COLLOQUIUM MATHEMATICUM

VOL. 123 2011 NO. 1

OPTIMAL EMBEDDINGS OF GENERALIZED HOMOGENEOUS SOBOLEV SPACES

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Abstract. We prove optimal embeddings of homogeneous Sobolev spaces built over function spaces in $\mathbb{R}^n$ with K-monotone and rearrangement invariant norm into other rearrangement invariant function spaces. The investigation is based on pointwise and integral estimates of the rearrangement or the oscillation of the rearrangement of f in terms of the rearrangement of the derivatives of f.

1. Introduction. Let L_{loc} be the space of all locally integrable functions f on $\mathbb{R}^n$, $n \geq 2$, with the Lebesgue measure and let f^* be the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \qquad t > 0,$$

where μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|_n,$$

$|\cdot|_n$ denoting Lebesgue n-measure. Define $f^{**}(t) := t^{-1}\int_0^t f^*(s)\,ds$.

Let L be the space of all locally integrable functions $g \geq 0$ on $(0,\infty)$ with the Lebesgue measure that are in $L^1 + L^\infty$ and have $g^*(\infty) = 0$.

We shall consider rearrangement invariant spaces E, continuously embedded in $L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$, such that the norm $\|f\|_E$ in E is generated by a norm ρ_E defined on L with values in $[0,\infty]$, in the sense that $\|f\|_E = \rho_E(f^*)$. In this way equivalent norms ρ_E give the same space E. We suppose that E is nontrivial.

We say that the norm ρ_E is *K-monotone* (cf. [5, Definition 1.16, p. 305]) if

$$(1.1) \qquad \int_0^t g_1^*(s)ds \leq \int_0^t g_2^*(s)\,ds \quad \text{implies} \quad \rho_E(g_1^*) \leq \rho_E(g_2^*), \quad g_1, g_2 \in L.$$

Then ρ_E is *monotone*, i.e. $g_1 \leq g_2$ implies $\rho_E(g_1) \leq \rho_E(g_2)$. If ρ_E is K-monotone, then $\|f\|_E$ satisfies the triangle inequality, since $(f+g)^{**} \leq (f^* + g^*)^{**}$.

2010 *Mathematics Subject Classification*: Primary 46E35; Secondary 46E30.
Key words and phrases: Sobolev homogeneous spaces, optimal embeddings, rearrangement invariant spaces.

DOI: 10.4064/cm123-1-1

We use the notation $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$, and we then say that a_1 is equivalent to a_2.

We say that the norm ρ_E *satisfies the Minkowski inequality* if

$$\rho_E\Big(\sum g_j\Big) \le \sum \rho_E(g_j), \qquad g_j \in L. \tag{1.2}$$

For example, if E is a rearrangement invariant Banach function space as in [5], then by the Luxemburg representation theorem, $\|f\|_E = \rho_E(f^*)$ for some norm ρ_E satisfying (1.1) and (1.2). A more general example is given by the Riesz–Fischer monotone spaces as in [5, p. 305].

Recall the definition of the *lower* and *upper Boyd indices* α_E and β_E. Let

$$h_E(s) = \sup\left\{\frac{\rho_E(g_s^*)}{\rho_E(g^*)} : g \in L\right\}, \qquad g_s(t) := g(t/s),$$

be the dilation function generated by ρ_E. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

If ρ_E is monotone, then the function h_E is submultiplicative, increasing, $h_E(1) = 1$, $1 \le h_E(s)h_E(1/s)$, therefore $0 \le \alpha_E \le \beta_E$. If ρ_E is K-monotone, then by interpolation (analogously to [5, p. 148]), we see that $h_E(s) \le \max(1, s)$. Hence $0 \le \alpha_E \le \beta_E \le 1$.

If $\beta_E < 1$ we have, analogously to [5, p. 150],

$$\rho_E(f^*) \approx \rho_E(f^{**}). \tag{1.3}$$

The condition $\beta_E < 1$ is equivalent to (see [5, p. 147])

$$\int_0^1 h_E(1/s)\, ds < \infty. \tag{1.4}$$

For example, consider the classical Lorentz spaces $\Lambda^q(w)$, $1 \le q \le \infty$, with w a positive weight, i.e. a positive function from L; $f \in \Lambda^q(w)$ if $\|f\|_{\Lambda^q_w} := \rho_{w,q}(f^*) < \infty$, where $\rho_{w,q}(g) := (\int_0^\infty [g(t)w(t)]^q\, dt/t)^{1/q}$. In general, the functional $f \mapsto \|f\|_{\Lambda^q_w}$ is not a norm. But in many cases we can find an equivalent norm. Consider the so-called Γ spaces, $\Gamma^q(w)$, with the norm $\|f\|_{\Gamma^q(w)} := \rho_{w,q,\Gamma}(f^*)$, where $\rho_{w,q,\Gamma}(g) := (\int_0^\infty [g^{**}(t)w(t)]^q\, dt/t)^{1/q}$. The following condition should be satisfied (otherwise the space will be trivial):

$$\Big(\int_0^\infty \min(1, t^{-q})[w(t)]^q\, dt/t\Big)^{1/q} < \infty.$$

Then this space is continuously embedded in the sum $L^1 + L^\infty$. Using this embedding the completeness of the space can be established in a standard way. The space $E = \Gamma^q(w)$ with $\rho_E = \rho_{w,q,\Gamma}$ satisfies the conditions (1.1), (1.2).

In some cases the Lorentz space $E = \Lambda^q(w)$, $1 \le q < \infty$, also satisfies the conditions (1.1), (1.2). For example, if $[w(t)]^q/t$ is not increasing, then (see [5, p. 72]), the functional $\rho_{w,q}$ is a K-monotone norm. It is easy to check that this space is continuously embedded in $L^q + L^\infty$.

We have the equivalence

$$\|f\|_{\Lambda^q(w)} \approx \|f\|_{\Gamma^q(w)} \tag{1.5}$$

in the following cases.

If $1 \le q < \infty$ then (1.5) is satisfied if and only if w is such that (see [2])

$$t^q \int_t^\infty s^{-q}[w(s)]^q \, ds/s \lesssim \int_0^t [w(s)]^q \, ds/s. \tag{1.6}$$

If $q = \infty$ then (1.5) is valid if and only if (see [8])

$$\frac{1}{t}\int_0^t \frac{1}{w(s)}\, ds \lesssim \frac{1}{w(t)}, \quad \text{where} \quad w(t) := \int_0^t v(s)\, ds \quad \text{for some } v. \tag{1.7}$$

For weights satisfying $\int_0^t w(s)\, ds/s \lesssim w(t)$, the condition (1.6) with $q = 1$ is equivalent to

$$\int_t^\infty \frac{w(s)}{s}\, ds/s \lesssim \frac{w(t)}{t}. \tag{1.8}$$

Indeed, (1.8) implies (1.6) with $q = 1$ by integration and Fubini's theorem. Conversely, (1.8) follows from (1.6) with $q = 1$ if $\int_0^t w(s)\, ds/s \preceq w(t)$.

Note also that if $E = \Gamma^q(w)$ and $\rho_E = \rho_{w,q,\Gamma}$, then (1.5) is equivalent to $\beta_E < 1$ (see [5, p. 150]).

Let C_0^∞ be the space of all infinitely differentiable functions f on $\mathbb{R}^n$ with compact support and let $|D^k f| := \sum_{|\alpha|=k} |D^\alpha f|$. Let

$$M_k = \{f \in L^1 + L^\infty : |D^j f|^*(\infty) = 0,\ 0 \le j \le k,\ \rho_E(|D^k f|^*) < \infty\}.$$

Definition 1.1. The *generalized homogeneous Sobolev norm* is the functional $\|f\|_{w^k E} := \rho_E(|D^k f|^*)$, defined on M_k, $k < n$.

The main goal of this paper is to prove optimal embeddings of $w^k E$ into rearrangement invariant function spaces G with a norm $\|f\|_G \approx \rho_G(f^*)$, where ρ_G is a monotone norm. Observe that we have two well known limiting embeddings: $w^k L^1 \hookrightarrow \Lambda^1(t^{1-k/n})$ and $w^k \Lambda^1(t^{k/n}) \hookrightarrow L^\infty$. For this reason we shall suppose that the domain space E and the target space G satisfy $E \hookrightarrow L^1 + \Lambda^1(t^{k/n})$ and $G \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty$. In particular, $\alpha_E > 0$ and $f \in w^k E$ implies $\int_1^\infty u^{k/n} |D^k f|^{**}(u)\, du/u < \infty$. It is convenient to introduce the following classes of norms:

- $N_{d,0}$ consists of all norms ρ_E that are K-monotone, rearrangement invariant and $\beta_E < 1$;
- $N_{d,1}$ consists of all norms ρ_E that are K-monotone, rearrangement invariant, satisfy the Minkowski inequality and $\alpha_E > (k-2)/n$, $\beta_E < 1$;
- $N_{d,2}$ consists of all norms ρ_E that are K-monotone, rearrangement invariant, satisfy the Minkowski inequality and $\alpha_E > (k-2)/n$, $\beta_E = 1$;
- N_d is a shorter notation for any of the above classes;
- N_t consists of all norms ρ_G that are monotone;
- $N_{t,1}$ consists of all norms ρ_G that are monotone, satisfy the Minkowski inequality and $\beta_G < 1 - k/n$.

Definition 1.2. We say that the couple $\rho_E \in N_d$, $\rho_G \in N_t$ (or E, G) is *admissible* if the following a priori estimate is valid:

$$\rho_G(f^*) \lesssim \rho_E(|D^k f|^*), \qquad f \in M_k. \tag{1.9}$$

If E and G are Banach spaces, then (1.9) allows us to define the Sobolev space $w^k E$ as the closure of C_0^∞; then we have the continuous embedding $w^k E \hookrightarrow G$. If (1.9) is true, then ρ_E (resp. E) is called the *domain norm* (resp. *domain space*), and ρ_G (resp. G) is called the *target norm* (resp. *target space*). We shall reserve the letter E for the domain space and ρ_E for the domain norm, while the letter G will be reserved for the target space and ρ_G for the target norm.

For example, by Theorem 3.1 or 3.2 below, the couple $E = \Gamma^q(t^{k/n} w)$, $G = \Lambda_0^q(v)$, $1 \le q \le \infty$, is admissible if v is related to w by the Muckenhoupt condition [25]:

$$\Big(\int_0^t [v(s)]^q \, ds/s\Big)^{1/q} \Big(\int_t^\infty [w(s)]^{-r} \, ds/s\Big)^{1/r} \lesssim 1, \qquad 1/q + 1/r = 1. \tag{1.10}$$

We denote by $\Lambda_0^q(v)$ the closure of C_0^∞ in the corresponding norm.

Now we recall the definition of optimal norms (see for example [13]).

Definition 1.3. Given the domain norm $\rho_E \in N_d$, the *optimal target norm*, denoted by $\rho_{G(E)}$, is the strongest target norm in N_t, i.e.

$$\rho_G(g^*) \lesssim \rho_{G(E)}(g^*), \qquad g \in L, \tag{1.11}$$

for any target norm $\rho_G \in N_t$ such that the couple ρ_E, ρ_G is admissible.

Definition 1.4. Given the target norm $\rho_G \in N_t$, the *optimal domain norm*, denoted by $\rho_{E(G)}$, is the weakest domain norm in N_d, i.e.

$$\rho_{E(G)}(g^*) \lesssim \rho_E(g^*), \qquad g \in L, \tag{1.12}$$

for any domain norm $\rho_E \in N_d$ such that the couple ρ_E, ρ_G is admissible.

Definition 1.5. The admissible couple $\rho_E \in N_d$, $\rho_G \in N_t$ is said to be *optimal* if $\rho_E = \rho_{E(G)}$ and $\rho_G = \rho_{G(E)}$.

The optimal norms are uniquely determined up to equivalence, while the corresponding optimal Banach spaces are unique.

We give a characterization of all admissible couples, optimal target norms, optimal domain norms, and optimal couples. It is convenient to consider the subcritical and critical cases separately.

DEFINITION 1.6. The *subcritical case* is defined by the condition

$$\int_0^1 s^{-k/n-1} h_E(s)\,ds < \infty, \quad \text{or equivalently} \quad k/n < \alpha_E. \tag{1.13}$$

For example, if $E = L^p$, $1 \le p < \infty$, then we get the classical homogeneous Sobolev space w_p^k and the condition (1.13) means that $k < n/p$.

In the subcritical case we prove that the optimal target norm satisfies $\rho_{G(E)}(g) \approx \rho_E(t^{-k/n} g(t))$, $g \in L$. Moreover, the couple $\rho_E, \rho_{G(E)}$ is optimal. For example, if $E = \Gamma^q(w)$, $\alpha_E > k/n$, $1 \le q \le \infty$, then $G(E) = \Lambda_0^q(t^{-k/n} w)$, and this couple is optimal (see Theorem 3.13 below).

In the critical case, i.e. $k/n = \alpha_E < 1$, we use real interpolation similarly to [10], but in a simpler way, and consider the domain norms

$$\rho_E(g) := \rho_H((t^{k/n} b(t) g^{**}(t))_\mu^{**}), \tag{1.14}$$

where ρ_H is a K-monotone norm on $(0,\infty)$ satisfying (1.4), and h_μ^* means the rearrangement of h with respect to the Haar measure on $(0,\infty)$, $d\mu := dt/t$, $h_\mu^{**}(t) := t^{-1} \int_0^t h_\mu^*(s)\,ds$. In this case the optimal target norm $\rho_{G(E)}$ is

$$\rho_{G(E)}(g) := \rho_H((cg)_\mu^{**}). \tag{1.15}$$

Here b and c belong to a large class of Muckenhoupt slowly varying weights (see Theorem 3.18 below).

Recall that w is *slowly varying on* $(1,\infty)$ (in the sense of Karamata) if for all $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to a nondecreasing function, and the function $t^{-\varepsilon} w(t)$ is equivalent to a nonincreasing function. By symmetry, we say that w is *slowly varying* on $(0,1)$ if the function $t \mapsto w(1/t)$ is slowly varying on $(1,\infty)$. Finally, w is *slowly varying* if it is slowly varying on $(0,1)$ and $(1,\infty)$.

For example, if $\rho_H(g) := (\int_0^\infty [g(t)]^q\,dt)^{1/q}$, $1 < q \le \infty$, then $\beta_H = 1/q < 1$, and

$$\rho_E(g) \approx \Big(\int_0^\infty [(t^{k/n} b(t) g^*(t))_\mu^*(s)]^q\,ds\Big)^{1/q} = \Big(\int_0^\infty [t^{k/n} b(t) g^*(t)]^q\,dt/t\Big)^{1/q}.$$

Hence $E = \Lambda^q(t^{k/n} b(t))$ and $G(E) = \Lambda_0^q(c)$.

The problem of optimal embeddings for inhomogeneous Sobolev spaces has been treated in several papers by somewhat different methods. In [14], [13], [17], [20], [24], [21], [15] the case of bounded domains is considered—and in [9], the case of second order Sobolev spaces. A different method, based on the theory of capacities, is applied in [16], [23]. The case of homogeneous Sobolev spaces is treated in [22] in the class of rearrangement invariant Banach function spaces as in [5]. Our domain spaces are more general. In particular, we do not use the Fatou property and duality arguments. The construction of the optimal target space in the subcritical case is rather simple and gives an optimal couple (see Theorem 3.13 below). In the critical case we construct a large class of domain spaces for which the corresponding optimal target space is found. In [22] the optimal target set is not linear (see also Theorem 3.4 below). The main results in a slightly different form are announced in [1].

2. Pointwise estimates for the rearrangement

LEMMA 2.1 ([12]). *For $k = 1$ and $k = 2$,*

$$f^{**}(t) - f^{**}(2t) \lesssim t^{k/n}|D^k f|^{**}(t), \qquad f \in C_0^\infty, \tag{2.1}$$

where $|D^k f| = \sum_{|\alpha|=k} |D^\alpha f|$.

When $n = 1$, $k = 1$ the estimate (2.1) is equivalent to one given in [18, Lemma 5]. For $k = 1$ it was proved in [3] using another method.

Proof. Let $t > 0$ and let B_t be the ball in $\mathbb{R}^n$ with centre 0, radius h and measure $2t$. Let $u \in \mathbb{R}^n$, $|u| \le h$. Let $\Delta_u f(x) := f(x+u) - f(x)$. Then

$$|f(x)| \le |\Delta_u f(x)| + |f(x+u)|,$$

and, integrating with respect to u over B_t,

$$2t|f(x)| \le \int_{B_t} |\Delta_u f(x)|\, du + \int_0^{2t} f^*(s)\, ds.$$

Now integrate with respect to x over a subset S of $\mathbb{R}^n$ with Lebesgue n-measure t and take the supremum over all such sets S. This gives (see [5, p. 53, Proposition 2.3.3])

$$2t[f^{**}(t) - f^{**}(2t)] \le \int_{B_t} (\Delta_u f)^{**}(t)\, du. \tag{2.2}$$

To estimate the right-hand side of this, we note that

$$|(\Delta_u f)(x)| = \left|\int_0^1 \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x+su)u_j\, ds\right| \le \int_0^1 |\nabla f(x+su)|\,|u|\, ds.$$

Integrate with respect to x over a subset E of $\mathbb{R}^n$ with Lebesgue measure t and take the supremum over all such subsets E. Then by [5, p. 53, Proposition 2.3.3],

$$(\Delta_u f)^{**}(t) \le \int_0^1 |\nabla f|^{**}(t)|u|\, ds = |\nabla f|^{**}(t)|u|.$$

Hence

$$f^{**}(t) - f^{**}(2t) \le \frac{1}{2t} \int\limits_{B_t} |\nabla f|^{**}(t)|u|\,du \lesssim t^{1/n}|\nabla f|^{**}(t),$$

as required.

To cover the case $k = 2$ we use $\Delta_u^2 f(x) := f(x+2u) - 2f(x+u) + f(x)$, whence

$$|f(x)| \le \frac{1}{2}|\Delta_u^2 f(x-u)| + \frac{1}{2}[|f(x+u)| + |f(x-u)|].$$

Integration with respect to u over B_t gives

$$2t|f(x)| \le \frac{1}{2} \int\limits_{B_t} |\Delta_u^2 f(x-u)|\,du + \int\limits_0^{2t} f^*(s)\,ds.$$

Hence taking the norm f^{**} we have

$$2t[f^{**}(t) - f^{**}(2t)] \le \frac{1}{2} \int\limits_{B_t} (\Delta_u^2 f)^{**}(t)\,du. \tag{2.3}$$

On the other hand,

$$|(\Delta_u^2 f)(x)| \lesssim \int\limits_0^1\int\limits_0^1 \left| \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x + s_1 u + s_2 u) u_i u_j \right| ds_1\,ds_2,$$

whence $(\Delta_u^2 f)^{**}(t) \lesssim |D^2 f|^{**}(t)$. Now (2.1) for $k = 2$ follows from (2.3). ∎

Lemma 2.2. *If $f \in C_0^\infty$ then*

$$f^{**}(t) \lesssim \int\limits_t^\infty u^{k/n}|D^k f|^{**}(u)\,\frac{du}{u}. \tag{2.4}$$

and

$$\delta f^{**}(t) := f^{**}(t) - f^*(t) \lesssim t^{2/n} \int\limits_t^\infty u^{(k-2)/n}|D^k f|^{**}(u)\,\frac{du}{u}. \tag{2.5}$$

Proof. In [22] the weaker estimate

$$\delta f^{**}(t) \lesssim t^{1/n} \int\limits_t^\infty u^{(k-1)/n}|D^k f|^{**}(u)\,\frac{du}{u}$$

is proved using (2.1) for $k = 1$ and induction.

We prove (2.5) by induction for $k > 2$. (If $k = 1$ or $k = 2$, then it follows from (2.1).) First we note that since $\frac{d}{dt} f^{**}(t) = (f^*(t) - f^{**}(t))t^{-1}$, we have

$$f^{**}(t) = \int\limits_t^\infty \delta f^{**}(u)\,\frac{du}{u} \tag{2.6}$$

and also $\delta f^{**}(t) \lesssim f^{**}(t) - f^{**}(2t)$. Using (2.1) and (2.6) we can write

$$\delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty \delta |D^2 f|^{**}(u) \, \frac{du}{u} \lesssim t^{2/n} \int_t^\infty u^{1/n} |D^3 f|^{**}(u) \, \frac{du}{u},$$

i.e. (2.5) for $k = 3$. By induction and (2.6), we have

$$\begin{aligned} \delta f^{**}(t) &\lesssim t^{2/n} \int_t^\infty u^{(k-2)/n} |D^k f|^{**}(u) \, \frac{du}{u} \\ &\lesssim t^{2/n} \int_t^\infty u^{(k-2)/n} \int_u^\infty \delta |D^k f|^{**}(s) \, \frac{ds}{s} \, \frac{du}{u}. \end{aligned}$$

Using also (2.1) for $k = 1$, we get

$$\delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty u^{(k-2)/n} \int_u^\infty s^{1/n} |D^{k+1} f|^{**}(s) \, \frac{ds}{s} \, \frac{du}{u}$$

and applying Fubini's theorem, we obtain the desired estimate (2.5) where k is replaced by $k + 1$. Finally, from (2.5) and (2.6) again using Fubini's theorem, we derive (2.4). ■

The results of Lemmas 2.1 and 2.2 are sufficient in order to cover the case $\beta_E < 1$. The limiting case $\beta_E = 1$ is more difficult. Then we shall use, instead of (2.1), the sharper estimate (3.6) below, proved in [22] for $k = 1$ and in [9] for $k = 2$.

3. Optimal Sobolev embeddings

3.1. Admissible couples. Here we give a characterization of all admissible couples $\rho_E \in N_d$, $\rho_G \in N_t$. Similar results are proved in [22] if E is a rearrangement invariant space as in [5].

THEOREM 3.1 (Case $\rho_E \in N_{d,0}$, $\rho_G \in N_t$). *The couple $\rho_E \in N_{d,0}$, $\rho_G \in N_t$ is admissible if and only if*

$$\rho_G(Tg) \lesssim \rho_E(g), \qquad g \in M, \tag{3.1}$$

where

$$Tg(t) := \int_t^\infty s^{k/n} g(s) \, ds/s, \qquad t > 0. \tag{3.2}$$

and

$$M := \{g \in L : t^m g(t) \text{ is increasing for some } m > 0 \text{ and } Tg(t) < \infty\}.$$

Proof. Evidently, (1.9) follows from (2.4) and (3.1). Now we prove that (1.9) implies (3.1). The proof given below is valid without the restriction

$\beta_E < 1$. Moreover, we can replace the class M by $M_0 = \{g \in L : g$ decreasing and $Tg(t) < \infty\}$. To this end we choose the test function in the form

$$f(x) = \int_0^\infty u^k g(u^n)\psi(|x/u|)\,\frac{du}{u}, \tag{3.3}$$

where $g \in M$ and $\psi \geq 0$ is a smooth function with compact support such that $\psi(|x|) = 1$ if $|x| \leq c$ and the constant c is chosen so that the ball $B_t := \{x : |x| \leq ct^{1/n}\}$ has volume t. Then $\psi(|x/u|) \geq 1$ if $x \in B_t$ and $u > t^{1/n}$. In particular

$$f^*(t) \geq \int_{t^{1/n}}^\infty u^k g(u^n)\,\frac{du}{u} = \frac{1}{n}\int_t^\infty u^{k/n} g(u)\,\frac{du}{u} = \frac{1}{n}Tg(t). \tag{3.4}$$

Since for a certain continuous function φ with compact support in $(0,\infty)$,

$$|D^\alpha f(x)| \leq \int_0^\infty g(u^n)\varphi(|x/u|)\,\frac{du}{u} = \int_0^\infty g((|x|/s)^n)\varphi(s)\,\frac{ds}{s}, \quad |\alpha| = k \geq 1,$$

we have, by the monotonicity condition on g,

$$|D^k f(x)| \lesssim g(C|x|^n).$$

Therefore,

$$\rho_E(|D^k f|^*) \lesssim \rho_E(g^*) = \rho_E(g). \tag{3.5}$$

Thus, if (1.9) is given, then (3.4) and (3.5) imply (3.1). ■

In the case $\beta_E = 1$ we require more properties of the domain norm ρ_E.

THEOREM 3.2 (Case $\rho_E \in N_{d,2}$, $\rho_G \in N_t$). *The couple $\rho_E \in N_{d,2}$, $\rho_G \in N_t$ is admissible if and only if the condition* (3.1) *is satisfied for all $g \in M$.*

A similar result is proved in [22] under the condition $k < 1 + n\alpha_E$, $n > 1$, if E is a rearrangement invariant Banach function space as in [5].

Proof. We only need to prove sufficiency. We start with the following estimate, proved in [22] for $k = 1$:

$$\int_0^t s^{-k/n}\delta f^{**}(s)\,ds \lesssim \int_0^t |D^k f|^*(s)\,ds, \qquad f \in C_0^\infty. \tag{3.6}$$

If $k = 2$ and $n > 2$ this estimate is also valid. It follows from

$$\int_0^t (s^{1-2/n}(-f^*(s))')^*(u)\,du \lesssim \int_0^t |D^2 f|^*(s)\,ds, \qquad n > 2, \tag{3.7}$$

which is proved in [9]. Indeed, let $g(t) := t^{-2/n}\int_0^t u(-f^*(u))'du$. Since $g(t) = t^{1-2/n}\delta f^{**}(t)$ we find, using also (2.1), that $g(0) = 0$. Now we can integrate by parts:

$$\int_0^t s^{-2/n}\delta f^{**}(s)\,ds = \int_0^t g(s)\,ds/s = -ng(t)/2 + n\int_0^t s^{1-2/n}(-f^*(s))'\,ds/2,$$

thus (3.6) for $k = 2$ follows.

Since $\alpha_E > 0$, inequalities (3.6) for $k = 1$ or $k = 2$ imply

$$\rho_E(t^{-k/n}\delta f^{**}(t)) \lesssim \rho_E(|D^k f|^*). \tag{3.8}$$

Indeed, the argument is similar to [22, proof of Lemma 2]. Introduce the Hardy operators

$$Pg(t) := \frac{1}{t}\int_0^t g(s)\,ds, \qquad Qg(t) := \int_t^\infty g(s)\,ds/s,$$

which commute. Then (3.6) gives

$$\int_0^t Q(s^{-k/n}\delta f^{**}(s))\,ds \lesssim \int_0^t Q(|D^k f|^*(s))\,ds, \tag{3.9}$$

whence by K-monotonicity of ρ_E,

$$\rho_E(Q(t^{-k/n}\delta f^{**}(t))) \lesssim \rho_E(Q(|D^k f|^*)). \tag{3.10}$$

We need the estimate

$$\rho_E(t^{-a}Qg(t)) \lesssim \rho_E(t^{-a}g(t)) \quad \text{if } \alpha_E > a,\ 0 \le a < 1,\ g \in M. \tag{3.11}$$

The proof is standard: we just have to use the fact that ρ_E satisfies (1.2), the monotonicity properties of $g \in M$ and the fact that $\alpha_E > a$ is equivalent to $\int_0^1 s^{-a}h_E(s)\,ds/s < \infty$ (cf. [5]).

From (3.10) we get (3.8) since $\alpha_E > 0$ implies the boundedness of Q, while the monotonicity of $t\delta f^{**}(t)$, Q and ρ_E give the needed estimate from below.

By induction, we will now prove (3.8) for all $k > 2$, provided $\alpha_E > (k-2)/n$. Let $h_k(t) = t^{-k/n}\delta f^{**}(t)$. If (3.8) is true for some $j > 2$ with $\alpha_E > (j-2)/n$, then by (2.1) and (2.6),

$$\rho_E(h_{j+1}) \lesssim \rho_E(t^{-(j-1)/n}|D^2 f|^{**}(t)) = \rho_E(t^{-(j-1)/n}Q(\delta|D^2 f|^{**}(t))),$$

and if $(j-1)/n < \alpha_E$ then by (3.11),

$$\rho_E(t^{-(j-1)/n}Q(\delta|D^2 f|^{**}(t))) \lesssim \rho_E(t^{-(j-1)/n}\delta|D^2 f|^{**}(t)) \lesssim \rho_E(|D^{j+1}f|^*).$$

Hence

$$\rho_E(h_{j+1}) \lesssim \rho_E(|D^{j+1}f|^*), \qquad \alpha_E > (j-1)/n.$$

Thus (3.8) is proved. Finally, since $f^{**} = Th_k$, $h_k(t) = t^{-k/n}\delta f^{**}(t)$, we get from (3.1) and (3.8) the estimate (1.9). ∎

3.2. Optimal norms. Here we give a characterization of the optimal domain and optimal target norms. Before constructing an optimal target norm, it is convenient first to prove an embedding in a target set that is rearrangement invariant but may not be a linear space [4], [19], [3]. We put for brevity $N_{d,3} := N_{d,1} \cup N_{d,2}$ and let $N_{t,3}$ be the subset of N_t consisting of all norms satisfying the Minkowski inequality.

DEFINITION 3.3. For a given domain norm $\rho_E \in N_{d,3}$, we define a rearrangement invariant *target set* G_E (not necessarily a linear space) by the condition

$$\rho_{G_E}(f^*) < \infty, \quad \text{where} \quad \rho_{G_E}(g) := \rho_E(t^{-k/n}(g^{**}(t) - g^*(t))). \tag{3.12}$$

THEOREM 3.4. *If $\rho_E \in N_{d,3}$, then*

$$\rho_{G_E}(f^*) \lesssim \rho_E(|D^k f|^*). \tag{3.13}$$

This embedding is proved in [22] under the condition $k < 1+n\alpha_E$, $n > 1$, if E is a rearrangement invariant Banach function space as in [5].

Proof. Let first $\beta_E < 1$. Then the relation (3.13) for $k = 1$ or $k = 2$ follows immediately from (2.1). If $k > 2$, from Lemma 2.2 we derive

$$\rho_{G_E}(f^*) \lesssim \rho_E\biggl(t^{(2-k)/n} \int_t^\infty u^{(k-2)/n} |D^k f|^{**}(u) \, \frac{du}{u}\biggr).$$

Since $(k-2)/n < \alpha_E$, we can apply (3.11) and (1.3), which gives (3.13). If $\beta_E = 1$, then (3.13) coincides with (3.8). ∎

We can define an optimal target norm by using Theorem 3.4. Namely, we have to take the linear and rearrangement invariant hull of the functional ρ_{G_E}.

DEFINITION 3.5. By definition, the *linear and rearrangement invariant hull* is the norm

$$\rho_l(g) := \inf \sum \rho_{G_E}(g_j), \qquad g_j \in L, \tag{3.14}$$

where the infimum is taken with respect to all finite sums: $g^{**} \le \sum g_j^{**}$.

PROPOSITION 3.6. *If $\rho_E \in N_{d,3}$, then ρ_l is a K-monotone and rearrangement invariant norm, the couple ρ_E, ρ_l is admissible and ρ_l is optimal, i.e. $\rho_l \approx \rho_{G(E)}$.*

Proof. The first two properties follow directly from the definition of ρ_l, while admissibility is a consequence of (3.13). To prove that ρ_l is an optimal target norm, let $\rho_G \in N_t$ be such that the couple ρ_E, ρ_G is admissible. Then by the results of the previous subsection we have $\rho_G(Tg) \lesssim \rho_E(g)$ for all $g \in M$. On the other hand, (2.6) gives

$$f^{**}(t) = Th_k(t), \qquad h_k(u) := u^{-k/n} \delta f^{**}(u). \tag{3.15}$$

Hence

$$\rho_G(f^{**}) \lesssim \rho_E(h_k) = \rho_{G_E}(f^*).$$

If $f^{**} \le \sum g_j^{**}$ as in the definition of ρ_l, then

$$\rho_G(f^*) \le \sum \rho_G(g_j^{**}) \lesssim \sum \rho_{G_E}(g_j),$$

therefore, taking the infimum, we get $\rho_G(f^*) \lesssim \rho_l(f^*)$. This finishes the proof. ∎

We can define an equivalent norm by using the results of the previous subsection, which will be useful below.

PROPOSITION 3.7. *If $\rho_E \in N_{d,3}$ then the monotone norm*

$$\rho(g) := \inf\{\rho_E(h) : g \le Th,\ h \in M\} \tag{3.16}$$

is equivalent to the norm ρ_l, i.e. $\rho(g^*) \approx \rho_l(g^*)$ for all $g \in L$. Thus ρ is an optimal target norm, i.e. $\rho \approx \rho_{G(E)}$ and $\alpha_{G(E)} \ge \alpha_E - k/n$, $\beta_{G(E)} \le \beta_E - k/n$. In addition,

$$E \hookrightarrow L^1 + \Lambda^1(t^{k/n}) \quad \textit{implies} \quad G(E) \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty. \tag{3.17}$$

Proof. Let $g^{**} \le \sum g_j^{**}$. We can write

$$g_j^{**} = Th_j, \qquad h_j := t^{-k/n}\delta g_j^{**},$$

hence

$$g^* \le g^{**} \le Th, \qquad h = \sum h_j.$$

This means that $\rho(g^*) \le \rho_E(h) \le \sum \rho_E(h_j) = \sum \rho_{G_E}(g_j)$ and taking the infimum we conclude that $\rho(g^*) \le \rho_l(g) = \rho_l(g^*)$.

For the reverse, if $g^* \le Th$ then using the fact that the couple ρ_E, ρ_l is admissible, we have $\rho_l(g^*) \le \rho_l(Th) \lesssim \rho_E(h)$. Hence, taking the infimum we obtain $\rho_l(g^*) \lesssim \rho(g^*)$.

To prove (3.17) let $f \in G(E)$. Then $f^* \le Th$, $h \in M$ and using also (3.17) we get $\int_0^1 t^{-k/n} f^*(t)\,dt \lesssim \rho_E(h)$. It remains to take the infimum with respect to h and use the equivalence (see for example [6])

$$\int_0^1 t^{-k/n} f^*(t)\,dt \approx \|f\|_{\Lambda^1(t^{1-k/n})+L^\infty}. \ ∎$$

If $\rho_G \in N_{t,1}$ we can construct an optimal domain norm in the class $N_{d,1}$.

THEOREM 3.8. *Let $\rho_G \in N_{t,1}$ be rearrangement invariant. Then the couple $\rho_{E(G)}, \rho_G$, where $\rho_{E(G)}(g) = \rho_G(Tg^{**})$, is optimal in the class $N_{d,1}$, $N_{t,1}$. The norm in the optimal domain space $E(G)$ is given by $\|f\|_{E(G)} := \rho_G(Tf^{**})$.*

Proof. The norm $\rho_{E(G)}$ is K-monotone, rearrangement invariant, satisfies the Minkowski inequality and its upper Boyd index is smaller than

$k/n + \beta_G < 1$, while its lower Boyd index is greater than k/n. Hence $\rho_{E(G)} \in N_{d,1}$. In particular, $\rho_{E(G)}(g) \approx \rho_G(Tg^*)$. If ρ_E, ρ_G is an admissible couple, then $\rho_G(Tg^*) \lesssim \rho_E(g^*)$. Therefore $\rho_{E(G)}(g^*) \lesssim \rho_E(g^*)$. This proves the optimality of the domain norm $\rho_{E(G)}$ in $N_{d,1}$.

It remains to check that ρ_G is an optimal target norm. To this end we need to prove that $\rho(g^*) \lesssim \rho_G(g^*)$, where ρ is defined by (3.16). Since $g^{**} = Th$ with $h(t) = t^{-k/n}\delta g^{**}(t) \in M$, we can write

$$\rho(g^*) \le \rho_{E(G)}(h) = \rho_G(Th^{**}).$$

Since $h^* \lesssim Qh$, we have $h^{**} = Ph^* \lesssim QPh$, therefore $Th^{**} \lesssim TQ(Ph) \lesssim T(Ph)$. Also $T(Ph) \approx Th + t^{k/n}Ph$ and $Ph \le h^{**}$. Therefore,

$$\rho(g^*) \lesssim \rho_G(Th) + \rho_G(t^{k/n}h^{**}).$$

Consider the norm $h \mapsto \rho_G(t^{k/n}h^{**}(t))$. It is K-monotone and its upper Boyd index is $k/n + \beta_G < 1$, hence

$$\rho(g^*) \lesssim \rho_G(Th) + \rho_G(t^{k/n}h^*(t)).$$

Since $h(t) \le t^{-k/n}g^{**}(t)$ we have $h^*(t) \le t^{-k/n}g^{**}$, thus $\rho_G(t^{k/n}h^*(t)) \le \rho_G(g^{**})$. Therefore

$$\rho(g^*) \lesssim \rho_G(g^{**}) \approx \rho_G(g^*).$$

The proof is complete. ∎

Now we give some examples.

Example 3.9. Consider the space $G = \Lambda_0^1(v)$ and let $\beta_G < 1 - k/n$. This is true in the particular case when v is slowly varying, since then $\alpha_G = \beta_G = 0$. Using Theorem 3.8, we can construct the optimal couple E, G, where $\rho_E(g) = \rho_G(Tg^{**}) = \int_0^\infty t^{k/n}w(t)g^{**}(t)\,dt/t$ and $w(t) = \int_0^t v(s)\,ds/s$. Hence $E = \Lambda^1(t^{k/n}w) = \Gamma^1(t^{k/n}w)$. Also $\alpha_E = \beta_E = k/n$ if v is slowly varying.

Example 3.10. If $G = C_0$ consists of all bounded functions such that $f^*(\infty) = 0$ and $\rho_G(g) = g^*(0) = g^{**}(0)$. Then $\alpha_G = \beta_G = 0$ and $\rho_{E(G)}(g) \approx \int_0^\infty t^{k/n}g^{**}(t)\,dt/t$, i.e. $E = \Lambda^1(t^{k/n}) = \Gamma^1(t^{k/n})$ and the couple E, G is optimal.

Example 3.11. Let $G = \Lambda_0^\infty(v)$ with $\beta_G < 1 - k/n$ and let

$$\rho_E(g) = \sup v(t) \int_t^\infty s^{k/n}g^{**}(s)\,ds/s.$$

Then by Theorem 3.8, the couple E, G is optimal and $\beta_E < 1$. In particular, this is true if v is slowly varying since then $\alpha_G = \beta_G = 0$ and $\alpha_E = \beta_E = k/n < 1$.

EXAMPLE 3.12. Let G be as in the previous example. Since

$$\rho_E(g) \le \sup t^{k/n} w(t) g^{**}(t), \qquad \frac{1}{v(t)} = \int_t^\infty \frac{1}{w(s)} \frac{ds}{s},$$

it follows that the couple $E_1 = \Gamma^\infty(t^{k/n}w), G = \Lambda_0^\infty(v)$ is admissible. Let w be slowly varying. In order to prove that ρ_G is optimal, take any $g \in L$, and define h from $t^{k/n}w(t)h(t) = \sup_{0<s\le t} v(s)g^*(s)$. Then $t^{k/n}w(t)h(t) \le \rho_G(g)$, therefore $t^{k/n}w(t)h^*(t) \lesssim \rho_G(g)$, whence $\rho_{E_1}(h) \lesssim \rho_G(g)$. On the other hand

$$Th(t) = \int_t^\infty \sup_{0<s\le u} v(s)g^*(s) \frac{1}{w(u)}\, du/u \ge \sup_{0<s\le t} v(s)g^*(s) \frac{1}{v(t)} \ge g^*(t).$$

Hence $\rho(g^*) \le \rho_{E_1}(h) \lesssim \rho_G(g)$, therefore ρ_G is optimal. But the couple ρ_{E_1}, ρ_G is not optimal. Indeed, suppose otherwise, i.e. $\rho_E \approx \rho_{E_1}$. Then $\rho_G(Tg) \gtrsim \rho_{E_1}(g)$ for all nonnegative and decreasing g. To contradict this inequality, we choose $w(t) = (1 - \log t)^3$ if $0 < t < 1$ and $w(t) = (1 + \log t)^2$ for $t > 1$. Then $v(t) \approx 1$ for $0 < t < 1/2$. Let $h(t) = t^{k/n} g(t) = (1 - \log t)^{-2}$ if $0 < t < 1$. Then $Tg \approx 1$ for $0 < t < 1/2$. Since $w(t)h(t) = 1 - \log t$, $v(t)Tg(t) \approx 1$ for $0 < t < 1/2$, we get a contradiction.

3.3. Subcritical case. Here we suppose that $k/n < \alpha_E$.

THEOREM 3.13. *Let $\rho_E \in N_{d,3}$. Then we define an optimal target norm $\rho_{G(E)}$ by*

$$\rho_{G(E)}(g) := \rho_E(t^{-k/n} g). \tag{3.18}$$

Moreover, the couple $\rho_E, \rho_{G(E)}$ is optimal and $\beta_{G(E)} = \beta_E - k/n$, $\alpha_{G(E)} = \alpha_E - k/n$.

Proof. Optimality of the target norm (3.18) is known [22] for rearrangement invariant Banach function spaces as in [5]. According to our previous results, the couple $\rho_E, \rho_{G(E)}$ is admissible if

$$\rho_E\Big(t^{-k/n} \int_t^\infty s^{k/n} g(s)\, ds/s\Big) \lesssim \rho_E(g), \qquad g \in M.$$

But this follows from (3.11) since $k/n < \alpha_E$. Further, let ρ_E, ρ_G be an admissible couple. Then, as before,

$$\rho_G(g^*) \le \rho_G(g^{**}) = \rho_G(T(t^{-k/n}\delta g^{**})) \lesssim \rho_E(t^{-k/n} g^{**}(t)).$$

Now consider the norm $g \mapsto \rho_E(t^{-k/n} g^{**})$. Since its upper Boyd index equals $\beta_E - k/n$, it follows that this index is smaller than one, hence the above gives $\rho_G(g^*) \lesssim \rho_E(t^{-k/n} g^*(t)) = \rho_{G(E)}(g^*)$. This proves the optimality of the target norm (3.18).

Finally, the couple $\rho_E, \rho_{G(E)}$ is optimal, since for any admissible couple $\rho_F, \rho_{G(E)}$, we have $\rho_{G(E)}(Tg) \lesssim \rho_F(g)$ for all $g \in M_0$. Then

$$\rho_F(g) \gtrsim \rho_{G(E)}(t^{k/n}g) \approx \rho_E(g). \ \blacksquare$$

Consider some examples.

EXAMPLE 3.14. The couple $E = \Lambda^q(tw)$, $G = \Lambda_0^q(t^{1-k/n}w)$, where $1 \leq q < \infty$, $k < n$, $t^{q-1}[w(t)]^q$ is nonincreasing and such that $k/n < \alpha_E$, is optimal. Indeed, $\rho_E := \rho_{tw,q}$ is K-monotone. It remains to apply Theorem 3.13. In particular, this is true if $q = 1$ and in addition w is slowly varying, since then $\alpha_E = \beta_E = 1$.

EXAMPLE 3.15. The couple $E = \Gamma^q(w)$, $G = \Lambda_0^q(t^{-k/n}w)$, where $k/n < \alpha_E$ and $\beta_E < 1$, is optimal.

EXAMPLE 3.16. Consider the Γ space $E = \Gamma^q(tw)$, where $k < n$, $1 < q \leq \infty$, w is a slowly varying weight and $\rho_E := \rho_{tw,q,\Gamma}$. In this case $\alpha_E = \beta_E = 1$ and Theorem 3.13 shows that

$$\rho_{G(E)}(g) = \left(\int_0^\infty [tw(t)]^q ([s^{-k/n}g(s)]^{**}(t))^q \, \frac{dt}{t} \right)^{1/q},$$

hence the space $G(E)$ is smaller than $\Gamma_0^q(t^{1-k/n}w)$.

If the target norm is from the class $N_{t,2} := \{\rho_G \in N_{t,1} : \alpha_G > 0\}$, then we can simplify the formula for the optimal domain norm constructed in Theorem 3.8.

THEOREM 3.17. *Let $\rho_G \in N_{t,2}$ be rearrangement invariant and define the norm ρ_E by*

$$\rho_E(g) = \rho_G(t^{k/n}g^{**}(t)). \tag{3.19}$$

Then the couple ρ_E, ρ_G is optimal in the class $N_{d,1}$, $N_{t,2}$.

Proof. According to Theorem 3.8 we only have to prove that

$$\rho_G(t^{k/n}g^{**}) \approx \rho_G(Tg^{**}), \qquad g \in L.$$

This follows from $Tg(t) \gtrsim t^{k/n}g(t)$ for all $g \in M$ and from

$$\rho_G(Tg) \lesssim \rho_G(t^{k/n}g(t)), \quad g \in M, \qquad \text{provided } \alpha_G > 0$$

(see (3.11)). ■

3.4. Critical case. Here we are going to use real interpolation for normed spaces, similarly to [11], [10]. First we recall some basic definitions. Let (A_0, A_1) be a couple of normed spaces (see [6], [7]) and let

$$K(t,f) = K(t,f;A_0,A_1) = \inf_{f=f_0+f_1} [\|f_0\|_{A_0} + t\,\|f_1\|_{A_1}], \qquad f \in A_0 + A_1,$$

be the K-functional of Peetre (see [6]). By definition, the *K-interpolation space* $A_\Phi = (A_0, A_1)_\Phi$ has the norm

$$\|f\|_{A_\Phi} = \|K(t, f)\|_\Phi \,,$$

where Φ is a normed function space with a monotone norm on $(0, \infty)$ with the Lebesgue measure and such that $\min\{1, t\} \in \Phi$. Then (see [7])

$$A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1.$$

Now we construct the needed couples of Muckenhoupt weights. Suppose

$$b \text{ is nondecreasing, slowly varying on } (0,\infty) \text{ with } b(t^2) \approx b(t) \tag{3.20}$$

and

$$(1+\log t)^{-1-\varepsilon} b(t),\ t>1, \text{ is increasing for some } \varepsilon > 0. \tag{3.21}$$

Let

$$c(t) = \frac{b(t)}{1+|\log t|}. \tag{3.22}$$

Then

$$\int_t^\infty \frac{1}{b(s)}\, ds/s \lesssim \frac{1}{c(t)}, \qquad t > 0. \tag{3.23}$$

Indeed, if $0 < t < 1$ we can write

$$\int_t^\infty \frac{1}{b(u)} \frac{du}{u} = \int_t^1 \frac{1}{b(u)} \frac{du}{u} + \int_1^\infty \frac{(1+\log u)^{-1-\varepsilon}}{b(u)(1+\log u)^{-1-\varepsilon}} \frac{du}{u}.$$

Using the monotonicity properties (3.20), (3.21) and $c(t) \lesssim 1$ for $0 < t < 1$, we get (3.23). The case $t > 1$ is analogous, but simpler.

THEOREM 3.18. *Let H be a rearrangement invariant space on* $(0, \infty)$ *with the Lebesgue measure, with a K-monotone rearrangement invariant norm ρ_H that satisfies the Minkowski inequality, and let $\beta_H < 1$. Let b, c be given by* (3.20)–(3.22). *Let ρ_E be defined by*

$$\rho_E(g) := \rho_F(t^{k/n} b(t) g^{**}(t)), \qquad k < n, \tag{3.24}$$

set

$$F := (L^1_*, L^\infty_*)_{H(1/t)}, \tag{3.25}$$

and suppose $H(1/t)$ has the norm $\|g\|_{H(1/t)} := \rho_H(g(t)/t)$. Then the optimal target norm is given by

$$\rho_{G(E)}(g) := \rho_F(gc). \tag{3.26}$$

Moreover, $E \hookrightarrow L^1 + \Lambda^1(t^{k/n})$ and $G(E) \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty$.

Proof. We denote by $L^r_*(v)$, $1 \le r \le \infty$, v a positive weight, the weighted Lebesgue space on $(0, \infty)$ with the Haar measure $d\mu = dt/t$ and norm

$$\|g\|_{L^r_*(v)} := \Big(\int_0^\infty |g(t)v(t)|^r \, dt/t\Big)^{1/r}.$$

We write L^r_* when $v = 1$.

Let L^r_v be the weighted Lebesgue space on $(0, \infty)$ with the Lebesgue measure and norm

$$\|g\|_{L^r_v} := \Big(\int_0^\infty |g(t)v(t)|^r \, dt\Big)^{1/r}.$$

Then the operator T defined by (3.2) is bounded in the following couple of spaces:

$$T : L^1_*(t^{k/n}b(t)) \to L^\infty_b \quad \text{and} \quad T : L^\infty_*(t^{k/n}b(t)) \to L^\infty_c, \tag{3.27}$$

where b, c are given by (3.20), (3.22), since they satisfy (3.23).

Define F by (3.25). It is well known ([6]) that

$$\rho_F(g) = \rho_H(g^{**}_\mu) \approx \rho_H(g^*_\mu), \tag{3.28}$$

where $g^{**}_\mu(t) = t^{-1}\int_0^t g^*_\mu(s)\, ds$. The equivalence in (3.28) is true because $\beta_H < 1$.

By interpolation,

$$T : E_1 \to G_1, \tag{3.29}$$

where

$$E_1 := (L^1_*(t^{k/n}b(t)), L^\infty_*(t^{k/n}b(t)))_{H(1/t)}, \qquad G_1 := (L^\infty_b, L^\infty_c)_{H(1/t)}. \tag{3.30}$$

Denote the norm in E_1 by ρ_1 and let $\rho_E(g) = \rho_1(g^{**})$. We have

$$\rho_E(g) = \rho_F(t^{k/n}b(t)g^{**}(t)) = \rho_H((t^{k/n}b(t)g^{**}(t))^{**}_\mu). \tag{3.31}$$

Hence ρ_E is rearrangement invariant, K-monotone norm with both Boyd indices equal to $k/n < 1$ (here we are using the fact that b is slowly varying). Therefore

$$\rho_E(g) \approx \rho_F(t^{k/n}b(t)g^*(t)) \approx \rho_H((t^{k/n}b(t)g^*(t))^{**}_\mu). \tag{3.32}$$

Since ρ_H satisfies the Minkowski inequality, so does ρ_E. Now we prove the property

$$\int_0^1 h(t)\, dt + \int_1^\infty t^{k/n-1}h(t)\, dt \lesssim \rho_E(h), \qquad h \in M_0, \tag{3.33}$$

which implies the embedding $E \hookrightarrow L^1 + \Lambda^1(t^{k/n})$. To prove (3.33), we notice that

$$L^1_*(t^{k/n}b) \hookrightarrow L^1 + L^1(t^{k/n-1}), \qquad L^\infty_*(t^{k/n}b) \hookrightarrow L^1 + L^1(t^{k/n-1}),$$

whence using $\rho_E(h) = \rho_1(h^{**}) \gtrsim \rho_1(h)$ for $h \in M_0$ and (3.30) we get (3.33).

Now we characterize the space G_1. Since (see [6])

$$K(t,g;L_b^\infty,L_c^\infty)=tK(1/t,g;L_c^\infty,L_b^\infty)=t\sup_s|g(s)|\min(c(s),b(s)/t),$$

we get the formula

$$\rho_{G_1}(g)=\rho_H(h_g),\qquad h_g(u):=\sup_s|g(s)|\min(c(s),b(s)/u). \tag{3.34}$$

Also, since $L_b^\infty\hookrightarrow L_c^\infty$ it follows that $h_g(u)\approx\sup|g(s)|c(s)$ if $0<u<1$. Let

$$H_g(t):=h_g(1+|\log t|),\qquad 0<t<\infty. \tag{3.35}$$

Then $(H_g)^*_\mu(t)\le h_g(t/2)$, hence by (3.28) and (3.34),

$$\rho_F(H_g)\lesssim\rho_{G_1}(g). \tag{3.36}$$

Note that $H_g\gtrsim gc$, hence, if we define the norm $\rho_G(g):=\rho_F(gc)$, we get the relation

$$\rho_G(Tg)\lesssim\rho_{G_1}(Tg)\lesssim\rho_E(g),\qquad g\in M. \tag{3.37}$$

Since $\beta_E<1$ Theorem 3.1 shows that the couple ρ_E,ρ_G is admissible.

Now we want to prove that ρ_G is an optimal target norm. It is sufficient to see that

$$\rho_G(g)\approx\rho(g),\qquad g\in L,\ g\text{ decreasing}, \tag{3.38}$$

where ρ is defined by (3.16); and since the norm ρ is optimal, we need only prove that $\rho(g)\lesssim\rho_G(g)$, $g\in L$, g decreasing. To this end first for any such g we construct an $h\in M$ such that $g\lesssim Th$ and $\rho_E(h)\lesssim\rho_G(g)$. Let $t^{k/n}b(t)h_1(t)=g_1(t)$, where $g_1(t)=g^{**}(t^2/e)c(t^2)$ for $0<t<1$ and $g_1(t)=g^{**}(\sqrt{t/e})c(\sqrt{t})$ if $t>1$. Note that $h_1\approx h_1^*$. Then $\rho_E(h_1)\approx\rho_F(t^{k/n}b(t)h_1^*(t))\approx\rho_F(gc)=\rho_G(g)$. On the other hand, for $0<t<1$,

$$Th_1(t)\ge\int_t^{\sqrt{te}}g(s^2/e)\frac{c(s^2)}{b(s)}\,ds/s\ge g(t)A(t)\gtrsim g(t),$$

since

$$A(t)=\int_t^{\sqrt{te}}\frac{c(s^2)}{b(s)}\,ds/s\approx\int_t^{\sqrt{te}}\frac{1}{1+|\log s|}\,ds/s\gtrsim 1.$$

Similarly, for $t>1$ we obtain

$$Th_1(t)\ge\int_t^{et^2}g(\sqrt{s/e})\frac{1}{1+\log s}\,ds/s\gtrsim g(t).$$

Thus $Th_1\gtrsim g$ and $\rho_E(h_1)\approx\rho_G(g)$, therefore we can find $h\approx h_1$ with the required properties. Then by the definition of ρ we get $\rho(g)\lesssim\rho_G(g)$.

Finally, from (3.33) and Proposition 3.7 the embedding $G(E)\hookrightarrow\Lambda(t^{1-k/n})+L^\infty$ follows. ■

Example 3.19. Let $G = \Lambda_0^q(c)$, $1 < q < \infty$, $E = \Gamma^q(t^{k/n}b(t))$, where b is slowly varying on $(0,\infty)$, $b(t^2) \approx b(t)$, $b(t) \lesssim (1 + |\log t|)c(t)$ and

$$\Big(\int_0^t [c(s)]^q \, ds/s\Big)^{1/q} \Big(\int_t^\infty [b(s)]^{-r} \, ds/s\Big)^{1/r} \lesssim 1, \quad 1/q + 1/r = 1.$$

Then the couple E, G is admissible and using the same argument as above, we see that G is an optimal target space. In particular, we can take $b(t) = 1$, $0 < t < 1$ and $b(t) = (1 + \log t)^2$, $t > 1$. Then $c(t) = (1 - \log t)^{-1}$, $0 < t < 1$ and $c(t) = 1 + \log t$, $t > 1$. But we cannot take $b(t) = 1$ for all $t > 0$. This means that the Lebesgue space $L^{n/k}$ is not allowed as a domain. It is not embedded in $L^1 + \Lambda^1(t^{k/n})$. This contrasts with the well known limiting embedding

$$\int_0^1 \Big(\frac{f^*(t)}{1 - \log t}\Big)^{n/k} \frac{dt}{t} \lesssim \int_0^1 (|D^k f|^*(t))^{n/k} \, dt, \quad f \in C_0^\infty(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^n$ (see [16]). Of course, by Theorem 3.4 (see also [22]) we have the optimal embedding in the rearrangement invariant nonlinear set:

$$\int_0^\infty (\delta f^{**}(t))^{n/k} \frac{dt}{t} \lesssim \int_0^\infty (|D^k f|^*(t))^{n/k} \, dt.$$

Acknowledgements. We are grateful to the referee whose suggestions improved the paper.

Research of G. E. Karadzhov was partially supported by the Abdus Salam School of Mathematical Sciences, GC University Lahore, and by a grant from HEC, Pakistan.

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Received 16 October 2009;
revised 24 July 2010 (5284)