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ON PAIRS OF BANACH SPACES WHICH ARE ISOMORPHIC TO COMPLEMENTED SUBSPACES OF EACH OTHER

BY

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Abstract. We establish the existence of Banach spaces E and F isomorphic to complemented subspaces of each other but with $E^m \oplus F^n$ isomorphic to $E^p \oplus F^q$, $m, n, p, q \in \mathbb{N}$, if and only if $m = p$ and $n = q$.

1. Introduction. For the sake of clarity we start with the notation. Throughout the note X and Y are real ($\mathbb{R}$) or complex ($\mathbb{C}$) Banach spaces. We write $X \overset{c}{\hookrightarrow} Y$ if X is isomorphic to a complemented subspace of Y, and $X \sim Y$ if X is isomorphic to Y. If $n \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$, then X^n denotes the sum of n copies of X. It is useful to define $X^0 = \{0\}$. By $\sum X$ we denote the infinite zero sum of X [4]. Now we are ready to present the motivation for the question which we consider here.

Suppose that X and Y are isomorphic to complemented subspaces of each other, that is,

$$X \overset{c}{\hookrightarrow} Y \quad \text{and} \quad Y \overset{c}{\hookrightarrow} X. \tag{1}$$

In 1996 W. T. Gowers [12] solved the so-called Schroeder–Bernstein Problem for Banach spaces by showing that X is not necessarily isomorphic to Y (see also [6], [8], and [13]). Moreover, in a recent paper [7], the author showed that one cannot conclude that some finite sum of X, X^n, $n \in \mathbb{N}^*$, is isomorphic to some finite sum of Y, Y^m, $m \in \mathbb{N}^*$.

However, it is well known that Pełczyński's decomposition method [4, p. 64] implies that X and Y satisfy the following equation which involves infinite sums of X and Y:

$$\sum X \sim \sum Y. \tag{2}$$

Hence, it is natural to ask whether X and Y also satisfy some non-trivial equation which involves only finite sums of X and Y. More precisely, is it true that there exist $m, n, p, q \in \mathbb{N}$ with $m \neq p$ or $n \neq q$ satisfying the

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equation below?

$$X^m \oplus Y^n \sim X^p \oplus Y^q. \tag{3}$$

The aim of this note is to answer this question in the negative, in other words, we prove what is announced in the abstract [Theorem 2.3]. Therefore, equations like (2) are in some sense the best ones that we can obtain from (1).

The construction of the Banach spaces E and F and the proofs of their properties are based on some recent developments in the theory of hereditarily indecomposable Banach spaces. In fact, our theorem is an application of a universal property of reflexive hereditarily indecomposable Banach spaces recently proved by S. A. Argyros [3, Theorem 1.1]. This result states that every separable Banach space containing an isomorphic copy of any reflexive hereditarily indecomposable Banach spaces also contains an isomorphic copy of any separable Banach space. We recall that a Banach space H is *hereditarily indecomposable* (H.I.) if no closed subspace E of H contains a pair of infinite-dimensional closed subspaces M and N such that $E = M \oplus N$ [11]. The H.I. spaces have been used to provide negative answers to several questions in Banach spaces; see for example [2], [7], [11], [12] and [14];

The main tool used in the proof of Theorem 2.3 is the theory of essentially incomparable Banach spaces. Thus, we also need to recall some definitions concerning operator theory. Let $L(X,Y)$ be the Banach space of all continuous linear operators from X into Y. An operator $T \in L(X,Y)$ is *Fredholm* if its kernel is finite-dimensional and its range is finite-codimensional. T is *inessential* ($T \in \mathrm{In}(X,Y)$) if $I_X - ST$ is Fredholm for every $S \in L(Y,X)$. If $L(X,Y) = \mathrm{In}(X,Y)$, then the spaces X and Y are said to be *essentially incomparable* [1].

2. The result. We begin with a simple lemma that will be used several times in this work.

LEMMA 2.1. *Suppose that X and Y satisfy* (1) *and* (3) *for some $m,n,p,q \in \mathbb{N}$.*

(a) *If $d \in \mathbb{N}, d \le n$ and $d \le q$, then $X^{m+d} \oplus Y^{n-d} \sim X^{p+d} \oplus Y^{q-d}$.*
(b) *If $d \in \mathbb{N}, d \le m$ and $d \le p$, then $X^{m-d} \oplus Y^{n+d} \sim X^{p-d} \oplus Y^{q+d}$.*

Proof. By symmetry it suffices to prove (a). Let A be a Banach space such that $X \sim Y \oplus A$. Hence, if $1 \le n$ and $1 \le q$, then

$$\begin{aligned} X^{m+1} \oplus Y^{n-1} &\sim X^m \oplus Y^n \oplus A \sim X^p \oplus Y^q \oplus A \sim X^p \oplus Y^{q-1} \oplus Y \oplus A \\ &\sim X^{p+1} \oplus Y^{q-1}. \end{aligned}$$

Analogously, if $2 \le n$ and $2 \le q$, we have $X^{m+2} \oplus Y^{n-2} \sim X^{p+2} \oplus Y^{q-2}$. Therefore, if $d \in \mathbb{N}, d \le n$ and $d \le q$, then after d steps, we obtain $X^{m+d} \oplus Y^{n-d} \sim X^{p+d} \oplus Y^{q-d}$. ∎

The following lemma plays an important role in the proof of our theorem.

LEMMA 2.2. *Suppose that X and Y satisfy* (1) *and* (3) *for some m,n, $p,q \in \mathbb{N}$. If $p < m$ and $n < q$, then $X^{2u(m-p)} \sim Y^{2u(q-n)}$ for some $u \in \mathbb{N}^*$.*

Proof. First we show that there exist $m_1, n_1, p_1, q_1 \in \mathbb{N}$ with $2n_1+p_1 \leq q_1$ and $2p_1 + n_1 \leq m_1$ such that

$$X^{m_1} \oplus Y^{n_1} \sim X^{p_1} \oplus Y^{q_1}. \tag{4}$$

In order to do this, we define $M = m + n$, $P = p + n$ and $Q = q - n$. Since (1) and (3) hold, it follows from Lemma 2.1(a) with $d = n$ that

$$X^M \sim X^P \oplus Y^Q. \tag{5}$$

We observe that $Q > 0$, therefore there exist $u, v \in \mathbb{N}^*$ such that $P+v \leq 2uQ$ and $uQ \leq v$. By (5), we obtain

$$Y^v \oplus X^P \sim Y^{v-Q} \oplus X^P \oplus Y^Q \sim Y^{v-Q} \oplus X^M. \tag{6}$$

Moreover,

$$Y^{v-Q} \oplus X^M \sim Y^{v-2Q} \oplus Y^Q \oplus X^{M-P} \oplus X^P \sim Y^{v-2Q} \oplus X^{M+(M-P)}. \tag{7}$$

It follows from (6) and (7) that

$$Y^v \oplus X^P \sim Y^{v-2Q} \oplus X^{M+(M-P)}. \tag{8}$$

Thus, after u steps, we have

$$Y^v \oplus X^p \sim Y^{v-uQ} \oplus X^{M+(u-1)(M-P)}. \tag{9}$$

Finally, we define $m_1 = M + (u-1)(M-P)$, $n_1 = (v - uQ)$, $p_1 = P$ and $q_1 = v$. By the choice of u and v, we know that $2n_1 + p_1 \leq q_1$ and $2p_1 + n_1 \leq m_1$. This finishes the proof of (4).

Now, by applying Lemma 2.1(a) with $d = n_1$ and Lemma 2.1(b) with $d = p_1$ in (4), we conclude that

$$X^{m_1+n_1} \sim Y^{n_1+p_1} \oplus Y^{q_1-n_1} \quad \text{and} \quad Y^{p_1+q_1} \sim X^{m_1-p_1} \oplus Y^{n_1+p_1}. \tag{10}$$

From (10) we deduce that

$$\begin{aligned} X^{2(m_1-p_1)} &\sim X^{m_1-n_1-2p_1} \oplus X^{m_1+n_1} \sim X^{m_1-p_1} \oplus Y^{q_1-n_1} \\ &\sim Y^{q_1-p_1-2n_1} \oplus Y^{p_1+q_1} \sim Y^{2(q_1-n_1)}. \end{aligned}$$

Hence the proof of the lemma is complete, since $m_1 - p_1 = u(m-p)$ and $q_1 - n_1 = u(q-n)$. ∎

THEOREM 2.3. *There exist Banach spaces E and F which are isomorphic to complemented subspaces of each other and such that $E^m \oplus F^n$ is isomorphic to $E^p \oplus F^q$, with $m,n,p,q \in \mathbb{N}$, if and only if $m = p$ and $n = q$.*

Proof. Let X and Y be the separable Banach spaces considered in [7]. So, X and Y satisfy (1) and X^t is not isomorphic to Y^t, for every $t \in \mathbb{N}^*$ [7, Theorem 4].

Claim 1. *Let $p \in \mathbb{R}$, $1 < p < 2$. Then X contains no subspace isomorphic to l_p.*

First we recall that X is a sequence space and the support of a vector $x = (x_n)_{n=1}^{\infty}$ in X, written $\mathrm{supp}(x)$, is $\{n : x_n \neq 0\}$. We write $x < y$ to mean $i < j$ for every $i \in \mathrm{supp}(x)$ and $j \in \mathrm{supp}(y)$. We say that $(x_n)_{n=1}^{\infty}$ in X is a *sequence of successive vectors* if $x_1 < x_2 < x_3 < \cdots$.

Suppose now that $T : l_p \to X$ is an isomorphism onto its image. Let $(e_n)_{n=1}^{\infty}$ stand for the unit vector basis of l_p. Since $(e_n)_{n=1}^{\infty}$ converges weakly to zero, by standard arguments, we may perturb T slightly in such a way that $x_n = T(e_n)$, $n \in \mathbb{N}$, are successive vectors.

Let $K, L > 0$ be such that $K\|x\| \le \|T(x)\| \le L\|x\|$ for every $x \in l_p$. By [7, Lemma 2], for every sequence of real numbers $(a_n)_{n=1}^{m}$, $m \in \mathbb{N}$, we have

$$(11) \quad K\Big(\sum_{i=1}^{m} |a_i|^p\Big)^{1/p} \le \Big\|\sum_{i=1}^{m} a_i x_i\Big\| \le \Big(\sum_{i=1}^{m} \|a_i x_i\|^2\Big)^{1/2} \le L\Big(\sum_{i=1}^{m} |a_i|^p\Big)^{1/p},$$

which of course gives a contradiction because l_p is not isomorphic to l_2. Thus, Claim 1 is proved.

Hence, X is not universal for the class of separable Banach spaces. Thus, according to the theorem of S. A. Argyros mentioned in the introduction, there exists a H.I. space H which is not isomorphic to any subspace of X.

Claim 2. *H and X are essentially incomparable.*

Assume on the contrary that H and X are not essentially incomparable. By [1, Proposition 4.11] and [1, Theorem 23], $X \sim H_1 \oplus Z$ for some Banach spaces H_1 and Z, where H_1 is an infinite-dimensional complemented subspace of H. Suppose that Z is finite-dimensional. Since $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$, it follows that $H_1 \oplus Z \sim Y \oplus A$ and $Y \sim H_1 \oplus Z \oplus B$ for some Banach spaces A and B. Therefore, $H_1 \oplus Z \sim H_1 \oplus Z \oplus A \oplus B$. Consequently, $H_1 \sim H_1 \oplus A \oplus B$. Hence, by [11, Corollary 23], $A = \{0\}$ and $B = \{0\}$. It follows immediately that $X \sim Y$, which is a contradiction. Hence Z is infinite-dimensional.

Since H is a H.I. space, $H \sim H_1 \oplus W$ for some finite-dimensional space W. Therefore, writing $Z \sim W \oplus Z_1$ for some Banach space Z_1, we have $X \sim H_1 \oplus W \oplus Z_1 \sim H \oplus Z_1$, contrary to the choice of H. This completes the proof of Claim 2.

Now we denote by $\operatorname{dens} X^*$ the density character of the dual space of X, that is, the smallest cardinal number δ such that there exists a set of cardinality δ dense in X^*. Fix a regular ordinal α with $\operatorname{dens} X^* < \overline{\alpha}$, where $\overline{\alpha}$ indicates the cardinality of α.

Next we consider the Banach space X^α of continuous X-valued functions defined on the interval $[1,\alpha]$ of ordinals and equipped with the supremum norm [6].

Finally, we define $E = H \oplus X^\alpha$ and $F = H \oplus Y^\alpha$.

Clearly $E \overset{c}{\hookrightarrow} F$ and $F \overset{c}{\hookrightarrow} E$. Suppose that $E^m \oplus F^n \sim E^p \oplus F^q$ for some $m,n,p,q \in \mathbb{N}$, that is,

$$H^{m+n} \oplus X^{\alpha m} \oplus Y^{\alpha n} \sim H^{p+q} \oplus X^{\alpha p} \oplus Y^{\alpha q}. \tag{12}$$

We must show that $m = p$ and $n = q$. We note that if $m = 0$ and $n = 0$, then, by (12), $p = 0$ and $q = 0$. Assume next that $m > 0$; then, again by (12), either $p > 0$ or $q > 0$. We will consider the case $p > 0$; the other case is similar.

Claim 3. *H and $X^{\alpha p}$ are essentially incomparable.*

Indeed, otherwise, by [1, Proposition 4.11] and [1, Theorem 23], there exists an infinite-dimensional complemented subspace H_2 of H such that $H_2 \overset{c}{\hookrightarrow} X^{\alpha p}$. Therefore, according to [9, Theorem 2.4], $H_2 \overset{c}{\hookrightarrow} X^r$ for some $r \in \mathbb{N}$, $r \geq 1$. Hence, [10, Observation 1.b] and [1, Theorem 23] imply that H_2 and X are not essentially incomparable. Consequently, the same applies to H and X, contrary to Claim 2. So, Claim 3 is proved.

Therefore $(H^{m+n}, X^{\alpha m} \oplus Y^{\alpha n})$ and $(H^{p+q}, X^{\alpha p} \oplus Y^{\alpha q})$ are also pairs of essentially incomparable spaces [10, Observation 1.b]. Thus, by (12) and [8, Remark 3.3], there exist $r, s \in \mathbb{N}$ such that

$$H^{m+n} \oplus \mathbb{R}^r \sim H^{p+q} \oplus \mathbb{R}^s. \tag{13}$$

Suppose first that $m + n < p + q$. Let H_3 be an infinite-dimensional Banach space such that $H \sim H_3 \oplus \mathbb{R}^r$. Adding H_3 to both sides of (13), we have

$$H^{m+n+1} \sim H^{p+q} \oplus \mathbb{R}^s \oplus H_3. \tag{14}$$

In particular, (14) implies that H^{m+n+1} is isomorphic to a direct sum of $p + q + 1$ infinite-dimensional subspaces, which is a contradiction because $m + n + 1 < p + q + 1$ [5, Corollary 2].

By the same argument, we cannot have $m+n > p+q$. Therefore, $m+n = p + q$. If $m = p$, then $n = q$ and the proof is complete. Otherwise, without loss of generality we may assume that $p < m$ and therefore $n < q$.

Hence, by (12) and Lemma 2.2, there exists $t \in \mathbb{N}^*$, $t = 2u(m - p) = 2u(q - n)$ for some $u \in \mathbb{N}^*$, such that $E^t \sim F^t$, that is,

$$H^t \oplus X^{\alpha t} \sim H^t \oplus Y^{\alpha t}. \tag{15}$$

Notice that $(H^t, X^{\alpha t})$ and $(H^t, Y^{\alpha t})$ are pairs of essentially incomparable spaces. Thus, by (15) and [8, Remark 3.3], there exist u and v in $\mathbb{N}$ such that

$$\mathbb{R}^u \oplus X^{\alpha t} \sim \mathbb{R}^v \oplus Y^{\alpha t}. \tag{16}$$

Since $X^{\alpha t}$ and $Y^{\alpha t}$ are isomorphic to any of their respective hyperplanes, it follows from (16) that

$$X^{\alpha t} \sim Y^{\alpha t}. \tag{17}$$

Finally, it suffices to apply [6, Corollary 2.8] to (17) to conclude that $X^t \sim Y^t$, which contradicts the choice of X and Y. Thus $m = p$ and $n = q$, yielding the theorem. ∎

3. Remarks and questions. Pełczyński's decomposition method [4, p. 64] states that if the equation $X \sim \sum X$ is added to (1), then $X \sim Y$. Thus, in view of Lemma 2.2, it is natural to ask whether there exists a non-trivial equation that involves only finite sums of X and Y in such a way that when added to (1), it yields $X \sim Y$. To be more precise:

QUESTION 3.1. Do there exist $m, n, p, q \in \mathbb{N}$ with (m, n, p, q) different from $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$ such that if X and Y satisfy (1) and (3), then $X \sim Y$?

REMARK 3.2. The answer to Question 3.1 is negative for some m, n, p and q. For instance:

(a) *For any $m, n, p, q \in \mathbb{N}$ with $m \le p$ and $n < q$, there exist non-isomorphic Banach spaces X and Y satisfying* (1) *and* (3).

Suppose first that $m > 0$. Then we write $d = m - 1 \ge 0$, $r = p - d > 0$ and $s = q - n > 0$. Take $t = r + 2s - 1 \ge 2$ and let X_t be the Banach space constructed by W. T. Gowers and B. Maurey in [13, p. 563]. That is, $X_t^u \sim X_t^v$ with $u, v \in \mathbb{N}^*$ if and only if u and v are equal modulo t. We define $X = X_t$ and $Y = X_t^2$. Then X and Y are not isomorphic, they satisfy (1) and moreover,

$$X = X_t \sim X_t^{t+1} \sim X_t^{r+2s} \sim X_t^r \oplus (X_t^2)^s = X^r \oplus Y^s. \tag{18}$$

Hence, if $X^d \oplus Y^n$ is added to both sides of (18), we conclude that (3) is also satisfied.

Suppose now that $m = 0$. Then $n \ge 1$. By what we have just proved, there exist Banach spaces X and Y which satisfy (1) as well as the following equation:

$$X \oplus Y^{n-1} \sim X^{p+1} \oplus Y^{q-1}. \tag{19}$$

Therefore, if we apply Lemma 2.1(b) with $d = 1$ to (19), once again we obtain (3).

(b) *For some $m, n, p, q \in \mathbb{N}$ with $p < m$ and $n < q$, there exist non-isomorphic Banach spaces X and Y satisfying* (1) *and* (3).

Denote by W the complex Banach space introduced by W. T. Gowers and B. Maurey in [13, Section 4.3]. Then, by [13, Theorem 19], $X = W$ is not isomorphic to $Y = W \oplus \mathbb{C}$, these spaces satisfy (1) and furthermore,

$$X^3 \oplus Y \sim W^3 \oplus (W \oplus \mathbb{C}^2) \oplus \mathbb{C} \sim W \oplus (W \oplus \mathbb{C})^3 \sim X \oplus Y^3. \tag{20}$$

When $p < m$ and $n < q$, we do not even know the answer to the simplest case of Question 3.1, that is:

QUESTION 3.3. Suppose that X and Y satisfy (1) and $X^{p+1} \oplus Y^q \sim X^p \oplus Y^{q+1}$ for some $p, q \in \mathbb{N}$. Does it follow that $X \sim Y$?

It is interesting to observe that the answer to the above question is affirmative for every $p, q \in \mathbb{N}$ if and only if the following question has an affirmative answer for every $p \in \mathbb{N}$, $p \geq 2$.

QUESTION 3.4. Suppose that X and Y satisfy (1) and $X^n \sim Y^n$ for every $n \geq p$, with $p \geq 2$. Is it true that $X \sim Y$?

Indeed, assume that Question 3.4 has an affirmative answer for every $p \in \mathbb{N}$, $p \geq 2$, and let X and Y satisfy (1) and $X^{p+1} \oplus Y^q \sim X^p \oplus Y^{q+1}$ for some $p, q \in \mathbb{N}$. According to Lemma 2.1(a) with $d = q$, we have $X^{p+q+1} \sim X^{p+q} \oplus Y$. Therefore $X^{p+q+1+n} \sim X^{p+q+n} \oplus Y$ for every $n \in \mathbb{N}$. Hence, applying Lemma 2.1(a) with $d = 1$, $p+q+n$ times to the previous equation, we conclude that

$$X^{p+q+1+n} \sim X^{p+q+n} \oplus Y \sim X^{p+q+n-1} \oplus Y^2 \sim \cdots \sim Y^{p+q+1+n}$$

for every $n \in \mathbb{N}$. Consequently, $X \sim Y$.

Conversely, assume that Question 3.3 has an affirmative answer for every $p, q \in \mathbb{N}$, and suppose $X^n \sim Y^n$ for every $n \geq p$, with $p \geq 2$, $X \sim Y \oplus A$ and $Y \sim X \oplus B$ for some Banach spaces A and B. Thus $X^p \sim X^{p-1} \oplus Y \oplus A$ and $X^{p-1} \oplus Y \sim X^p \oplus B$. That is, $X^p \overset{c}{\hookrightarrow} X^{p-1} \oplus Y$ and $X^p \overset{c}{\hookrightarrow} X^{p-1} \oplus Y$. Moreover, $X^{p^2} \sim Y^{p^2} \sim Y^{p^2-1} \oplus Y \sim X^{p^2-1} \oplus Y$. That is, $(X^p)^p \sim (X^p)^{p-1} \oplus (X^{p-1} \oplus Y)$. Therefore, putting $q = 0$ in the hypothesis of Question 3.3 with X^p and $X^{p-1} \oplus Y$, we have $X^p \sim X^{p-1} \oplus Y$. Hence, again by our hypothesis, $X \sim Y$.

Finally, we observe that the answer to Question 3.3 is affirmative when X is isomorphic to some non-trivial finite sum of X and Y, that is, $X \sim X^r \oplus Y^s$ for some $r, s \in \mathbb{N}$, $r + s \geq 2$. This is a direct consequence of the fact that $X^{p+q+1} \sim X^{p+q} \oplus Y$ and the following remark:

REMARK 3.5. *Suppose that X and Y satisfy* (1) *and $X^p \sim X^{p-1} \oplus Y$ for some $p \in \mathbb{N}$, $p \geq 2$. If $X \sim X^r \oplus Y^s$ for some $r, s \in \mathbb{N}$, $r + s \geq 2$, then $X \sim Y$.*

Indeed, let B be a Banach space such that $Y \sim X \oplus B$. We first suppose that $r \geq 2$. Then there exists $n \in \mathbb{N}$ such that $r + (n-1)(r-1) \geq p$. Define $m = r + (n-1)(r-1)$ and $j = ns$. Now, adding $X^{r-1} \oplus Y^s$ to both sides of $X \sim X^r \oplus Y^s$ we obtain $X \sim X^r \oplus Y^s \sim X^{r+(r-1)} \oplus Y^{2s}$. Hence, by induction, $X \sim X^m \oplus Y^j$. Consequently,

$$\begin{aligned} Y &\sim X \oplus B \sim X^{m-p} \oplus X^{p-1} \oplus Y^j \oplus X \oplus B \\ &\sim X^{m-p} \oplus X^{p-1} \oplus Y^j \oplus Y \sim X^{m-p} \oplus X^p \oplus Y^j \sim X^m \oplus Y^j \sim X. \end{aligned}$$

Next we suppose that $r = 1$, that is, $X \sim X \oplus Y^s$, with $s \geq 1$. By Lemma 2.1(b) with $d = 1$, we have $Y \sim Y^{s+1}$. Moreover, since $X^p \sim X^{p-1} \oplus Y$, by Lemma 2.1(b) with $d = p-1$, $Y^p \sim Y^{p-1} \oplus X$. Thus, by the first case, $X \sim Y$.

Finally, assume that $r = 0$, that is, $X \sim Y^s$ with $s \geq 2$. We have

$$\begin{aligned} X &\sim Y^s \sim Y^{s-1} \oplus X \oplus B \sim Y^{s-1} \oplus Y^s \oplus B \sim Y^{2s-1} \oplus B \\ &\sim Y^{2s-2} \oplus X \oplus B \oplus B \sim Y^{2s-2} \oplus Y^s \oplus B^2 \sim Y^{3s-2} \oplus B^2. \end{aligned}$$

Hence, by induction, $X \sim Y^{ns-(n-1)} \oplus B^{n-1}$ for every $n \in \mathbb{N}$. In particular, $X \sim Y^{p(s-1)+s} \oplus B^p$.

On the other hand, if we apply Lemma 2.1(a) with $d = 1$, p times to $X^p \sim X^{p-1} \oplus Y$, we obtain $X^p \sim X^{p-1} \oplus Y \sim X^{p-2} \oplus Y^2 \sim \cdots \sim Y^p$. Therefore, it follows from $Y^p \sim X^p \oplus B^p$ that

$$\begin{aligned} X &\sim Y^{p(s-1)+s} \oplus B^p \sim Y^{p(s-1)+s-p} \oplus Y^p \oplus B^p \\ &\sim Y^{p(s-1)+s-p} \oplus X^p \oplus B^p \sim Y^{p(s-1)+s-p} \oplus Y^p \\ &\sim Y^{p(s-1)+s} \sim (Y^p)^{s-1} \oplus Y^s \sim (X^p)^{s-1} \oplus X \sim X^{p(s-1)+1}. \end{aligned}$$

Hence, once again by the first case, $X \sim Y$.

REFERENCES

[1] P. Aiena and M. González, *On inessential and improjective operators*, Studia Math. 131 (1998), 271–287.

[2] G. Allexandrov, D. Kutzarova and A. Plichko, *A separable space with no Schauder decomposition*, Proc. Amer. Math. Soc. 127 (1999), 2805–2806.

[3] S. A. Argyros, *A universal property of reflexive hereditarily indecomposable Banach spaces*, Proc. Amer. Math. Soc. 129 (2001), 3231–3239.

[4] P. G. Casazza, *The Schroeder–Bernstein property of Banach space*, in: Contemp. Math. 85, Amer. Math. Soc., 1989, 61–77.

[5] V. Ferenczi, *Hereditarily finitely decomposable Banach spaces*, Studia Math. 123 (1997), 135–149.

[6] E. M. Galego, *How to generate new Banach spaces non-isomorphic to their cartesian squares*, Bull. Polish Acad. Sci. Math. 47 (1999), 21–25.

[7] —, *Banach spaces complemented in each other without isomorphic finite sums*, ibid. 50 (2002), 1–9.

[8] —, *On solutions to the Schroeder–Bernstein problem for Banach spaces*, Arch. Math. (Basel) 79 (2002), 299–307.

[9] —, *On subspaces and quotients of Banach spaces $C(K, X)$*, Monatsh. Math. 136 (2002), 87–97.

[10] M. González, *On essentially incomparable Banach spaces*, Math. Z. 215 (1994), 621–629.
[11] W. T. Gowers, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.
[12] —, *A solution to the Schroeder–Bernstein problem for Banach spaces*, Bull. London Math. Soc. 28 (1996), 297–304.
[13] W. T. Gowers and B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. 307 (1997), 543–568.
[14] N. Tomczak-Jaegermann, *A solution of the homogeneous Banach space problem*, in: Canadian Mathematical Society, 1945–1995, Vol. 3, Canadian Math. Soc., 1996, 267–286.

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