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AN EXTENSION WHICH IS RELATIVELY TWOFOLD MIXING BUT NOT THREEFOLD MIXING

BY

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Abstract. We give an example of a dynamical system which is mixing relative to one of its factors, but for which relative mixing of order three does not hold.

1. Factors, extensions and relative mixing

1.1. *Factors, extensions and Rokhlin cocycle.* We are interested in dynamical systems $(X, \mathscr{A}, \mu, T)$, where T is an ergodic automorphism of the Lebesgue space $(X, \mathscr{A}, \mu)$. We will often designate such a system by simply the symbol T. A *factor* of T is a sub-σ-algebra $\mathscr{H}$ of $\mathscr{A}$ such that $\mathscr{H} = T^{-1}\mathscr{H}$.

The canonical example of a system with a factor is given by the *skew product*, constructed from a dynamical system $(X_H, \mathscr{A}_H, \mu_H, T_H)$ (called the *base* of the skew product) and a measurable map $x \mapsto S_x$ from X_H to the group of automorphisms of some Lebesgue space $(Y, \mathscr{B}, \nu)$ (such a map is called a *Rokhlin cocycle*). The transformation is defined on the product space $(X_H \times Y, \mathscr{A}_H \otimes \mathscr{B}, \mu_H \otimes \nu)$ by

$$\widetilde{T}(x, y) = (T_H x, S_x y).$$

In this context, the sub-σ-algebra $\mathscr{A}_H \otimes \{Y, \emptyset\}$ is clearly a factor of $\widetilde{T}$.

Since the work of Abramov and Rokhlin [1], this kind of construction is known to be the general model for a system with a factor: If $\mathscr{H}$ is a factor of T, then there exists an isomorphism φ between T and a skew product $\widetilde{T}$ constructed as above, with $\varphi(\mathscr{H}) = \mathscr{A}_H \otimes \{Y, \emptyset\}$. In such a situation, we say that T is an *extension* of T_H.

1.2. *Mixing relative to a factor.* To understand precisely the way a factor is embedded in the dynamical system, one is led to study the behaviour of the system *relative to the factor*; to this end, relative properties are defined which are generalizations of absolute properties of dynamical systems. For example, one can define weak mixing relative to a factor (see e.g. [2]), or the property of being a K-system relative to a factor [4].

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In this work we are interested in the property of mixing relative to a factor.

DEFINITION 1.1. Let $\mathscr{H}$ be a factor of the system $(X, \mathscr{A}, \mu, T)$. Then T is said to be *$\mathscr{H}$-relatively mixing* if

$$(1)\quad \forall A, B \in \mathscr{A}, \quad \mu(A \cap T^{-k}B \mid \mathscr{H}) - \mu(A \mid \mathscr{H})\mu(T^{-k}B \mid \mathscr{H}) \xrightarrow[k\to+\infty]{\text{prob}} 0.$$

As for absolute mixing, it is possible to define mixing relative to a factor of any order $n \geq 2$. The property described by (1) corresponds to relative mixing of order 2 (*twofold relative mixing*); for relative mixing of order 3 (*threefold relative mixing*), (1) should be replaced by

$$(2)\quad \forall A, B, C \in \mathscr{A},$$
$$\mu(A\cap T^{-j}B\cap T^{-k}C \mid \mathscr{H}) - \mu(A \mid \mathscr{H})\mu(T^{-j}B \mid \mathscr{H})\mu(T^{-k}C \mid \mathscr{H}) \xrightarrow[j,k-j\to+\infty]{\text{prob}} 0.$$

Whether (absolute) twofold mixing implies threefold mixing is a well known open problem in ergodic theory. The main goal of this work is to show that as far as relative mixing is concerned, twofold does not necessarily imply threefold.

THEOREM 1.1. *There is a dynamical system $(X, \mathscr{A}, \mu, T)$ with a factor $\mathscr{H}$ such that T is $\mathscr{H}$-relatively twofold mixing but not $\mathscr{H}$-relatively threefold mixing.*

2. An extension which is relatively twofold mixing but not relatively threefold mixing

2.1. *The base.* The dynamical system announced in Theorem 1.1 is constructed as a skew product, whose base $(X_H, \mathscr{A}_H, \mu_H, T_H)$ is obtained as follows: Take $X_H := [0, 1[$ equipped with the Lebesgue measure μ_H on the Borel σ-algebra $\mathscr{A}_H$. The transformation T_H can be viewed as a triadic version of the von Neumann–Kakutani transformation; we now describe its construction by the *cutting and stacking* method (see Figure 1).

We begin by splitting X_H into three subintervals of length $1/3$; we set $B_1 := [0, 1/3[$. The transformation T_H translates B_1 onto $T_HB_1 := [1/3, 2/3[$, and translates T_HB_1 onto $T_H^2B_1 := [2/3, 1[$. At this first step, T_H is not yet defined on $T_H^2B_1$. In general, after the nth step of the construction, X_H has been split into 3^n intervals of the same length: $B_n, T_HB_n, \dots, T_H^{3^n-1}B_n$. These intervals form a so-called *Rokhlin tower* with base B_n and height 3^n. Such a tower is usually represented by putting the intervals on top of one another, the transformation T_H mapping each point to the one exactly above. At this step, the transformation is not yet defined on $T_H^{3^n-1}B_n$. Step $n + 1$ starts by chopping the base B_n into three subintervals of the same length, the first of which is denoted by B_{n+1}. The nth Rokhlin tower is thus split into three columns, which are stacked together to get the $n+1$st tower. This amounts to mapping $T_H^{3^n-1}B_{n+1}$ onto the second piece of B_n by a translation, and $T_H^{2\times 3^n-1}B_{n+1}$ onto the third piece of B_n. T_H is now defined everywhere except on $T_H^{3^{n+1}-1}B_{n+1}$.

The iteration of this construction for all $n \geq 1$ defines T_H everywhere on X_H. The transformation obtained in this way preserves Lebesgue measure, and it is well known that the dynamical system is ergodic.

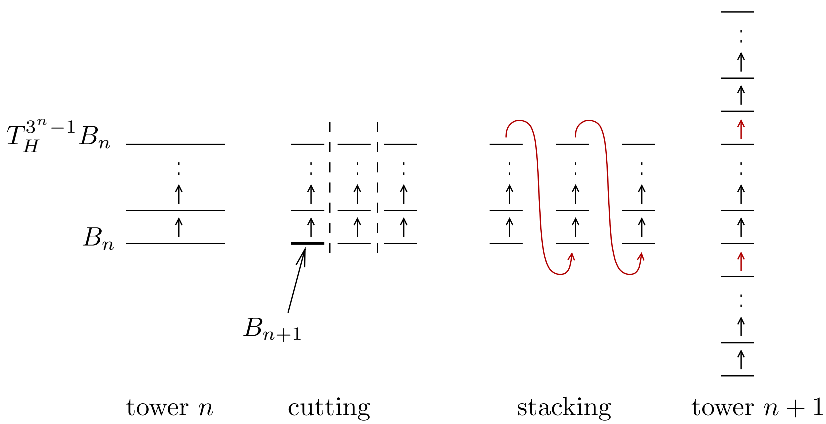

Fig. 1. Construction of T_H by cutting and stacking

2.2. *The extension.* In order to construct the extension of T_H, we will now define a Rokhlin cocycle $x \mapsto S_x$ from X_H into the group of automorphisms of $(Y, \mathscr{B}, \nu)$, where $Y := \{-1,1\}^{\mathbb{N}}$, $\mathscr{B}$ is the Borel σ-algebra of Y, and ν is the probability measure on Y which makes the coordinates independent and identically distributed, with $\nu(y_k = 1) = \nu(y_k = -1) = 1/2$ for each $k \geq 0$.

If $y = (y_k)_{k\in\mathbb{N}} \in Y$ and $0 \leq i \leq j$, we denote by $y|_i^j$ the finite word $y_i y_{i+1} \dots y_j$. For each $n \geq 0$, we define an *n-block* to be a word of length 2^n on the alphabet $\{-1,1\}$. The *first n-block of y* is thus $y|_0^{2^n-1}$. If $w_1 = y_0 \dots y_{2^n-1}$ and $w_2 = z_0 \dots z_{2^n-1}$ are two n-blocks, we denote by $w_1 w_2$ the $(n+1)$-block obtained by the concatenation of w_1 and by w_2, and $w_1 .\!\times w_2$ the n-block defined by the termwise product of w_1 and w_2:

$$w_1 w_2 := y_0 \dots y_{2^n-1} z_0 \dots z_{2^n-1}, \qquad w_1 .\!\times w_2 := (y_0 \times z_0) \dots (y_{2^n-1} \times z_{2^n-1}).$$

For each $n \geq 1$, we now define a transformation τ_n of Y which will be useful for the construction of the Rokhlin cocycle. This transformation only affects the first n-block of y: if this first n-block is $w_1 w_2$ (where w_1 and w_2 are $(n-1)$-blocks), then the first n-block of $\tau_n y$ is $w_2(w_1 .\!\times w_2)$.

Coordinates with indices at least 2^n of $\tau_n y$ remain unchanged. The following two properties of τ_n are easy to verify:

- τ_n preserves the probability ν,
- $\tau_n^3 = \mathrm{Id}_Y$.

For every $x \in X_H$, we denote by $n(x)$ the smallest integer $n \geq 1$ such that x does not belong to the top of tower n. In other words, $n(x)$ is the integer $n \geq 1$ such that $T_H x$ is defined at step n of the construction of T_H. We then set

$$S_x \;:=\; \tau_{n(x)} \circ \tau_{n(x)-1} \circ \cdots \circ \tau_1.$$

From the properties of τ_n, it is easy to derive that S_x is always an automorphism of $(Y, \mathscr{B}, \nu)$. From now on, we denote by T the skew product on $X_H \times Y$ equipped with the product measure $\mu_H \otimes \nu$ defined by

$$T(x, y) \;:=\; (T_H x, S_x y).$$

Let $\mathscr{H}$ be the factor of T given by the σ-algebra $\mathscr{A}_H \otimes \{Y, \emptyset\}$.

2.3. *Relative twofold mixing which is not threefold.* Let $n \geq 1$, and $(x, y) \in X_H \times Y$ with x in the base B_n of the nth tower. For each $k \geq 0$, we denote by $y^{(k)}$ the point of Y defined by $T^k(x, y) = (T_H^k x, y^{(k)})$. From the construction of the Rokhlin cocycle, while $T_H^k x$ has not reached the top of tower n, y is only transformed by some τ_j with $j \leq n$. Therefore, in the sequence $y^{(0)}, y^{(1)}, \ldots, y^{(3^n-1)}$ (corresponding to the climb of x upward tower n), only the first n-block is modified and these modifications do not depend on the coordinates of y with indices at least 2^n.

We are particularly interested in the sequence $y_0^{(0)} y_0^{(1)} \ldots y_0^{(3^n-1)}$ of coordinates with zero index, which we see as a random colouring of the climb of x upward tower n. From the preceding remark, this colouring only depends on the first n-block of y. Therefore there exists some map $\gamma_n \colon \{-1, 1\}^{2^n} \to \{-1, 1\}^{3^n}$ such that

$$y_0^{(0)} y_0^{(1)} \ldots y_0^{(3^n-1)} = \gamma_n(y|_0^{2^n-1}).$$

LEMMA 2.1. *Assume further that x lies in the base of the first or second column in tower n (i.e. $x \in B_{n+1}$ or $x \in T_H^{3^n} B_{n+1}$). Then*

$$y^{(3^n)} = \tau_{n+1} y.$$

Proof. This is easily checked by induction on n, using the fact that $\tau_n^3 = \mathrm{Id}_Y$. ∎

Lemma 2.1 gives a relation between γ_n and γ_{n+1}. Indeed, if x lies in B_{n+1}, the climbing of x upward tower $n+1$ can be seen as three successive climbings of x upward tower n, whose colourings are given by $y^{(0)} = y$, $y^{(3^n)} = \tau_{n+1} y$ and $y^{(2\times 3^n)} = \tau_{n+1}^2 y$. It follows that the colouring of the first climbing of x upward tower n is coded by the first n-block $y|_0^{2^n-1}$ of y, the colouring of the second climbing of x upward tower n is coded by the second n-block $y|_{2^n}^{2^{n+1}-1}$, and the colouring of the third climbing of x upward tower n is coded by their termwise product $y|_0^{2^n-1} .\!\times y|_{2^n}^{2^{n+1}-1}$. Hence, if w is an $(n+1)$-block which is the concatenation of the two n-blocks w_1w_2, we have

$$\gamma_{n+1}(w) = \gamma_n(w_1)\gamma_n(w_2)\gamma_n(w_1 .\!\times w_2). \tag{3}$$

Therefore, the sequence $(\gamma_n)_{n\geq 1}$ of coding maps is entirely determined by

$$\gamma_1 \colon ab \mapsto ab(a\times b)$$

and the recurrence relation (3). The proof of the following lemma follows easily:

LEMMA 2.2. *Let w_1 and w_2 be two n-blocks. Then*

$$\gamma_n(w_1 .\!\times w_2) = \gamma_n(w_1) .\!\times \gamma_n(w_2).$$

From the preceding observations, we can deduce some properties of the conditional law of the colouring process knowing x.

PROPOSITION 2.1. *Let $x \in X_H$ and $n \geq 1$. Let $j \geq 0$ be the smallest integer such that $T_H^{-j}x \in B_{n+1}$. Denote by C_1^n, C_2^n and C_3^n the random colourings of the three successive climbings of x upward tower n. The conditional law of (C_1^n, C_2^n, C_3^n) knowing $\mathscr{H}$ has the following properties*:

- *C_1^n, C_2^n and C_3^n are identically distributed*;
- *C_1^n, C_2^n and C_3^n are pairwise independent*;
- *$C_3^n = C_1^n .\!\times C_2^n$.*

Proof. Since $\mathscr{H}$ is a T-invariant σ-algebra, we can always assume to simplify notation that $j=0$ (i.e. $x \in B_{n+1}$). It follows from what has been seen before that C_1^n, C_2^n and C_3^n are given respectively by $\gamma_n(y|_0^{2^n-1})$, $\gamma_n(y|_{2^n}^{2^{n+1}-1})$ and $\gamma_n(y|_0^{2^n-1} .\!\times y|_{2^n}^{2^{n+1}-1})$. But the three n-blocks $y|_0^{2^n-1}$, $y|_{2^n}^{2^{n+1}-1}$ and $y|_0^{2^n-1} .\!\times y|_{2^n}^{2^{n+1}-1}$ are identically distributed and pairwise independent. Therefore, the three colourings are themselves identically distributed and pairwise independent. The equality $C_3^n = C_1^n .\!\times C_2^n$ is a straightforward consequence of Lemma 2.2. ∎

It follows easily from Proposition 2.1 that the property (1) characterizing twofold mixing relatively to the factor $\mathscr{H}$ is true when A and B are measurable with respect to a finite number of coordinates of the colouring process $(y_0 \circ T^k)_{k\in\mathbb{Z}}$. Indeed, in that case we can find an integer n (depending on x) such that A and B are measurable with respect to one of the blocks C_i^n ($i = 1, 2$ or 3) defined in the previous proposition. Then, as soon as $k \geq 3^n$, A and $T^{-k}B$ are given by two blocks C_j^m (for some $m \geq n$) which are independent under the conditional law knowing $\mathscr{H}$.

Next, (1) extends by density to any sets A and B measurable with respect to the σ-algebra generated by $\mathscr{H}$ and the colouring process $(y_0 \circ T^k)_{k\in\mathbb{Z}}$. But this σ-algebra is easily shown to be the whole $\mathscr{A}_H \otimes \mathscr{B}$, since knowing x and $(y_0 \circ T^k)_{k\in\mathbb{Z}}$ we can always recover each coordinate y_n, $n \in \mathbb{N}$. (Details are left to the reader.) It follows that the system is $\mathscr{H}$-relatively twofold mixing.

However, the system is not $\mathscr{H}$-relatively threefold mixing: If

$$A = B = C := \{(x, y) : y_0 = 1\},$$

we have

$$\mu(A \mid \mathscr{H}) = \mu(B \mid \mathscr{H}) = \mu(C \mid \mathscr{H}) = 1/2,$$

but for each $n \geq 1$ and each x in the first column of tower n,

$$\mu(A \cap T^{-3^n} B \cap T^{-2\times 3^n} C \mid \mathscr{H}) = 1/4.$$

3. Comments and questions

Joinings. The question of the existence of a system which is twofold but not threefold mixing is strongly connected with the following question: Does there exist a joining of three copies of some weakly mixing, zero-entropy dynamical system which is pairwise independent but which is not the product measure? In [3], Lemańczyk, Mentzen and Nakada answer positively the *relative* version of this problem: They construct a relatively weakly mixing extension T of an ergodic rotation T_H, and a 3-joining λ of T identifying the three copies of T_H, which is pairwise but not threewise independent relative to T_H. However their construction does not seem to come from an extension which is twofold but not threefold relatively mixing.

Mixing in the base? The example which we have presented above can easily be modified in order to make the dynamical system in the base weakly mixing. Indeed, we can replace the triadic von Neumann–Kakutani by Chacon's transformation, whose construction is similar with only the following difference: In each step of the construction we add a supplementary *spacer* interval between the second and third column. The sequence (h_n) of the heights of the successive towers thus satisfies $h_{n+1} = 3h_n + 1$. It is well known that Chacon's transformation is weakly, but not strongly, mixing. Defining S_x in a similar way when x does not lie in some spacer, and $S_x := \mathrm{Id}$ in any spacer, we get the same conclusion concerning twofold but not threefold relative mixing. The lack of threefold relative mixing is checked by considering, for x in the first column of tower n, $\mu(A \cap T^{-h_n} B \cap T^{-(2h_n+1)} C \mid \mathscr{H})$.

Then it is natural to look for a similar result with the dynamical system in the base strongly mixing. Indeed, it is easily shown that if T_H is mixing and if T is $\mathscr{H}$-relatively mixing, then T is mixing. This would give some hope to get a transformation that is twofold but not threefold mixing. However, there seem to be serious obstacles to achieving the same kind of construction with a mixing base.

On the definition of relative mixing. In the present work we have used the notion of relative mixing defined by the convergence to zero in probability (or equivalently in L^1) of the sequence

$$\mu(A \cap T^{-k}B \,|\, \mathscr{H}) - \mu(A \,|\, \mathscr{H})\mu(T^{-k}B \,|\, \mathscr{H}). \tag{4}$$

Another possible definition of relative mixing is used by Rahe in his work on factors of Markov processes [5]: In that paper, a process $(x_k)_{k\in\mathbb{Z}}$ (with $x_k = x_0 \circ T^k$) is called *$\mathscr{H}$-relatively mixing* if, for all A and B measurable with respect to a finite number of cooordinates of the process (x_k), the convergence of (4) to zero holds almost surely.

The difference between these two definitions is discussed in a recent work of Rudolph [6], where it is shown that there exists a system which is relatively mixing with respect to one of its factors in the L^1 sense, but not in the almost sure sense. Rudolph also shows that checking almost sure convergence of (4) to zero for a dense class of subsets A and B (as in Rahe's definition) implies that the same convergence holds for *every* A and B.

It is not difficult to see that, for the example we presented here, the same results concerning twofold and threefold relative mixing hold if we replace L^1 convergence by almost sure convergence.

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