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A REMARK ON THE IDENTITY PRINCIPLE FOR ANALYTIC SETS

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Abstract. We present a version of the identity principle for analytic sets, which shows that the extension theorem for separately holomorphic functions with analytic singularities follows from the case of pluripolar singularities.

1. Introduction. Main result. The classical identity principle for analytic sets asserts that if S_0, S are analytic subsets of an open set $G \subset \mathbb{C}^n$ such that S_0 is irreducible and $\emptyset \neq S_0 \cap \Omega \subset S$, where $\Omega \subset G$ is open, then $S_0 \subset S$ (cf. [Chi 1989, §5.3]). Our starting point was a problem from the theory of continuation of separately holomorphic functions with singularities (see below). To solve this problem we needed an identity principle for certain domains G but with the assumption that $\emptyset \neq S_0 \cap T \subset S$, where T is a certain "thin" subset of G. Such an identity principle will be presented in Theorem 1.4, which is the main result of the paper.

Throughout the paper we work in the following geometric context. Fix an integer $N \geq 2$ and let D_j be a (connected) *Riemann domain of holomorphy* over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$. Let $\emptyset \neq A_j \subset D_j$ be *locally pluriregular*, $j = 1, \dots, N$.

To simplify notation, for $B_j \subset D_j$, $j = 1, \dots, N$, we write $B'_j := B_1 \times \dots \times B_{j-1}$, $j = 2, \dots, N$, $B''_j := B_{j+1} \times \dots \times B_N$, $j = 1, \dots, N-1$. Similarly, for $a = (a_1, \dots, a_N)$ we put $a'_j := (a_1, \dots, a_{j-1})$, $j = 2, \dots, N$, $a''_j := (a_{j+1}, \dots, a_N)$, $j = 1, \dots, N-1$. We define an *N-fold cross*

$$\boldsymbol{X} = \mathbb{X}((D_j, A_j)_{j=1}^N) := \bigcup_{j=1}^N A'_j \times D_j \times A''_j,$$

where $A'_1 \times D_1 \times A''_1 := D_1 \times A''_1$, $A'_N \times D_N \times A''_N := A'_N \times D_N$. One can prove that $\boldsymbol{X}$ is connected.

We say that a function $f : \boldsymbol{X} \to \mathbb{C}$ is *separately holomorphic on $\boldsymbol{X}$* (we write $f \in \mathcal{O}_s(\boldsymbol{X})$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j$, the function $D_j \ni z_j \mapsto f(a'_j, z_j, a''_j)$ is holomorphic in D_j.

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Separately holomorphic functions have been studied by many authors (under various additional assumptions) during the last 100 years, which finally led to the following extension theorem.

THEOREM 1.1 (Cross theorem, cf. [Ber 1912], [Cam-Sto 1966], [Sic 1968], [Sic 1969a], [Sic 1969b], [Akh-Ron 1973], [Zah 1976], [Sic 1981], [Shi 1989], [Ngu-Zer 1991], [Ngu-Zer 1995], [NTV 1997], [Ale-Zer 2001]). *Every function $f \in \mathcal{O}_s(\boldsymbol{X})$ extends holomorphically to the domain of holomorphy*

$$\widehat{\boldsymbol{X}} := \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : h^*_{A_1,D_1}(z_1) + \dots + h^*_{A_N,D_N}(z_N) < 1\},$$

where h_{A_j,D_j} denotes the relative extremal function of A_j in D_j, $j=1,\dots,N$, and * stands for the upper semicontinuous regularization.

Recall that

$$h_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \le 1,\ u|_A \le 0\}, \qquad A \subset D.$$

We are interested in the extension theory with singularities. Let $M \subset \boldsymbol{X}$ be *relatively closed*. We say that a function $f : \boldsymbol{X} \setminus M \to \mathbb{C}$ is *separately holomorphic on $\boldsymbol{X} \setminus M$* if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j$, the function $D_j \setminus M_{(a'_j,\cdot,a''_j)} \ni z_j \mapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$ is holomorphic in $D_j \setminus M_{(a'_j,\cdot,a''_j)}$, where $M_{(a'_j,\cdot,a''_j)} := \{z_j \in D_j : (a'_j, z_j, a''_j) \in M\}$ is the *fiber of M over (a'_j, a''_j)*; in the above situation we write $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$.

In the case of analytic singularities the following result is known.

THEOREM 1.2 (Cross theorem with analytic singularities, cf. [Jar-Pfl 2003a]). *Let $S \subsetneq U$ be an analytic subset of an open connected neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$ and let $M := S \cap \boldsymbol{X}$. Then there exists an analytic set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:*

- *$\widehat{\boldsymbol{X}} \setminus \widehat{M}$ is a domain of holomorphy (consequently, either $\widehat{M} = \emptyset$ or $\widehat{M}$ is of pure codimension one),*
- *$\widehat{M} \cap U_0 \subset S$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$,*
- *for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$,*
- *if $U = \widehat{\boldsymbol{X}}$, then $\widehat{M}$ coincides with the union of all irreducible components of S of pure codimension one.*

For *pluripolar* sets $\Sigma_j \subset A'_j \times A''_j$, $j = 1, \dots, N$, we define an *N-fold generalized cross*

$$\boldsymbol{T} = \mathbb{T}((D_j, A_j, \Sigma_j)_{j=1}^N) := \bigcup_{j=1}^N \{(a'_j, z_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j\}.$$

We say that $\boldsymbol{T}$ *is generated by* $\Sigma_1, \dots, \Sigma_N$. Obviously, $\boldsymbol{X} = \mathbb{T}((D_j, A_j, \emptyset)_{j=1}^N)$.

Observe that any 2-fold generalized cross is in fact a 2-fold cross, namely

$$\mathbb{T}((D_j, A_j, \Sigma_j)_{j=1}^2) = \mathbb{X}((D_1, A_1 \setminus \Sigma_2), (D_2, A_2 \setminus \Sigma_1)).$$

For $N \geq 3$ the geometric structure of $\boldsymbol{T}$ is essentially different.

In the case of pluripolar singularities we have the following extension theorem.

THEOREM 1.3 (Cross theorem with pluripolar singularities, cf. [Jar-Pfl 2003b], [Jar-Pfl 2007]). *Let $M \subset \boldsymbol{X}$ be a relatively closed set such that for every $j \in \{1, \dots, N\}$ there exists a pluripolar set $\Sigma_j^0 \subset A_j' \times A_j''$ such that the fiber $M_{(a_j', \cdot, a_j'')}$ is pluripolar for all $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$. Then there exist an N-fold generalized cross $\boldsymbol{T}$ generated by pluripolar sets $\Sigma_j \subset A_j' \times A_j''$ with $\Sigma_j^0 \subset \Sigma_j$, $j = 1, \dots, N$, and a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:*

- *$\widehat{\boldsymbol{X}} \setminus \widehat{M}$ is a domain of holomorphy,*
- *$\widehat{M} \cap \boldsymbol{T} \subset M$,*
- *for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\boldsymbol{T} \setminus M$,*
- *if for any $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$, the fiber $M_{(a_j', \cdot, a_j'')}$ is thin, then $\widehat{M}$ is analytic in $\widehat{\boldsymbol{X}}$ (consequently, either $\widehat{M} = \emptyset$ or $\widehat{M}$ is of pure codimension one).*

It has been conjectured that Theorem 1.2 follows from Theorem 1.3. This would give a uniform presentation of the cross theorems with singularities.

Observe that if S is as in Theorem 1.2, then $M := S \cap \boldsymbol{X}$ satisfies the assumption of Theorem 1.3 with $\Sigma_j^0 := \{(a_j', a_j'') \in A_j' \times A_j'' : S_{(a_j', \cdot, a_j'')} = D_j\}$, $j = 1, \dots, N$. Then Theorem 1.3 gives an analytic set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that $\widehat{M} \cap \boldsymbol{T} \subset S$ and for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\widehat{f} = f$ on $\boldsymbol{T} \setminus S$. Now, an elementary reasoning shows that in order to get Theorem 1.2 (in its full generality) we only need to show that $\widehat{M} \cap U_0 \subset S$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$.

Summarizing, the main problem is to prove the following identity principle for analytic sets, which may be also interesting for other questions.

THEOREM 1.4. *Let S_0, S be analytic subsets of an open connected neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$ such that:*

- *S_0 is of pure codimension one,*
- *there exists an N-fold generalized cross $\boldsymbol{T} \subset \boldsymbol{X}$ (generated by pluripolar sets $\Sigma_1, \dots, \Sigma_N$) for which $S_0 \cap \boldsymbol{T} \subset S$.*

Then there exists an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$ such that $S_0 \cap U_0 \subset S$. Moreover, if $U = \widehat{\boldsymbol{X}}$, then $S_0 \subset S$.

2. Proof of Theorem 1.4

STEP 1. *Reduction to the case where $U = \widehat{\boldsymbol{X}}$.*

Suppose that Theorem 1.4 is true for $U = \widehat{\boldsymbol{X}}$ (with other elements arbitrary).

Now, let $U \subsetneq \widehat{\boldsymbol{X}}$ be arbitrary. It suffices to show that for every $a \in \boldsymbol{X}$ there exists an open neighborhood $U_a \subset U$ such that $S_0 \cap U_a \subset S$.

We may assume that $a = (a_1, \dots, a_N) = (a'_N, a_N) \in A'_N \times D_N$. Let $G_N \Subset D_N$ be a domain of holomorphy such that $G_N \cap A_N \neq \emptyset$ and $a_N \in G_N$. Since $\{a'_N\} \times G_N \Subset U$, there exist domains of holomorphy $G_j \Subset D_j$, $a_j \in G_j$, $j = 1, \dots, N-1$, with $G_1 \times \dots \times G_N \subset U$. Note that $A_j \cap G_j$ is locally pluriregular, $j = 1, \dots, N$. Consider the N-fold cross $\boldsymbol{Y}_a := \mathbb{X}((G_j, A_j \cap G_j)_{j=1}^N) \subset \boldsymbol{X}$, $a \in \boldsymbol{Y}_a$. We have $\widehat{\boldsymbol{Y}}_a \subset G_1 \times \dots \times G_N \subset U$. Consequently, the analytic sets $S_0 \cap \widehat{\boldsymbol{Y}}_a$ and $S \cap \widehat{\boldsymbol{Y}}_a$ satisfy all the assumptions of Theorem 1.4 with $(U, \boldsymbol{T})$ replaced by

$$(\widehat{\boldsymbol{Y}}_a, \mathbb{T}((G_j, A_j \cap G_j, \Sigma_j \cap (G'_j \times G''_j))_{j=1}^N)).$$

Hence, $S_0 \cap U_a \subset S$ for an open neighborhood $U_a \subset \widehat{\boldsymbol{Y}}_a$ of $\boldsymbol{Y}_a$.

From now on we assume that $U = \widehat{\boldsymbol{X}}$.

STEP 2. *Let S'_0 be an irreducible component of S_0 and let $\boldsymbol{T}' \subset \boldsymbol{T}$ be an arbitrary generalized N-folds cross (generated by pluripolar sets $\Sigma'_j \subset A'_j \times A''_j$ with $\Sigma_j \subset \Sigma'_j$, $j = 1, \dots, N$). Then $S'_0 \cap \boldsymbol{T}' \neq \emptyset$.*

Indeed, suppose that $S'_0 \cap \boldsymbol{T}' = \emptyset$. Since S'_0 is of pure codimension one, the set $\widehat{\boldsymbol{X}} \setminus S'_0$ is a domain of holomorphy. Thus, there exists a $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus S'_0)$ such that $\widehat{\boldsymbol{X}} \setminus S'_0$ is the domain of existence of g. Since $\boldsymbol{T}' \subset \widehat{\boldsymbol{X}} \setminus S'_0$, we conclude that $f := g|_{\boldsymbol{T}'} \in \mathcal{O}_s(\boldsymbol{T}') \cap \mathcal{C}(\boldsymbol{T}')$. Here $f \in \mathcal{O}_s(\boldsymbol{T}')$ means that for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma'_j$, the function $D_j \ni z_j \mapsto f(a'_j, z_j, a''_j)$ is holomorphic. Then, by [Jar-Pfl 2003b], [Jar-Pfl 2007], there exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{T}'$. Since $\boldsymbol{T}'$ is non-pluripolar, we get $\widehat{f} = g$ on $\widehat{\boldsymbol{X}} \setminus S'_0$. Thus g extends holomorphically to $\widehat{\boldsymbol{X}}$; a contradiction.

STEP 3. *Let S'_0 be an irreducible component of S_0. Then there exists an open set $\Omega \subset \widehat{\boldsymbol{X}}$ such that $\emptyset \neq S'_0 \cap \Omega \subset S$ (and consequently, by the classical identity principle, $S'_0 \subset S$).*

Indeed, for every point $a = (a_1, \dots, a_N) \in S'_0$ there exist an open neighborhood U_a and a defining function $g_a \in \mathcal{O}(U_a)$ for $S'_0 \cap U_a$ (cf. [Chi 1989, §2.9]). We may assume that $U_a = U^1_{a_1} \times \dots \times U^N_{a_N}$, where $U^j_{a_j} \Subset D_j$ is a univalent neighborhood of a_j, $j = 1, \dots, N$. Using the Lindelöf theorem, we find a countable set $I \subset S'_0$ such that $S'_0 \subset \bigcup_{a \in I} U_a$. Let

$$C_{j,a} = (\mathrm{pr}_{D'_j \times D''_j}(S'_0 \cap U_a)) \cap ((A'_j \times A''_j) \setminus \Sigma_j), \quad j = 1, \dots, N,\ a \in I.$$

Suppose that all the sets $C_{j,a}$ are pluripolar. Put $\Sigma'_j := \Sigma_j \cup \bigcup_{a\in I} C_{j,a}$. Then Σ'_j is pluripolar, $j = 1,\dots,N$. Let $\boldsymbol{T}' := \mathbb{T}((D_j, A_j, \Sigma'_j)_{j=1}^N)$. Observe that $S'_0 \cap \boldsymbol{T}' = \emptyset$, which contradicts Step 2. Thus there exists a pair (j,a) such that $C_{j,a}$ is not pluripolar.

Consequently, the proof is reduced to the following lemma.

LEMMA 2.1. *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be domains. Let $S'_0 \subset D \times G$ be an irreducible analytic set of pure codimension one. Let $g \in \mathcal{O}(D \times G)$ be a defining function for S'_0 and let $S := \{(z,w) \in D \times G : h_1(z,w) = \dots = h_k(z,w) = 0\}$, where $h_1,\dots,h_k \in \mathcal{O}(D \times G)$. Assume that there exists a non-pluripolar set $A \subset \operatorname{pr}_D S'_0$ such that $S'_0 \cap (A \times G) \subset S$. Then there exists an open set $\Omega \subset D \times G$ such that $S'_0 \cap \Omega \subset S$.*

Proof. Let $V := \{z \in D : g(z,\cdot) \equiv 0\}$. Then $V \subsetneq D$ is an analytic set. Hence $A_0 := A \setminus V$ is not pluripolar. Fix a pluriregular point $a_0 \in A_0$ and let $(a_0, b_0) \in S'_0$. Write $b_0 = (b'_0, b_{0,q})$. Using a biholomorphic mapping of the form $\mathbb{C}^p \times \mathbb{C}^q \ni (z,w) \mapsto (z, b_0 + \Phi(w - b_0)) \in \mathbb{C}^p \times \mathbb{C}^q$, where Φ is a suitable unitary transformation, one can easily reduce the problem to the case where $g(a_0, b'_0, \cdot) \not\equiv 0$ in a neighborhood of $b_{0,q}$. Consequently, there exist neighborhoods P of (a_0, b'_0) and Q of $b_{0,q}$ such that $P \times Q \subset D \times G$ and the projection $\operatorname{pr}_P : S'_0 \cap (P \times Q) \to P$ is proper. Thus, there exists an analytic set $\Delta \subsetneq P$ such that

$$\operatorname{pr}_P : (S'_0 \cap (P \times Q)) \setminus \operatorname{pr}_P^{-1}(\Delta) \to P \setminus \Delta$$

is an analytic covering. Observe that the set $B := (A_0 \times \mathbb{C}^{q-1}) \cap (P \setminus \Delta)$ is not pluripolar. Fix a pluriregular point $c \in B$. Then there exists an open set $\Omega \subset P \times Q$, an open connected neighborhood $W \subset P$ of c, and a holomorphic function $\varphi : W \to Q$ such that $S'_0 \cap \Omega = \{(z, w', \varphi(z,w')) : (z,w') \in W\}$. In particular, $h_j(z, w', \varphi(z,w')) = 0$, $(z,w') \in B \cap W$. Since $B \cap W$ is not pluripolar, we conclude that $h_j(z, w', \varphi(z,w')) = 0$, $(z,w') \in W$, $j = 1,\dots,k$, which implies that $S'_0 \cap \Omega \subset S$. ∎

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