand support* of X, consisting of all $B \in \mathcal{A}_{\mathrm{o}}$ such that n_B is nonzero, where $F_\bullet X \simeq M_n$.

This suggests restricting the investigation of $\operatorname{mod} \overline{R}$ to some of its parts. For any $\mathcal{U} \subset \mathcal{A}_{\mathrm{o}}$ one can study the full subcategory $\operatorname{mod}_{\mathcal{U}} \overline{R}$ of $\operatorname{mod} \overline{R}$ consisting of all X in $\operatorname{mod} \overline{R}$ such that $\operatorname{dss}(X) \subset \mathcal{U}$.

The set $\mathcal{A}$ splits naturally into the disjoint union $\mathcal{A} = \mathcal{A}^{\mathrm{f}} \cup \mathcal{A}^{\infty}$, where $\mathcal{A}^{\mathrm{f}}$ (resp. $\mathcal{A}^{\infty}$) is the subset of all finite (dimensional) (resp. infinite (dimensional)) G-atoms. It is well known that if G acts freely on $(\operatorname{ind} R)/\simeq$ then the above splitting induces the splitting

$$(*) \qquad \operatorname{mod} \overline{R} = \operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R} \vee \operatorname{mod}_{\mathcal{A}^{\infty}} \overline{R}$$

in the sense explained below (see [12, Lemma] and [14, 2.3]).

Let $\mathcal{C}$ be a Krull–Schmidt category and $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ full subcategories of $\mathcal{C}$ which are closed under direct sums, direct summands and isomorphisms. The notation $\mathcal{C} = \mathcal{C}_1 \vee \mathcal{C}_2$ means that the set of indecomposable objects in $\mathcal{C}$ splits into the disjoint union of the indecomposables in $\mathcal{C}_1$ and in $\mathcal{C}_2$. We denote by $[\mathcal{C}_0]$ the ideal of all morphisms in $\mathcal{C}$ which factor through an object from $\mathcal{C}_0$. For any ideal $\mathcal{I}$ in the category $\mathcal{C}$ the restriction of $\mathcal{I}$ to $\mathcal{C}_0$ is denoted by $\mathcal{I}_{\mathcal{C}_0}$.

1.4. The splitting as in $(*)$ occurs in other, more general situations, when we replace $\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R}$ by the category $\operatorname{mod}_{\mathcal{U}} \overline{R}$, for some special classes $\mathcal{U} \subset \mathcal{A}_{\mathrm{o}}$, also contained in $\mathcal{A}^{\infty}$.

Let B be a periodic G-atom B and ν_B an R-action of G_B on B. Then (B, ν_B) is in $\operatorname{Mod}_{\mathrm{f}}^{G_B} R$ and $F_\lambda B$ has the structure of a kG_B-$\overline{R}$-bimodule, which is finitely generated free as a left kG_B-module, where kG_B is the group algebra of G_B over k (see [14, 3.6] for the precise definition of this structure). Consequently, it induces two functors

$$\Phi^B = - \otimes_{kG_B} F_\lambda B : \operatorname{mod} kG_B \to \operatorname{mod}_B \overline{R}$$

and

$$\Psi^B = (\overline{\mathcal{H}}_R(B, F_\bullet(-))^{-1} : \operatorname{mod} \overline{R} \to \operatorname{mod} kG_B,$$

where $\overline{\mathcal{H}}_R = \operatorname{Hom}_R / \mathcal{J}_R$ (see [4, 2.3 and 2.4]).

Now we recall the concept of Galois covering reduction to stabilizers, introduced in [5] and developed in [8].

Let $\mathcal{U} = (\mathcal{U}, \nu)$ be a pair where $\mathcal{U} \subset \mathcal{P}_{\mathrm{o}}$ is a subset of periodic G-atoms and $\nu = (\nu_B)_{B \in \mathcal{U}}$ a fixed selection of R-actions of G_B on B. We denote by

$$\Phi^{\mathcal{U}} : \coprod_{B \in \mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \overline{R}$$

the functor defined by the family $(\Phi^B)_{B \in \mathcal{U}}$, and by

$$\Psi^{\mathcal{U}} : \operatorname{mod} \overline{R} \to \prod_{B \in \mathcal{U}} \operatorname{mod} kG_B$$

the functor induced by the family $(\Psi^B)_{B\in\mathcal{U}}$, where Φ^B and Ψ^B are defined by the pairs (B,ν_B) (note that $\operatorname{Im}\Psi^{\mathcal{U}} \subset \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$). Then the pair $(\Phi^{\mathcal{U}},\Psi^{\mathcal{U}})$ of functors

$$\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \overset{\Phi^{\mathcal{U}}}{\underset{\Psi^{\mathcal{U}}}{\rightleftarrows}} \operatorname{mod} \overline{R}$$

is called the *Galois covering reduction to stabilizers* (briefly, *GCS-reduction*) *with respect to* $\mathcal{U}$ (in fact with respect to $(\nu_B)_{B\in\mathcal{U}}$). It is used to describe (in suitable situations) the category $\operatorname{mod}_{\mathcal{U}} \overline{R}$ in terms of the module categories of the stabilizer group algebras.

It is proved in [5, Theorem 2.2] that for any family $\mathcal{U} \subset \overline{\mathcal{P}}_{\mathrm{o}}$ the functor $\Phi^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \overline{R}$ is a right quasi-inverse for $\Psi^{\mathcal{U}} : \operatorname{mod} \overline{R} \to \prod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ (therefore faithful) and is a representation embedding in the sense of [29] (i.e. yields an injection of the set of isoclasses of indecomposables in $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ into the set of isoclasses of indecomposables in $\operatorname{mod}_{\mathcal{U}} \overline{R}$). Note that $\Psi^{\mathcal{U}}$ induces a functor

$$\overline{\Psi}^{\mathcal{U}} : \operatorname{mod} \overline{R}/[\operatorname{mod}_{\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U}} \overline{R}] \to \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$$

$(\ker \Psi^{\mathcal{U}} \subset [\operatorname{mod}_{\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U}} \overline{R}]$ by [8, Proposition 6.1]), and that $\Phi^{\mathcal{U}}$ induces a faithful representation embedding functor

$$\overline{\Phi}^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B \to \operatorname{mod} \overline{R}/[\operatorname{mod}_{\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U}} \overline{R}],$$

which is a right quasi-inverse for $\overline{\Psi}^{\mathcal{U}}$.

Following [5, 8], the GCS-reduction $(\Phi^{\mathcal{U}},\Psi^{\mathcal{U}})$ with respect to $\mathcal{U}$ is said to be *full* provided $\Phi^{\mathcal{U}}$ and $\Psi^{\mathcal{U}}$ induce

(a) a splitting $\operatorname{mod} \overline{R} = \operatorname{mod}_{\mathcal{U}} \overline{R} \vee \operatorname{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})} \overline{R}$,
(b) a bijection between the sets of isoclasses of indecomposable objects in the categories $\coprod_{B\in\mathcal{U}} \operatorname{mod} kG_B$ and $\operatorname{mod}_{\mathcal{U}} \overline{R}$.

The GCS-reduction $(\Phi^{\mathcal{U}},\Psi^{\mathcal{U}})$ with respect to $\mathcal{U}$ is called *strictly full* [8] provided the pair $(\overline{\Phi}^{\mathcal{U}},\overline{\Psi}^{\mathcal{U}})$ yields an equivalence of categories.

The possibility and efficiency of applying GCS-reduction with respect to a fixed set of G-atoms $\mathcal{U}$ usually depends on the splitting properties of G-atoms from $\mathcal{U}$.

Let B be a periodic G-atom together with an R-action ν_B of G_B on B. Following [8] we say that $B = (B,\nu_B)$ *splits* (resp. *splits properly*) an object $M = (M,\mu)$ in $\operatorname{Mod}^H R$ provided both embeddings $\mathcal{J}_R(B,M) \subset \operatorname{Hom}_R(B,M)$ and $\mathcal{J}_R(M,B) \subset \operatorname{Hom}_R(B,M)$ are splittable (resp. splittable, proper) monomorphisms in $\operatorname{MOD}(kG_B)^{\mathrm{op}}$ (see 1.2 for the definition of the G_B-action defining the kG_B-module structure).

The general result [8, Theorem A] concerning GCS-reductions asserts the following:

Let R be a locally bounded k-category and $G \subset \mathrm{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on $\operatorname{ob} R$. *Suppose that $\mathcal{U} \subset \overline{\mathcal{P}}_{\mathrm{o}}$ is a family of G-atoms together with a selection $(\nu_B)_{B\in\mathcal{U}}$ of R-actions of G_B on B such that each (B,ν_B) splits* $\mathrm{Mod}_{\mathrm{f}}^G R$, *for $B \in \mathcal{U}$. Then the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ with respect to $\mathcal{U}$ is full.*

1.5. As usual, we denote by $\mathcal{A}^1$ the set of all G-atoms $B \in \mathcal{A}$ (in fact infinite G-atoms) such that G_B is an infinite cyclic group, and by $\mathcal{A}^{1'}$ the subset of all $B \in \mathcal{A}^\infty$ such that G_B has an infinite cyclic subgroup of finite index. Observe that $\mathcal{A}^1 \subset \mathcal{P}$ and that for any $B \in \mathcal{A}^1$ the group algebra kG_B is isomorphic to the Laurent polynomial algebra $k[T,T^{-1}]$. It is shown in [7] that $\mathcal{A}^\infty$ coincides with $\mathcal{A}^1$ provided R is a representation-tame category over an algebraically closed field and the group G is torsion-free. Moreover, $\mathcal{A}^{1'} = \mathcal{A}^1$ provided G is torsion-free (see [9, Corollary 6.3]).

Following [8] for any $B \in \mathcal{A}^1$ we denote by $\mathcal{A}^1(B)$ (resp. $\mathcal{A}^{1'}(B)$) the set of all $B' \in \mathcal{A}^1$ (resp. $B' \in \mathcal{A}^{1'}$) satisfying the following conditions:

(a) $\operatorname{supp} B' \subset \widehat{\operatorname{supp}} B$,
(b) $G_{B'} \cap G_B \neq \{e\}$,
(c) $\operatorname{supp} B' \cap \operatorname{supp} B \neq \emptyset$.

Here for any subcategory L of R, $\widehat{L}$ denotes the full subcategory of R consisting of all $y \in R$ such that $R(x,y)$ or $R(y,x)$ is nonzero for some $x \in L$ (see [13]). Note that if (b) and (c) hold then $\operatorname{supp} B' \cap \operatorname{supp} B$ is infinite since so is $G_B \cap G_{B'}$.

We recall from [8, proof of Proposition 6.3] that for a given $B \in \mathcal{A}^1$ only G-atoms from $\mathcal{A}^{1'}(B)$ are important when $\overline{R}$-modules containing B in their direct summand support are considered. Moreover, the splitting properties of G-atoms $B \in \mathcal{A}^1$ often depend on the properties of the homomorphism spaces between B and G-atoms $B' \in \mathcal{A}^{1'}(B)$, which are expressed in terms of factorization through direct sums of R-modules which belong to a fixed class (see [8]). In [8] the class of finite-dimensional modules is considered; in the present paper we consider a larger class of indecomposable R-modules called halflines.

An R-module M in $\operatorname{Ind} R$ is called a *halfline* provided there exists a torsion-free element $h \in G$ and a finite full subcategory D of R such that $\operatorname{supp} M \subset \bigcup_{n\in\mathbb{N}} h^n D$ (see 3.1 and 2.1 for the precise definitions).

We are interested in the properties of the ideal generated by halflines, especially in some injectivity property (see Theorem 3.3). One of our main results is the following theorem, which is a generalization of [8, Theorem B].

Main Theorem. *Let R be a locally bounded k-category, $G \subset \mathrm{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on* $\mathrm{ob}\, R$, *and $\mathcal{U}$ be a subset of $\overline{\mathcal{A}}_{\mathrm{o}}^{1}$ together with a selection $\{\nu_B\}_{B\in\mathcal{U}}$ of R-actions of G_B on B. Assume that for any $B \in \mathcal{U}$ and $B' \in \mathcal{A}^{1'}(B)$ each R-nonisomorphism $f : B \to B'$ (resp. $f : B' \to B$) factors through a direct sum of halflines. Then the Galois covering reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ to stabilizers with respect to $\mathcal{U}$ is strictly full and the functors $\Phi^{\mathcal{U}} : \coprod_{B\in\mathcal{U}} \mathrm{mod}\, kG_B \to \mathrm{mod}\, \overline{R}$ and $\Psi^{\mathcal{U}} : \mathrm{mod}\, \overline{R} \to \prod_{B\in\mathcal{U}} \mathrm{mod}\, kG_B$ defined by the families $(\Phi^B)_{B\in\mathcal{U}}$ and $(\Psi^B)_{B\in\mathcal{U}}$ induce the following equivalence*:

$$\coprod_{B\in\mathcal{U}} \mathrm{mod}\, k[T,T^{-1}] \simeq \mathrm{mod}\, \overline{R}/[\mathrm{mod}_{\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U}}\, \overline{R}] \simeq \mathrm{mod}_{\mathcal{U}}\, \overline{R}/[\mathrm{mod}_{\mathcal{A}^{\mathrm{f}}}\, \overline{R}]_{\mathrm{mod}_{\mathcal{U}}\, \overline{R}}.$$

In particular, the functors $\Phi^{\mathcal{U}}$ and $\Psi^{\mathcal{U}}$ induce:

(i) *a splitting* $\mathrm{mod}\, \overline{R} = \mathrm{mod}_{\mathcal{U}}\, \overline{R} \vee \mathrm{mod}_{(\mathcal{A}_{\mathrm{o}}\setminus\mathcal{U})}\, \overline{R}$,

(ii) *a bijection between the sets of isoclasses of indecomposables in* $\mathrm{mod}_{\mathcal{U}}\, \overline{R}$ *and in* $\coprod_{B\in\mathcal{U}} \mathrm{mod}\, k[T,T^{-1}]$.

In case the group G acts freely on $(\mathrm{ind}\, R)/\simeq$ *the above equivalence has the form*

$$\coprod_{B\in\mathcal{U}} \mathrm{mod}\, k[T,T^{-1}] \simeq \underline{\underline{\mathrm{mod}}}\,_{\mathcal{U}}\overline{R},$$

where $\underline{\underline{\mathrm{mod}}}\,_{\mathcal{U}}\overline{R}$ is defined below.

Suppose the group G acts freely on $(\mathrm{ind}\, R)/\simeq$. We denote by $\mathrm{mod}_1\, \overline{R}$ the full subcategory of $\mathrm{mod}\, \overline{R}$ consisting of the $\overline{R}$-modules of the first kind, i.e. those of the form $F_\lambda(M)$ for some M in $\mathrm{mod}\, R$ (see [14, 4, 5]). We denote by $\underline{\underline{\mathrm{mod}}}\, \overline{R}$ the factor category $\mathrm{mod}\, \overline{R}/[\mathrm{mod}_1\, \overline{R}]$. For any subset $\mathcal{U} \subset \mathcal{A}$ we denote by $\underline{\underline{\mathrm{mod}}}\,_{\mathcal{U}}\overline{R}$ the image of $\mathrm{mod}_{\mathcal{U}}\, \overline{R}$ in the factor category $\underline{\underline{\mathrm{mod}}}\, \overline{R}$.

2. Injectivity of the modules $\mathcal{H}^+$ and $\mathcal{H}^-$

2.1. The full subcategory L of R is called a *generalized line* if there exists a torsion-free element $h \in G$ and a finite full subcategory D of R such that $L \subset \bigcup_{n\in\mathbb{Z}} h^n D$ and the intersection $L \cap h^n D$ is nontrivial, for every $n \in \mathbb{Z}$. We then say that L is an H-*line*, where $H = \{h^n : n \in \mathbb{Z}\}$.

The subcategory L is called a *generalized halfline* (briefly, a *halfline*) if there exists a torsion-free element $h \in G$ and a finite full subcategory D of R such that $L \subset \bigcup_{n\in\mathbb{N}} h^n D$. We then say that L is an H'-*halfline*, where $H' = \{h^n : n \in \mathbb{N}\}$. The halfline L is called *proper* if $L \cap h^n D$ is nontrivial for almost all $n \in \mathbb{N}$.

Remark. L is a proper halfline if and only if there exists h and D as above such that $L \subset \bigcup_{n\in\mathbb{N}} h^n D$ and $L \cap h^n D$ is nontrivial for all $n \in \mathbb{N}$ (take $D' = \bigcup_{0\le n\le n_0} h^n D$, where $L \cap h^n D$ is nontrivial for all $n \geq n_0$).

Lemma. (a) *Let L be a subcategory of R, H an infinite cyclic group with a fixed generator $h \in G$, and H_m the subgroup generated by h^m for $m \in \mathbb{N} \setminus \{0\}$. Then L is an H-line if and only if L is an H_m-line.*

(b) *Let L be a subcategory of R, $H' = \{h^n : n \in \mathbb{N}\}$ for some torsion-free element $h \in G$, and $H'_m = \{h^{nm} : n \in \mathbb{N}\}$ for $m \in \mathbb{N} \setminus \{0\}$. Then L is an H'-halfline (resp. a proper H'-halfline) if and only if L is an H'_m-line (resp. a proper H'_m-halfline).*

(c) *Let L' be a halfline (resp. proper halfline) and L be an H-line, where H is an infinite cyclic group with a fixed torsion-free generator $h \in G$. Suppose that $L \cap L'$ is infinite. Then L' is an H'-halfline (resp. a proper H'-halfline), where $H' = \{h^n : n \in \mathbb{N}\}$ or $H' = \{h^{-n} : n \in \mathbb{N}\}$.*

Proof. (a) Assume that L is contained in the H-line $\bigcup_{n\in\mathbb{Z}} h^n D$ (with finite D) such that $h^n D \cap L$ is nontrivial for all $n \in \mathbb{Z}$. Then setting $D' = \bigcup_{0\le n<m} h^n D$ we have $L \subset \bigcup_{n\in\mathbb{Z}} h^{nm} D'$ $(= \bigcup_{n\in\mathbb{Z}} h^n D)$ and $L \cap h^{nm} D'$ is clearly nontrivial for all $n \in \mathbb{N}$, so L is an H_m-line.

Suppose now that L is an H_m-line contained in the H_m-line $\bigcup_{n\in\mathbb{Z}} h^{nm} D$ (with finite D) such that $L \cap h^{nm} D$ is nontrivial for every $n \in \mathbb{N}$. Then setting again $D' = \bigcup_{0\le n<m} h^n D$ we clearly have $L \subset \bigcup_{n\in\mathbb{Z}} h^n D'$ and $h^n D' \cap L$ is nontrivial for every $n \in \mathbb{N}$ ($m\mathbb{Z} \cap \{n, \ldots, n+m-1\} \neq \emptyset$!), so L is an H-line.

(b) The proof is analogous to that of (a).

(c) Assume that L is contained in the H-line $L_1 = \bigcup_{n\in\mathbb{Z}} h^n D$ and L' is contained in the halfline $\bigcup_{n\in\mathbb{N}} h'^n D'$ (h, h' and D, D' are as in the definition). Denote by L_2 the H_2-line $\bigcup_{n\in\mathbb{Z}} h^n D'$, where H_2 is the infinite cyclic group generated by h', and set $H_1 = H$. By assumption $L_1 \cap L_2$ is infinite; consequently, by [4, Lemma 3.6], $H_0 = H_1 \cap H_2$ is nontrivial. Since H_0 is an infinite cyclic group with a generator h_0, there exist $m, m' \in \mathbb{Z}$ such that $h_0 = h^m = (h')^{m'}$. Now the assertion follows easily from (b). ∎

2.2. Let M, N be a pair of modules in $\operatorname{Mod} R$. Set $L = \operatorname{supp} M \cap \operatorname{supp} N$. Assume that the intersection of the stabilizers $G_M \cap G_N$ contains an infinite cyclic group H such that L is contained in a finite number of H-orbits in R (in fact L).

Fix a pair of R-actions: μ of H on M and ν of H on N (always exist since H is a free group). Then the k-vector space $\mathcal{H} = \operatorname{Hom}_R(M, N)$ can be regarded as a left kH-module with the structure defined by the action $\operatorname{Hom}_R(\mu, \nu)$, which is given by the mapping $(h, f) \mapsto {}^h\nu_h \cdot {}^h f \cdot \mu_{h^{-1}}$ for $h \in H$ and $f \in \operatorname{Hom}_R(M, N)$.

Fix a generator h of the group H and denote by H^+ (resp. H^-) the subsemigroup $\{h^n : n \in \mathbb{N}\}$ (resp. $\{h^{-n} : n \in \mathbb{N}\}$).

Denote by $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) the subset of $\mathcal{H}$ formed by all $f \in \mathcal{H}$ such that $\operatorname{supp} f$ is an H^+-halfline (resp. H^--halfline). It is easily seen that both $\mathcal{H}^+$ and $\mathcal{H}^-$ are KH-submodules of $\mathcal{H}$. We can also regard $\mathcal{H}^+$ as a kH^+-submodule and $\mathcal{H}^-$ as a kH^--submodule of $\mathcal{H}$, where $kH^+ = k[h]$ $(= \bigoplus_{n\in\mathbb{N}} kh^n)$ and $kH^- = k[h^{-1}]$ $(= \bigoplus_{n\in\mathbb{N}} kh^{-n})$ are two different copies of the polynomial algebra in one variable over k contained in the algebra kH. We denote by $k[[h]]$ and $k[[h^{-1}]]$ the corresponding formal power series algebras.

THEOREM. *The kH-modules $\mathcal{H}^+$ and $\mathcal{H}^-$ are injective.*

The proof of this theorem (see 2.4) needs some preparation.

2.3. Recall from [6] that for R-modules M_1 and M_2 in $\mathrm{MOD}\,R$ a family $(f_i)_{i\in I}$ of homomorphisms in $\mathrm{Hom}_R(M_1, M_2)$ is said to be *summable* if for each $x \in R$ and $m \in M_1(x)$, $f_i(x)(m) = 0$ for almost all $i \in I$ (if M_1 is in $\mathrm{Mod}\,R$, this is equivalent to the condition that for each $x \in R$, $f_i(x) = 0$ for almost all $i \in I$). In this case the well defined R-homomorphism $f = \sum_{i\in I} f_i : M_1 \to M_2$, given by $f(x)(m) = \sum_{i\in I} f_i(x)(m)$ for any $x \in R$ and $m \in M_1(x)$, is called the *sum* of the family $(f_i)_{i\in I}$.

A subspace $\mathcal{W}$ of $\mathrm{Hom}_R(M_1, M_2)$ is called *summably closed* if the sum of any summable family $(f_i)_{i\in I}$ of R-homomorphisms in $\mathcal{W}$ belongs to $\mathcal{W}$. An ideal $\mathcal{I}$ of a full subcategory $\mathcal{C}$ of $\mathrm{MOD}\,R$ is said to be summably closed if the subspace $\mathcal{I}(M_1, M_2)$ of $\mathrm{Hom}_R(M_1, M_2)$ is summably closed for each pair M_1, M_2 of R-modules in $\mathcal{C}$.

Let M and N be as in 2.2. We denote by $\mathcal{H}_+$ the set of all $f \in \mathcal{H}$ such that $\{h^n f\}_{n\in\mathbb{N}}$ is summable in $\mathcal{H}$. For any $f \in \mathcal{H}_+$ and a formal series $a = \sum_{n\in\mathbb{N}} a_n h^n \in k[[h]]$, where $a_n \in k$ for $n \in \mathbb{N}$, we denote by $a \cdot f$ the sum $\sum_{n\in\mathbb{N}} a_n(h^n f)$ in $\mathcal{H}$ ($\{a_n(h^n f)\}_{n\in\mathbb{N}}$ is summable in $\mathcal{H}$). This yields a map

$$(*) \qquad \cdot : k[[h]] \times \mathcal{H}_+ \to \mathcal{H}.$$

Analogously, we denote by $\mathcal{H}_-$ the set of all $f \in \mathcal{H}$ such that $\{h^{-n} f\}_{n\in\mathbb{N}}$ is summable in $\mathcal{H}$, and for any $f \in \mathcal{H}_-$ and $a = \sum_{n\in\mathbb{N}} a_n h^{-n} \in k[[h^{-1}]]$ we denote by $a \cdot f$ the sum $\sum_{n\in\mathbb{N}} a_n(h^{-n} f)$ in $\mathcal{H}$. This furnishes a map

$$(**) \qquad \cdot : k[[h^{-1}]] \times \mathcal{H}_- \to \mathcal{H}.$$

LEMMA.

- (i) (a) *$\mathcal{H}_+$ is a kH-submodule* (*resp. kH^+-submodule*) *of $\mathcal{H}$.*
  - (b) *$\mathcal{H}_+$ is a $k[[h]]$-module* (*via the map* $(*)$).
  - (c) *If $W \subseteq \mathcal{H}_+$ is a summably closed subspace* (*in $\mathcal{H}$*) *which is a kH^+-submodule of $\mathcal{H}$ then W is a $k[[h]]$-submodule of $\mathcal{H}_+$.*
  - (d) *$\mathcal{H}^+$ is a $k[[h]]$-submodule of $\mathcal{H}_+$, in fact $\mathcal{H}^+ = \mathcal{H}_+$.*
- (ii) (a′) *$\mathcal{H}_-$ is a kH-submodule* (*resp. kH^--submodule*) *of $\mathcal{H}$.*
  - (b′) *$\mathcal{H}_-$ is a $k[[h^{-1}]]$-module* (*via the map* $(**)$).

(c′) *If $W \subseteq \mathcal{H}^-$ is a summably closed subspace (in $\mathcal{H}$) which is a kH_--submodule of $\mathcal{H}$ then W is a $k[[h^{-1}]]$-submodule of $\mathcal{H}_-$.*

(d′) *$\mathcal{H}^-$ is a $k[[h^{-1}]]$-submodule of $\mathcal{H}_+$, in fact $\mathcal{H}^- = \mathcal{H}_-$.*

Proof. We only prove (i); the proof of (ii) is completely analogous.

(a) Fix f in $\mathcal{H}_+$. For any $x \in R$ there exists the smallest natural number $n_x = n_x(f)$ such that $(h^n f)(x) = f(h^{-n}x) = 0$ for all $n \geq n_x$. Then for any $p \in \mathbb{Z}$ and $n \in \mathbb{N}$, $(h^n(h^p f))(x) = (h^{n+p} f)(x) = 0$ if $n + p \geq n_x$. Consequently, for $p \in \mathbb{Z}$, the family $(h^n(h^p f))_{n\in\mathbb{N}}$ is summable ($n_x(h^p f) = n_x - p$ if $n_x \geq p$, and $n_x(h^p f) = 0$, otherwise).

(b) We have to show that for any fixed $a = \sum_{n\in\mathbb{N}} a_n h^n \in k[[h]]$ and $f \in \mathcal{H}_+$ the element $a \cdot f$ belongs to $\mathcal{H}_+$ (i.e. $(h^m(a \cdot f))_{m\in\mathbb{N}}$ is summable). We start by observing that under the notation introduced above for any $p \in \mathbb{Z}$ and $x \in R$,

$$n_{h^p x} = \begin{cases} n_x + p & \text{if } n_x + p \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

since $(h^n f)(h^p x) = f(h^{p-n}x) = (h^{n-p} f)(x)$ for every $n \in \mathbb{N}$, and $(h^{n-p} f)(x) = 0$ if $n - p \geq n_x$. For any $m \in \mathbb{N}$, the family $(h^{m+n} f)_{n\in\mathbb{N}}$ is summable (see proof of (a)); therefore $h^m(a \cdot f)$ is well defined and we have

$$h^m(a \cdot f) = \sum_{n\in\mathbb{N}} a_n (h^{m+n} f)$$

since for any $x \in R$,

$$\begin{aligned}\Big(\sum_{n\in\mathbb{N}} a_n(h^n f)\Big)(h^{-m}x) &= \sum_{0\leq n\leq n_{h^{-m}x}} a_n (h^n f)(h^{-m}x) \\ &= \sum_{0\leq n\leq n_x - m} a_n f(h^{-m-n}x) = \sum_{0\leq n\leq n'_x} a_n (h^{n+m} f)(x)\end{aligned}$$

(by previous remarks $n_{h^{-m}x} = n_x - m = n'_x$, where $n'_x = n_x(h^m f)$). Consequently, $(h^m(a \cdot f))(x) = 0$ for all $m \geq n_x$ (then $(h^{m+n} f)(x) = 0$), the family $(h^m(a \cdot f))_{m\in\mathbb{N}}$ is summable ($n_x(a \cdot f) \leq n_x$) and $a \cdot f$ belongs to $\mathcal{H}_+$.

(c) Follows easily from (b).

(d) To show $\mathcal{H}^+ \subseteq \mathcal{H}_+$ fix $f \in \mathcal{H}^+$ with $\operatorname{supp} f \subset \bigcup_{n\in\mathbb{N}} h^n D$, where D is a finite subcategory of L. Then there exists $n_0 \in \mathbb{N}$ such that $D \cap h^n D = \emptyset$ for every $n \geq n_0$ (R is locally bounded and H acts freely on R). Therefore for any fixed $p \in \mathbb{N}$, $h^p D \cap h^n D = \emptyset$ for all $n \geq n_0 + p$. Consequently, $(h^m f)_{|\, h^p D} = 0$ for every $m \geq n_0 + p$, since $\operatorname{supp} h^m f \subset h^m(\operatorname{supp} f) \subset \bigcup_{n\in\mathbb{N}} h^{n+m} D \subset \bigcup_{n\geq n_0+p} h^n D$. Hence, $(h^m f)_{m\in\mathbb{N}}$ is summable and $f \in \mathcal{H}_+$. It is clear that $\operatorname{supp}(a\cdot f) \subset \bigcup_{n\in\mathbb{N}} h^n D$ for every $a \in k[[h]]$, and therefore $\mathcal{H}^+$ is a $k[[h]]$-submodule of $\mathcal{H}_+$.

To prove the last assertion ($\mathcal{H}_+ \subseteq \mathcal{H}^+$) fix $f \in \mathcal{H}_+$ and a (finite) set L_{o} of representatives of H-orbits in L. For any $x \in L_{\mathrm{o}}$ we denote by x' the object $x' = h^{-n_x}x$ of L, where $n_x = n_x(f)$ is as in the proof of (a). Then $n_{x'} = 0$ by the definition, and $f(h^{-n}x') = (h^n f)(x') = 0$ for all $n \in \mathbb{N}$. Consequently, $\operatorname{supp} f \subset \bigcup_{n\in\mathbb{N}} h^n D$, where D is a finite full subcategory of L formed by the set $\{x' : x \in L_{\mathrm{o}}\}$, and so $f \in \mathcal{H}^+$. ■

Remark. $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) is not necessarily summably closed since for any $f \in \mathcal{H}$ with $\operatorname{supp} f$ finite (equivalently, $f \in \mathcal{H}^+ \cap \mathcal{H}^-$) the family $(h^{-n}f)_{n\in\mathbb{N}}$ (contained in $\mathcal{H}^+$) is summably closed but $\sum_{n\in\mathbb{N}}(h^{-n}f) \in \mathcal{H}^+ \setminus \mathcal{H}^-$.

2.4. Denote by $k[t]$ the polynomial algebra in one variable t over k, by $k[t,t^{-1}]$ the Laurent polynomial algebra and by $k[[t]]$ the corresponding power series algebra. Since $k[t]$ can be naturally treated as a canonical subalgebra of $k[t,t^{-1}]$ (resp. of $k[[t]]$), each module M over $k[t,t^{-1}]$ (resp. $k[[t]]$) can be regarded as a module over $k[t]$.

Lemma. *Let W be a $k[[t]]$-module such that $t\cdot : W \to W$ is a k-isomorphism. Then W is injective as a module over $k[t]$, as well as over $k[t,t^{-1}]$.*

Proof. First observe that by the assumptions W can indeed be regarded as a $k[t,t^{-1}]$-module. Moreover, both $k[t]$ and $k[t,t^{-1}]$ are principal ideal domains, so they are factorial. Therefore to prove the injectivity of W it suffices to show that W is divisible, that is, $rW = W$ for any irreducible $r \in k[t]$ (resp. $r \in k[t,t^{-1}]$). It is well known that irreducible elements in $k[t,t^{-1}]$ are, up to invertibles (multiplicities of powers of t), polynomials (in $k[t]$) with nonzero constant term. Consequently, in the case of $k[t,t^{-1}]$ we can assume that the irreducible elements $r \in k[t,t^{-1}]$ belong to $k[t]$ and are invertible as elements of $k[[t]]$; hence, the required equality $rW = W$ holds automatically since W is a $k[[t]]$-module. Similarly, in the case of $k[t]$, any irreducible element $r \in k[t]$ is, up to an invertible (now a nonzero scalar), either equal to t or invertible in $k[[t]]$. Now, by the assumption and the argument as above we again obtain $rW = W$. ■

Proof of Theorem 2.2. To show that the kH-module $\mathcal{H}^+$ is injective observe that by Lemma 2.3, $\mathcal{H}^+$ is a $k[[h]]$-module which satisfies the assumption of Lemma 2.4. Consequently, $\mathcal{H}^+$ is injective as kH^+ and kH-module. The proof for $\mathcal{H}^-$ is analogous. ■

2.5. The following fact plays an essential role in further considerations and allows us to understand better the structure of lines.

Lemma (cf. [6, Lemma 2.4]). *Let L be a full subcategory of a locally bounded k-category R, and H be an infinite cyclic group of k-linear automorphisms of R acting freely on R, with a fixed generator $h \in H$. Assume that H stabilizes L (i.e. $gL = L$ for all $g \in H$) and L is contained in the* *union of a finite number of H-orbits in R (consequently, L is a line). Then for any finite subcategory D of L there exist full subcategories D^0, D^+ and D^- of R and a trisection*

$$L = D^- \vee D^0 \vee D^+$$

of the category L satisfying the following conditions:

(a) *D^0 is a finite subcategory containing D and intersecting nontrivially each H-orbit in L,*
(b) *D^+ and D^- are orthogonal,*
(c) *D^+ is a proper H^+-halfline in R contained in $\bigcup_{n\in\mathbb{N}} h^n D^0$,*
(d) *D^- is a proper H^--halfline in R contained in $\bigcup_{n\in\mathbb{N}} h^{-n} D^0$,*
(e) *$h^n(D^0\cup D^+)\subset D^+$ for $n \gg 0$,*
(f) *$h^{-n}(D^0\cup D^-)\subset D^-$ for $n \gg 0$.*

Proof. Fix any full subcategory L_{o} of L formed by a fixed set of representatives of H-orbits in L. Then $L=\bigcup_{n\in\mathbb{Z}} h^n L_{\mathrm{o}}$ so L is a line. We set $D' = D\cup L_{\mathrm{o}}$. Then clearly $L=\bigcup_{n\in\mathbb{Z}} h^n D'$. Since R is locally bounded there exists $n_0\in\mathbb{N}$ such that D' and $h^m D'$ are pairwise orthogonal for any $m\geq n_0$. Now it is easily seen that setting $D^0=\bigcup_{0<m<n_0} h^m D'$, $D^+=\bigcup_{n_0\leq m} h^m D'$ and $D^-=\bigcup_{m\leq 0} h^m D'$, we obtain the required trisection and conditions (a)–(f) are satisfied. ■

Remark. If L is a connected line then D^0 can also be chosen connected.

3. Properties of the ideal generated by halflines

3.1. An R-module M in $\operatorname{Ind} R$ is called a *halfline* (resp. *proper halfline*) if $\operatorname{supp} M$ is a halfline (resp. proper halfline).

Now we prove the following property of halflines (rather natural from the intuitive point of view).

Lemma. *Let the R-module M be a halfline. Then either M is in* $\operatorname{ind} R$ *or M is a proper halfline.*

Sublemma. *Let L be a full subcategory of R, and L_0, L_1, L_2 be three full nontrivial subcategories of L. Suppose that L admits a trisection $L = L_1\vee L_0\vee L_2$ such that L_1 and L_2 are orthogonal. If M is in* $\operatorname{Ind} R$ *with* $\operatorname{supp} M\subset L$ *and* $\operatorname{supp} M\cap L_0$ *is trivial then exactly one of the inclusions* $\operatorname{supp} M\subset L_2$ *or* $\operatorname{supp} M\subset L_1$ *holds.*

Proof. Follows immediately from the fact that the support of any indecomposable R-module forms a connected subcategory of R (apply for example [13, Lemma 2]). ■

Proof of Lemma. Let M be an infinite-dimensional halfline (in $\operatorname{Ind} R$) such that $S=\operatorname{supp} M$ is contained in the halfline $L^+=\bigcup_{n\in\mathbb{N}} h^n D$, where h and D are as in the definition. Without loss of generality we can assume that $D\cap S$ is nontrivial; fix $x \in D\cap S$. Set $L=\bigcup_{n\in\mathbb{Z}} h^nD$. Then there exists a trisection $L=D^-\vee D^0\vee D^+$ as in Lemma 2.5 (in particular $D\subset D^0$). Let n_0 be the smallest $m\in\mathbb{N}$ such that $h^mD^0\subset L^+$ (it exists, see the construction of D^0 in the proof of Lemma 2.5). Then clearly $h^mD^0\subset L^+$ for all $m\geq n_0$ (as $hL^+\subset L^+$). Moreover, h^mD^- contains x for almost all $m\geq n_0$, and consequently L^+ admits a splitting

$$L^+=(h^mD^-\cap L^+)\vee h^mD^0\vee h^mD^+$$

into three nontrivial subcategories (note that $x\in h^mD^-\cap L^+$, $h^mD^0\cap L^+=h^mD^0$ and $D^+\subset\bigcup_{n\in\mathbb{N}} h^nD^0$, so $h^mD^+\subset\bigcup_{n\geq n_0} h^nD^0\subset L^+$). Then for any m as above, S cannot be contained in $h^mD^-\cap L^+$ since the latter is finite, while S is an infinite subcategory of L^+. Hence by the Sublemma, $S\cap h^mD^0$ is nontrivial since $x\in h^mD^-\cap S$. Consequently, M is a proper halfline ($S\subset\bigcup_{n\in\mathbb{N}} h^nD\subset\bigcup_{n\in\mathbb{N}} h^nD^0$ and $S\cap h^mD^0$ is nontrivial for almost all $m\geq n_0$). ■

Corollary. *Let M in* $\operatorname{Mod} R$ *be such that* $\operatorname{supp} M$ *is contained in a halfline. Then each term of a decomposition of M into a direct sum of indecomposables is either in* $\operatorname{ind} R$ *or a proper halfline.*

3.2. For any M and N in $\operatorname{MOD} R$ we define the subspace

$$(*)\qquad \mathcal{H}alf(M,N)\subseteq \operatorname{Hom}_R(M,N)$$

to consist of all R-homomorphisms $f:M\to N$ having a factorization through a direct sum of halflines in $\operatorname{Ind} R$. It is easily seen that the subspaces $(*)$ define a two-sided ideal

$$(**)\qquad \mathcal{H}alf(\cdot,-)\subseteq \operatorname{Hom}_R(\cdot,-)$$

of the category $\operatorname{MOD} R$.

The following property of the restriction $\mathcal{H}alf_{\operatorname{Mod} R}$ of the ideal $\mathcal{H}alf$ to $\operatorname{Mod} R$ is essential for further considerations.

Lemma. *Let M, N in* $\operatorname{Mod} R$ *be as in* 2.2. *Then $\mathcal{H}alf(M,N)$ consists of all homomorphisms $f:M\to N$ factorizing through a locally finite-dimensional module $Z=\bigoplus_{i\in I} Z_i$ such that each Z_i is an* (*indecomposable*) *halfline with* $\operatorname{supp} Z_i$ *contained in the line* $L=\operatorname{supp} M\cap\operatorname{supp} N$.

Proof. Fix any $f\in\mathcal{H}alf(M,N)$ and a factorization

$$M\xrightarrow{f'}\bigoplus_{i\in I} Z_i\xrightarrow{f''} N$$

of f, where f' (resp. f'') is given by the family $(f'_i:M\to Z_i)_{i\in I}$ (resp. $(f''_i:Z_i\to N)_{i\in I}$) of R-homomorphisms and each Z_i, $i\in I$, is a halfline.

For every $i \in I$, we set $\overline{Z}_i = \operatorname{Im} f_i$, where $f_i = f_i'' f_i'$. Then f also has the factorization

$$M \xrightarrow{\overline{f}'} \bigoplus_{i\in I} \overline{Z}_i \xrightarrow{\overline{f}''} N,$$

where $\overline{f}'$ is given by $(f_i : M \to \operatorname{Im} f_i)_{i\in I}$ and $\overline{f}''$ by $(\operatorname{Im} f_i \hookrightarrow N)_{i\in I}$. Note that $\overline{Z} = \bigoplus_{i\in I} \overline{Z}_i$ is a locally finite-dimensional R-module since all $f_i : M \to \overline{Z}_i$, $i \in I$, are surjective and M is locally finite-dimensional. Now we decompose each $\overline{Z}_i$ into indecomposables, $\overline{Z}_i = \bigoplus_{j\in I_i} \overline{Z}_{i,j}$. Thus f factorizes through $\bigoplus_{i\in I} \bigoplus_{j\in I_i} \overline{Z}_{i,j}$. Moreover, by Lemma 2.1(c), each $\overline{Z}_{i,j}$ is a halfline with $\operatorname{supp} \overline{Z}_{i,j} \subset L$, because $\operatorname{supp} \overline{Z}_{i,j} \subset \operatorname{supp} \overline{Z}_i$ and $\operatorname{supp} \overline{Z}_{i,j} \subset \operatorname{supp} \overline{Z}_i \subset L$. ∎

Remark. If L' is a halfline contained in the H-line L (H is a cyclic group with a fixed generator h) then L' is an H^+- or H^--halfline.

3.3. Let M and N be as in 2.2, together with fixed R-actions μ and ν of H on M and N, respectively. Define $\mathcal{H}' = \mathcal{H}alf(M,N) \subseteq \mathcal{H}$ $(= \operatorname{Hom}_R(M,N))$. Observe that $\mathcal{H}'$ is a kH-submodule of $\mathcal{H}$.

Theorem. *$\mathcal{H}'$ is an injective kH-module.*

Proof. We show that $\mathcal{H}' = \mathcal{H}^+ + \mathcal{H}^-$, which implies the assertion. Note that $\mathcal{H}^+ + \mathcal{H}^-$ is divisible since both $\mathcal{H}^+$ and $\mathcal{H}^-$ are injective (see Theorem 2.2), so divisible; therefore it is injective (kH is a principal ideal domain).

Observe first that $\mathcal{H}^+ \subset \mathcal{H}'$ since by Corollary 3.1 for any $f \in \mathcal{H}'$ each term of a decomposition of $\operatorname{Im} f = \bigoplus_{i\in I} Z_i$ into a direct sum of indecomposable R-submodules ($\operatorname{Im} f$ belongs to $\operatorname{Mod} R$) is a halfline ($\operatorname{supp} f$ is a halfline). By analogous reasons $\mathcal{H}^- \subseteq \mathcal{H}'$, and consequently $\mathcal{H}^+ + \mathcal{H}^- \subseteq \mathcal{H}'$.

To prove the inverse inclusion, let $f \in \mathcal{H}'$ have a factorization

$$M \xrightarrow{f'} \bigoplus_{i\in I} Z_i \xrightarrow{f''} N,$$

where f' (resp. f'') is given by the family $(f_i' : M \to Z_i)_{i\in I}$ (resp. $(f_i'' : Z_i \to N)_{i\in I}$) of R-homomorphisms and each Z_i, $i \in I$, is an (indecomposable) halfline. Then by Lemma 3.2 we can assume that $Z = \bigoplus_{i\in I} Z_i$ is locally finite-dimensional and each Z_i is a halfline (by Lemma 3.1, either a finite-dimensional module or a proper halfline) with $\operatorname{supp} Z_i \subset L$. Now we fix a trisection $L = D^- \vee D^0 \vee D^+$ satisfying the assertions of Lemma 2.5. Then by Sublemma 3.1, the set I splits into the disjoint union

$$I = I^- \cup I_0 \cup I^+,$$

where $I_0 = \{i \in I : \operatorname{supp} Z_i \cap D^0 \neq \emptyset\}$, $I^+ = \{i \in I : \operatorname{supp} Z_i \subset D^+\}$, and $I^- = \{i \in I : \operatorname{supp} Z_i \subset D^-\}$. Observe that I_0 is finite since Z is a locally finite-dimensional R-module and D^0 is a finite subcategory.

Denote by I_0^+ (resp. I_0^-) the set of all $i \in I_0$ such that $\operatorname{supp} Z_i$ is a proper H^+-halfline (resp. H^--halfline), and set $I_0^0 = I_0 \setminus (I_0^+ \cup I_0^-)$ (note that $I_0^0 = \{i \in I_0 : \dim_k Z_i < \infty\}$). Then $f = \sum_{i\in I} f_i$ can be represented in the form

$$f = \sum_{i\in I^+ \cup I_0^+} f_i + \sum_{i\in I_0^- \cup I_0^-} f_i + \sum_{i\in I_0^0} f_i$$

(note that $(f_i)_{i\in I}$ is summable!). Observe that the first sum is in $\mathcal{H}^+$ since $f_i \in \mathcal{H}^+$ for all $i \in I_0^+$ (I_0^+ is finite) and $\operatorname{supp} \sum_{i\in I^+} f_i \subset D^+$, which is an H^+-halfline. Analogously one shows that the second sum is in $\mathcal{H}^-$. It is also easily seen that $\sum_{i\in I_0^0} f_i \in \mathcal{H}^+ \cap \mathcal{H}^-$; consequently, $f \in \mathcal{H}^+ + \mathcal{H}^-$. ∎

3.4. We need one more result on the behaviour of the ideal $\mathcal{H}alf$ with respect to the property of summable closedness.

PROPOSITION. *Let M and N be as in* 2.2. *Then for any decomposition $N = \bigoplus_{t\in T} N_t$ $(= \prod_{t\in T} N_t)$ into a direct sum of R-submodules the induced injections*

$$(*) \qquad \mathcal{H}alf(M, N) \to \prod_{t\in T} \mathcal{H}alf(M, N_t)$$

and

$$(**) \qquad \mathcal{H}alf(N, M) \to \prod_{t\in T} \mathcal{H}alf(N_t, M)$$

are k-isomorphisms.

Proof. It suffices to prove that one of the maps is a k-isomorphism. Indeed, we have the duality

$$(-)^* : \operatorname{Mod} R \to \operatorname{Mod} R^{\mathrm{op}}.$$

Moreover, for any X, Y in $\operatorname{Mod} R$ as in 2.2 the space $\mathcal{H}alf(X, Y)$ consists of all R-homomorphisms $f : X \to Y$ which factorize through a locally finite-dimensional module $Z = \bigoplus_{i\in I} Z_i$ such that all Z_i are halflines (see Lemma 3.2). Finally, $(-)^*$ preserves halflines and direct sums which are locally finite-dimensional.

We prove that $(*)$ is a k-isomorphism. Let $f_t \in \mathcal{H}alf(M, N_t)$, $t \in T$. We can regard each f_t as a map in $\operatorname{Hom}_R(M, N)$ (via the canonical embedding $N_t \subseteq N$); then $(f_t)_{t\in T}$ is a summable family in $\operatorname{Hom}_R(M, N)$, in fact in $\mathcal{H}alf(M, N)$. To prove our claim it suffices to show that $f = \sum_{t\in T} f_t \in \mathcal{H}alf(M, N)$. We fix a trisection $L = D^- \vee D^0 \vee D^+$ as in Lemma 2.5, where D is an arbitrary selection of representatives of H-orbits in L. Note that f factors through $\bigoplus_{t\in T} \operatorname{Im} f_t$ which is a locally finite-dimensional submodule N (each $\operatorname{Im} f_t$ is a submodule of N_t). There exists a finite subset $T_0 \subseteq T$ such that $(\operatorname{Im} f_t)_{|D^0} = 0$ for all $t \in T' = T \setminus T_0$. Consequently, by Sublemma 3.1, $\operatorname{Im} f_t$ admits a (unique) decomposition $\operatorname{Im} f_t = N_t^+ \oplus N_t^-$

such that $\operatorname{supp} N_t^+ \subset D^+$ and $\operatorname{supp} N_t^- \subset D^-$ (decompose $\operatorname{Im} f_t$ into a direct sum of indecomposables in $\operatorname{Ind} R$; then N_t^+, resp. N_t^-, is the direct sum of all summands in the decomposition whose support is contained in D^+, resp. D^-). The decomposition $\operatorname{Im} f_t = N_t^+ \oplus N_t^-$, $t \in T'$, induces two standard maps $f_t^+, f_t^- : M \to N$ such that $f_t = f_t^+ + f_t^-$. Note that $(f_t^+)_{t\in T'}$ (resp. $(f_t^-)_{t\in T'}$) is a summable family since so is $(f_t)_{t\in T'}$ and $f_t^+{}_{|D^+} = f_{t|D^+}$, $f_t^+{}_{|D^0\cup D^-} = 0$ (resp. $f_t^-{}_{|D^-} = f_{t|D^-}$, $f_t^-{}_{|D^0\cup D^+} = 0$). Moreover,

$$\sum_{t\in T'} f_t = \sum_{t\in T'} f_t^+ + \sum_{t\in T'} f_t^-.$$

Recall that $\mathcal{H}alf(M,N) = \mathcal{H}^+ + \mathcal{H}^-$ (the proof of Theorem 2.2) and therefore for any $t \in T_0$, f_t has a decomposition $f_t = f_t^+ + f_t^-$, where $f_t^+ \in \mathcal{H}^+$ and $f_t^- \in \mathcal{H}^-$. Consequently, $(f_t^+)_{t\in T}$ and $(f_t^-)_{t\in T}$ are summable families with sums $f^+ = \sum_{t\in T} f_t^+$ and $f^- = \sum_{t\in T} f_t^-$, and we have

$$f = f^+ + f^-$$

since

$$\sum_{t\in T_0\cup T'} f_t = \sum_{t\in T'} f_t^+ + \sum_{t\in T'} f_t^- + \sum_{t\in T_0} (f_t^+ + f_t^-) = \sum_{t\in T} f_t^+ + \sum_{t\in T} f_t^-.$$

Moreover, f^+ belongs to $\mathcal{H}^+$ since so do f_t^+, $t \in T_0$, and $\sum_{t\in T'} f_t^+$ ($\operatorname{supp} \sum_{t\in T'} f_t^+ \subset D^+$); analogously, f^- belongs to $\mathcal{H}^-$ since so do $f_t^-, t\in T_0$, and $\sum_{t\in T'} f_t^-$ ($\operatorname{supp} \sum_{t\in T'} f_t^- \subset D^-$). In conclusion, $f \in \mathcal{H}afl(M,N)$ and the proof is complete. ∎

Corollary. *Let M and N be as above. Then for any decompositions $M = \bigoplus_{s\in S} M_s$, $N = \bigoplus_{t\in T} N_t$ $(= \prod_{t\in T} N_t)$ the induced injective map*

$$\mathcal{H}alf(M,N) \to \prod_{s\in S} \prod_{t\in T} \mathcal{H}alf(M_s, N_t)$$

is a k-linear isomorphism.

Proof. Since by the proposition the standard map induces a k-isomorphism $\mathcal{H}alf(M,N) \simeq \prod_{t\in T} \mathcal{H}alf(M,N_t)$ it suffices to show that the standard injection induces isomorphisms

$$(*)_t \qquad \mathcal{H}alf(M,N_t) \simeq \prod_{s\in S} \mathcal{H}alf(M_s,N_t)$$

for all $t \in T$. For any $t \in T$, set $N_t' = N_t' = \bigoplus_{t'\in T\setminus\{t\}} N_t$. Then applying the above proposition and the standard isomorphism we have

$$\mathcal{H}alf(M,N) \simeq \prod_{s\in S} \mathcal{H}alf(M_s,N) \simeq \prod_{s\in S} \mathcal{H}alf(M_s,N_t) \oplus \prod_{s\in S} \mathcal{H}alf(M_s,N_t').$$

Composing the above isomorphism with

$$\mathcal{H}alf(M, N_t) \oplus \mathcal{H}alf(M, N'_t) \simeq \mathcal{H}alf(M, N)$$

and next looking at components we obtain the required isomorphism $(*)_t$ for every $t \in T$. ■

4. Proof of the main result and some specialization

4.1. *Proof of Main Theorem.* We proceed analogously to the proof of [8, Theorem B]. First we show that under the assumptions of the theorem, for a G-atom $B \in \mathcal{A}^1$ and a fixed R-action ν_B of G_B on B, (B, ν_B) splits properly (see Introduction) every $(M, \mu) \in \operatorname{Mod}^{G_B} R$ such that B is a direct summand of M (in $\operatorname{Mod} R$), provided $\mathcal{J}_R(B, B') = \mathcal{H}alf(B, B')$ and $\mathcal{J}_R(B', B) = \mathcal{H}alf(B', B)$ for every $B' \in \mathcal{A}^{1'}(B)$. This follows by repeating the arguments from the proof of [8, Proposition 6.3], where injectivity of the kG_B-modules $\mathcal{J}_R(B, M)$ and $\mathcal{J}_R(M, B)$ is proved (G_B is an infinite cyclic group). We just need to replace the ideal $\mathcal{P}u$ in that proof by $\mathcal{H}alf$, and references to [6, Theorem A] by references to Theorem 3.3 and Proposition 3.4. In this way we conclude that the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ with respect to $\mathcal{U}$ is full (see Introduction).

To prove the main assertion of the theorem, the equivalence

$$\coprod_{B \in \mathcal{U}} \operatorname{mod} k[T, T^{-1}] \simeq \operatorname{mod} \overline{R}/[\operatorname{mod}_{\mathcal{A}_{\mathrm{o}} \setminus \mathcal{U}} \overline{R}] \simeq \operatorname{mod}_{\mathcal{U}} \overline{R}/[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R}]_{\operatorname{mod}_{\mathcal{U}} \overline{R}}$$

(in particular, that the GCS-reduction $(\Phi^{\mathcal{U}}, \Psi^{\mathcal{U}})$ is strictly full), we again apply the arguments from [8, 6.3]. Since the inclusion

$$[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R}] \subset [\operatorname{mod}_{\mathcal{A}_{\mathrm{o}} \setminus \mathcal{U}} \overline{R}] \subset \ker \Psi^{\mathcal{U}}$$

is obvious (cf. [8, 6.3(d)]), we only have to show the inverse inclusion

$$\mathcal{I}_{\operatorname{mod}_{\mathcal{U}} \overline{R}} \subset [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R}]_{\operatorname{mod}_{\mathcal{U}} \overline{R}}$$

(cf. [8, 6.3(e)]; recall that by [8, Proposition 6.1], $(\ker \Psi^{\mathcal{U}})_{\operatorname{mod}_{\mathcal{U}} \overline{R}} = \mathcal{I}_{\operatorname{mod}_{\mathcal{U}} \overline{R}}$ and $\ker \Psi^{\mathcal{U}}(X, Y) = \operatorname{Hom}_{\overline{R}}(X, Y)$ if X or Y is not in $\operatorname{mod}_{\mathcal{U}} \overline{R}$). To show this inclusion fix any $f \in \mathcal{I}(X', X)$, where X and X' are indecomposables in $\operatorname{mod}_{\mathcal{U}} \overline{R}$ ($X \simeq \Phi^B V$, $X' \simeq \Phi^{B'} V'$ for some $B, B' \in \mathcal{U}$ and indecomposables V, V' in $\operatorname{mod} kG_B$). To prove that $f \in [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R}]$ we proceed as in [8, 6.3] to show that f (in fact, its composition with some $\overline{R}$-isomorphism u) factors through the $\overline{R}$-module $\operatorname{Hom}_{kG_B}(F_\lambda B^{kG_B}, \mathcal{C}_{B^*}(F_\bullet X')^{-1})$, where $\mathcal{C}_{B^*}(F_\bullet X')$ is as in [8, 4.4 and 5.1]. Note that since the kG_B-module $\mathcal{J}^B(F_\bullet X') = \mathcal{J}_R(F_\bullet X', B)$ is injective, the finitely generated kG_B-module $\mathcal{C}_{B^*}(F_\bullet X')$ is free ($\mathcal{C}_{B^*}(F_\bullet X')^* \simeq \mathcal{J}_R(F_\bullet X', B)$, see [8, 5.4]). Consequently, f factors through $F_\lambda B^m$ for some $m \in \mathbb{N}$, and a slight modification of [4, Lemma 4.4] yields $f \in [\mathrm{mod}_{\mathcal{A}^{\mathrm{f}}} \overline{R}]$. ■

4.2. Now we formulate an immediate consequence of the main result. We need some notation from [8].

For any subset $\mathcal{U} \subset \mathcal{P}_{\mathrm{o}}$ and $n \in \mathbb{N}$ we denote by $\mathcal{U}(n)$ the set of all $B \in \mathcal{U}$ such that the rank of the free kG_B-module $F_\lambda B$ is n.

THEOREM (cf. [8, Theorem 6.3]). *Let R be a locally bounded k-category over an algebraically closed field, and $G \subset \mathrm{Aut}_k(R)$ be a torsion free group of k-linear automorphisms of R acting freely on R such that $\overline{R}$ is a finite category. Assume R satisfies the following conditions*:

(i) $\mathcal{A}^\infty = \mathcal{A}^1$,
(ii) $\mathcal{J}_R(B_1, B_2) = \mathcal{H}alf\,(B_1, B_2)$ (*resp.* $\mathcal{J}_R(B_2, B_1) = \mathcal{H}alf\,(B_2, B_1)$ *for all* $B_1 \in \mathcal{A}^1_{\mathrm{o}}$ *and* $B_2 \in \mathcal{A}^1(B_1)$.

Then the functors $\Phi^{\mathcal{A}^1_{\mathrm{o}}}$ *and* $\Psi^{\mathcal{A}^1_{\mathrm{o}}}$ *induce an equivalence*

$$\coprod_{B \in \mathcal{A}^1_{\mathrm{o}}} \mathrm{mod}\, k[T, T^{-1}] \simeq \underline{\underline{\mathrm{mod}}}\, \overline{R},$$

and $\overline{R}$ *is tame if and only if* R *is tame and all sets* $\mathcal{A}^1_{\mathrm{o}}(n)$, $n \in \mathbb{N}$, *are finite.*

Proof. Note that $\mathcal{A}^{1'} = \mathcal{A}^1$ (see [9, Corollary 6.3]) and $\overline{\mathcal{A}} = \mathcal{A}$ (see [10, Proposition 7.5]). Then the assertions follow directly from the Main Theorem by applying [14, Lemma 2.2] and [13, Lemma 3, Proposition 2]. ■

REMARK. (!) Assume that R is tame (this is always the case if $\overline{R}$ is tame, see [13, Proposition 2]). Then (i) of Theorem 4.2 automatically holds under the assumptions preceding (i) (see [9, Main Theorem]).

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