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THE COMPOSITE OF IRREDUCIBLE MORPHISMS IN REGULAR COMPONENTS

BY

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Abstract. We study when the composite of n irreducible morphisms between modules in a regular component of the Auslander–Reiten quiver is non-zero and lies in the $n+1$-th power of the radical $\Re$ of the module category. We prove that in this case such a composite belongs to $\Re^\infty$. We apply these results to characterize those string algebras having n irreducible morphisms between band modules such that their composite is a non-zero morphism in $\Re^{n+1}$.

The composite of n irreducible morphisms between indecomposable modules over a finite-dimensional k-algebra A, with k an algebraically closed field, belongs to $\Re^n$, the n-th power of the radical $\Re$ of the module category. In general such a composite could be a non-zero morphism in $\Re^{n+1}$ (see for instance [5]–[7]). We are interested to know when the composite of n irreducible morphisms is a non-zero morphism in $\Re^{n+1}$. The purpose of this work is to study this problem for irreducible morphisms between indecomposable modules in regular components of the Auslander–Reiten quiver Γ_A of $\operatorname{mod} A$ of type ZA_∞ or stable tubes.

The case where the irreducible morphisms belong to a left (or right) almost sectional path was considered in [6]. Here, we will consider morphisms in stable tubes or components of type ZA_∞. We will prove that the existence of n irreducible morphisms with composite in $\Re^{n+1}$ implies the existence of n such irreducible morphisms in a left or in a right almost sectional path. Precisely, we prove:

THEOREM A. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a stable tube or a component of Γ_A of type ZA_∞. Then the following conditions are equivalent:*

(a) *There are an integer $n > 1$ and irreducible morphisms $h_1, \dots, h_n$ between modules in Γ such that $h_n \dots h_1 \neq 0$ and $h_n \dots h_1 \in \Re^{n+1}$.*

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(b) *There are an integer* $t \geq 1$ *and a simple regular module* W *in* Γ *such that* $\Re(W, \tau^{-t}W) \neq 0$.

(c) *There are an integer* $m > 1$ *and irreducible morphisms* $f_1, \ldots, f_m$ *over a left (right) almost sectional path starting (ending) at a simple regular module* W, *such that* $0 \neq f_m \ldots f_1 \in \Re^{m+1}$.

In Section 3, we will apply the above result to characterize string algebras which contain n irreducible morphisms between indecomposable band modules whose composite is a non-zero morphism in $\Re^{n+1}$.

We will prove that if the composite of n irreducible morphisms belongs to a power of the radical greater than n, then it is in the infinite radical. Actually, we prove the following theorem:

THEOREM B. *Let* A *be a finite-dimensional algebra over an algebraically closed field and* Γ *a component of* Γ_A *of type* ZA_∞ *or a stable tube. Let* $h_i : X_i \to X_{i+1}$ *be irreducible morphisms with* $X_i \in \Gamma$ *for* $i = 1, \ldots, n+1$. *Then* $0 \neq h_n \ldots h_1 \in \Re^{n+1}(X_1, X_{n+1})$ *if and only if* $0 \neq h_n \ldots h_1 \in \Re^{\infty}(X_1, X_{n+1})$.

The proof of Theorem B is given in Section 2. The notion of degree of an irreducible morphism, introduced by Liu in [11], is a very useful tool in the study of this problem, as shown in [5]–[7].

Recently a general solution to the problem of finding necessary and sufficient conditions for the existence of n irreducible morphisms with non-zero composite in $\Re^{n+1}$ is given in [8], where the results are proven using covering techniques.

Here, we use a different technique, namely the concept of degree of an irreducible morphism. We think that this notion could help to solve the problem in the more general context of artin algebras.

1. Preliminaries. Throughout this paper, all algebras are finite-dimensional algebras over an algebraically closed field. For such an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated left A-modules, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ consisting of one representative of each isomorphism class of indecomposable A-modules.

Let X be an indecomposable A-module. If X is not projective, we denote by $\alpha(X)$ the number of indecomposable summands of the middle term of the almost split sequence ending at X. Dually, if X is not injective, we denote by $\alpha'(X)$ the number of indecomposable summands of the middle term of the almost split sequence starting at X.

We denote the radical of the module category $\operatorname{mod} A$ by $\Re_A$, or just by $\Re$. We recall that, for $X, Y \in \operatorname{mod} A$, we denote by $\Re_A(X, Y)$ the set of all the morphisms $f : X \to Y$ such that, for all $M \in \operatorname{ind} A$, each $h : M \to X$ and each $h' : Y \to M$ the composite $h'fh$ is not an isomorphism. In particular, when $X, Y \in \operatorname{ind} A$, then $\Re_A(X, Y)$ is the set of all the morphisms f :

$X \to Y$ which are not isomorphisms. Inductively, the powers of $\Re_A(X, Y)$ are defined. By $\Re_A^\infty(X, Y)$ we denote the intersection of all powers $\Re_A^i(X, Y)$, $i \geq 1$, of $\Re_A(X, Y)$.

We denote by Γ_A the Auslander–Reiten quiver of A and by τ the Auslander–Reiten translation DTr.

An arrow $\alpha : M \to N$ has *valuation* (a, b) if there is a minimal right almost split morphism $aM \oplus X \to N$ where M is not a summand of X, and a minimal left almost split morphism $M \to bN \oplus Y$ where N is not a summand of Y. If $a = b = 1$ then we say that the arrow α has *trivial valuation.* A component Γ of Γ_A is said to have trivial valuation if all the arrows in Γ have trivial valuation.

A component Γ of Γ_A is said to satisfy the condition that $\alpha(\Gamma) \leq 2$ if $\alpha(X) \leq 2$ for every X in Γ.

For unexplained notions from representation theory we refer the reader to [2, 3, 13].

We recall the definition of left almost sectional path, given in [6]. Let $\xi : X_1 \to X_2 \to \cdots \to X_n \to X_{n+1}$ be a non-sectional path in Γ_A of length $n \geq 2$. We say that ξ is *a left almost sectional path* provided $X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n$ is sectional. Observe that if $\xi : X_1 \to X_2 \to \cdots \to X_n \to X_{n+1}$ is a left almost sectional path, then $\tau X_{n+1} \simeq X_{n-1}$.

Let $X \to Z$ be an arrow in a component Γ of Γ_A. Any path $X = Y_0 \to Y_1 \to \cdots \to Y_n = Z$ in Γ of length $n \geq 2$ with $Y_1 \neq Y_n$ and $Y_0 \neq Y_{n-1}$ is called a *bypass* of the arrow $X \to Z$. If this path is sectional, then it is called a *sectional bypass.* Otherwise, it is called a *non-sectional bypass* (see [1]).

Let $f : X \to Y$ be an irreducible morphism in $\operatorname{mod} A$, and assume that either X or Y is indecomposable. Following [11], we say that the left degree of f is infinite if for each positive integer n, for each $Z \in \operatorname{mod} A$ and each morphism $g \in \Re^n(Z, X) \setminus \Re^{n+1}(Z, X)$, we have $fg \notin \Re^{n+2}(Z, Y)$. Otherwise, the *left degree* of f is the smallest positive integer m such that there exists a morphism $g \in \Re^m(Z, X) \setminus \Re^{m+1}(Z, X)$, for some $Z \in \operatorname{mod} A$, such that $fg \in \Re^{m+2}(Z, Y)$. We denote the left degree of f by $d_l(f)$.

Dually, one defines the *right degree* $d_r(f)$ of f. We refer the reader to [9, 11, 12] for a detailed account on these degrees.

2. Regular components. In this section, we are going to look for necessary and sufficient conditions for the existence of n irreducible morphisms between modules in regular components of Γ_A of type ZA_∞ or stable tubes, with non-zero composite in $\Re^{n+1}$.

In [6, Proposition 4.3], the case where the irreducible morphisms belong to a left almost sectional path is considered. Precisely, the following result is proven:

Proposition 2.1 ([6, Proposition 4.3]). *Let A be a finite-dimensional algebra over an algebraically closed field. Let Γ be a component of Γ_A of type ZA_∞ or a stable tube. For a natural number l the following conditions are equivalent:*

(a) *There exist irreducible morphisms $f_1, \ldots, f_{l+1}$ over a left almost sectional path such that $f_{l+1} \ldots f_2 \notin \Re^{l+1}$ and $0 \neq f_{l+1} \ldots f_1 \in \Re^{l+2}(W, Y)$, where W is a simple regular module and Y is the codomain of f_{l+1}.*

(b) *There exists a simple regular module W such that $\Re(W, \tau^{-r}W) \neq 0$ for some r with $1 \leq r \leq l$.*

We are going to prove that the same characterization holds also when the irreducible morphisms do not belong to an almost sectional path. First, we study some properties of stable tubes and components of Γ_A of type ZA_∞.

For standard components Γ of Γ_A we can always represent arrows of Γ by irreducible morphisms in $\operatorname{mod} A$ satisfying the mesh relations. Although stable tubes are not always standard, and components of type ZA_∞ are never standard, the following result along these lines can be proven:

Proposition 2.2. *Let Γ be a component of Γ_A of type ZA_∞ or a stable tube. Then there exist a configuration of almost split sequences with modules in Γ of the form*

$$
\begin{array}{ccccccccccccccc}
\cdots\bullet & & & & \bullet & & & & \bullet & & & & \bullet & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet\cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots\bullet & & & & \bullet & & & & \bullet & & & & \bullet & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet\cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots\bullet & & & & \bullet & & & & \bullet & & & & \bullet & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet\cdots \\
 & & \vdots & & & & \vdots & & & & \vdots & & & &
\end{array}
$$

Proof. We start by considering vertices $M[i]$ in Γ, for i even, such that the graph

$$\begin{tikzcd}
 & M[i] \arrow[dr, "\beta_i"] & \\
N[i+1] \arrow[ur, "\alpha_i"] \arrow[dr, "\alpha_{i+1}"'] & & \tau^{-1}N[i+1] \\
 & M[i+2] \arrow[ur, "\beta_{i+1}"'] &
\end{tikzcd}$$

is a full and convex subquiver of Γ with $M[0]$ in the border of Γ.

For each arrow α_i we fix an irreducible morphism $f_i : N[i+1] \to M[i]$ in $\operatorname{mod} A$, and we choose a morphism $f_{i+1} : N[i+1] \to M[i+2]$ such that $(f_i, f_{i+1})^t : N[i+1] \to M[i] \oplus M[i+2]$ is a minimal left almost split morphism.

Let $g = (g_i, g_{i+1}) : M[i] \oplus M[i+2] \to \tau^{-1}N[i+1]$ be the cokernel of f_i, so that we have an almost split sequence

$$0 \to N[i+1] \xrightarrow{(f_i, f_{i+1})^t} M[i] \oplus M[i+2] \xrightarrow{(g_i, g_{i+1})} \tau^{-1}N[i+1] \to 0$$

for each $i = 0, 2, 4, \dots$.

In order to iterate this procedure we only need (g_{i+1}, g_{i+2}) to be a minimal left almost split morphism.

So far, we just know that both g_{i+1} and g_{i+2} are irreducible morphisms. Since Γ has trivial valuation, we obtain the desired result from our next lemma. ∎

Lemma 2.3. *Let $(f, g)^t : A \to B_1 \oplus B_2$ be a left minimal almost split morphism with B_1, B_2 indecomposable non-isomorphic A-modules, and let $\alpha \in k^*$ and $\mu \in \Re^2(A, B_2)$. Then the irreducible morphism $(f, \alpha g + \mu)^t : A \to B_1 \oplus B_2$ is also left minimal almost split.*

Proof. We may assume that $\alpha = 1$ and $\mu \neq 0$. Since $(f, g)^t : A \to B_1 \oplus B_2$ is a left minimal almost split morphism and $\mu : A \to B_2$ is not an isomorphism, there is a morphism $h : B_1 \oplus B_2 \to B_2$ such that $\mu = h\,(f, g)^t$. We write $h = (h_1, h_2)$, with $h_i : B_i \to B_2$ for $i = 1, 2$. Then $\mu = h_1 f + h_2 g$. We claim that $h_2 \in \Re(B_2, B_2)$. In fact, since $h_1 : B_1 \to B_2$ is not an isomorphism, we have $h_2 g = \mu - h_1 f \in \Re^2(A, B_2)$. Thus h_2 is not an isomorphism, because g is irreducible. Hence $\mathrm{id} + h_2 : B_2 \to B_2$ is an isomorphism.

Let

$$t = \begin{pmatrix} \mathrm{id} & 0 \\ h_1 & \mathrm{id} + h_2 \end{pmatrix} : B_1 \oplus B_2 \to B_1 \oplus B_2.$$

Then $(f, g + \mu)^t = t\,(f, g)^t$. One can easily verify that t is an isomorphism, with inverse

$$\begin{pmatrix} \mathrm{id} & 0 \\ -(\mathrm{id}+h_2)^{-1}h_1 & (\mathrm{id}+h_2)^{-1} \end{pmatrix}.$$

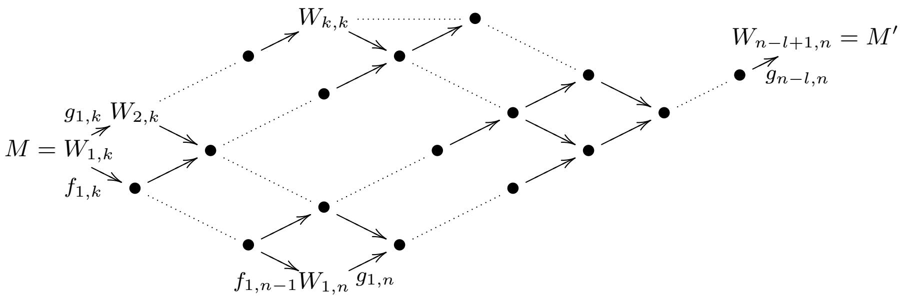
From this and the fact that $(f,g)^t$ is left minimal almost split it follows that $(f, g+\mu)^t$ is also left minimal almost split, proving the desired result. ■

Given a path $\nu : f_n \dots f_1 : X_1 \to X_{n+1}$ with each $f_i : X_i \to X_{i+1}$ irreducible, we say that $f_n, \dots, f_1$ *satisfy the mesh relations*, or that ν *is a path in the mesh*, if $f_n \dots f_1$ is a path of irreducible morphisms in the configuration of Proposition 2.2. Thus, morphisms in two paths in the mesh from X_1 to X_{n+1} have the same composite.

Now, we will prove some useful lemmas.

LEMMA 2.4. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a component of Γ_A of type ZA_∞ or a stable tube. Let $M, M' \in \Gamma$ be such that there is a path of irreducible morphisms in the mesh of length m from M to M' with zero composite. If $\Re^{m+1}(M,M') \neq 0$ then there are a simple regular module $W \in \Gamma$ and a positive integer r such that $\Re(W, \tau^{-r}W) \neq 0$.*

Proof. Let γ be a path of irreducible morphisms of length m from M to M' with zero composite. Then we can assume that there is a configuration of almost split sequences as follows:



with $n \geq 2$, and having at least an almost split sequence with indecomposable middle term. We will prove the result by induction on the sum $k+l$, assuming that M is in position $(1,k)$ of a ray and M' is in position $(n-l+1, n)$ of a co-ray in Γ. We denote such a sum by $n(\gamma)$.

If $n(\gamma) = 2$ then $k = l = 1$ and the modules M and M' are simple regular. Hence $M' = \tau^{-1}M$ and $0 \neq \Re^{m+1}(M,M') \subset \Re(M,M')$, proving the result in this case.

Assume that $n(\gamma) > 2$ and that the result holds for paths δ satisfying the hypothesis of the lemma, with $n(\delta) < n(\gamma)$. Consider $k > 1$, $f_{1,k-1} : W_{1,k-1} \to W_{1,k}$ an irreducible morphism and $0 \neq \varphi \in \Re^{m+1}(M,M')$.

If $\theta = \varphi f_{1,k-1} \neq 0$ then $\theta \in \Re^{m+2}(W_{1,k-1}, M')$ and $n(\gamma f_{1,k-1}) = k-1+l < n(\gamma)$. Since $\gamma f_{1,k-1} : W_{1,k-1} \to M'$ is a path of irreducible morphisms of length $m+1$ with zero composite, by the inductive hypothesis there is a simple regular module satisfying the result.

Now, assume that $\theta = \varphi f_{1,k-1} = 0$. Then the almost split sequence starting at $W_{1,k-1}$ is

$$0 \to W_{1,k-1} \xrightarrow{f_{1,k-1}} W_{1,k} \xrightarrow{g_{1,k}} W_{2,k} \to 0$$

if $k = 2$, and

$$0 \to W_{1,k-1} \xrightarrow{(f_{1,k-1}, g_{1,k-1})^t} W_{1,k} \oplus W_{2,k-1} \xrightarrow{(g_{1,k}, f_{2,k-1})} W_{2,k} \to 0$$

if $k > 2$. In the first case $\varphi f_{1,k-1} = 0$, and in the second case

$$(\varphi, 0)(f_{1,k-1}, g_{1,k-1})^t = 0.$$

In either case, there is a morphism $\varphi_1 : W_{2,k} \to M$ such that $0 \neq \varphi = \varphi_1 g_{1,k}$.

On the other hand, since $g_{1,k}$ belongs to a co-ray, we deduce from [9, Proposition 4.11] that $d_r(g_{1,k}) = \infty$. We conclude that $0 \neq \varphi_1 \in \Re^m(W_{2,k}, M')$, since $\varphi \in \Re^{m+1}(M, M')$.

Finally, we have to prove that there is a path of $m-1$ irreducible morphisms from $W_{2,k}$ to M' with zero composite. Since the path γ from M to M' has zero composite, it is not sectional (see [10, Appendix]). Furthermore, since $\alpha(\Gamma) \leq 2$, we may assume that $\gamma = \gamma' g_{1,k}$ where $\gamma' : W_{2,k} \to M'$ is a path of $m-1$ irreducible morphisms. Moreover, $\gamma' = 0$, because $g_{1,k}$ is an epimorphism. Since $n(\gamma') = k-1+l < n(\gamma)$ we can apply the inductive hypothesis to γ' and conclude that there is a simple regular module satisfying the statement, proving the result in the case $k > 1$.

If $k = 1$ and $l > 1$ the proof is analogous, analyzing separately the cases where $g_{n-l+1,n}\varphi = 0$ and $g_{n-l+1,n}\varphi \neq 0$. ■

The next result is essential for our considerations. We are going to prove that the existence of n irreducible morphisms with composite in $\Re^{n+1}$ implies the existence of a path of n irreducible morphisms through the same modules with zero composite. Even though this result follows from [8, Proposition 5.1], we will give an easier proof for our particular case, using the notion of degree of an irreducible morphism.

Lemma 2.5. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a stable tube or a component of type ZA_∞. Let $h_i : X_i \to X_{i+1}$ be irreducible morphisms for $i = 1, \ldots, n$, with $X_i \in \Gamma$ for $i = 1, \ldots, n+1$. If $0 \neq h_n \ldots h_1 \in \Re^{n+1}$ then $f_n \ldots f_1 = 0$ for any choice of irreducible morphisms $f_i : X_i \to X_{i+1}$ satisfying the mesh relations.*

Proof. Consider irreducible morphisms $f_i : X_i \to X_{i+1}$ satisfying the mesh relations of Γ. Since k is an algebraically closed field, we have $h_i = \alpha_i f_i + \mu_i$ with $\alpha_i \in k^*$ and $\mu_i \in \Re^2(X_i, X_{i+1})$ for each i. Then $h_n \dots h_1 = \alpha f_n \dots f_1 + \mu$ with $\alpha \in k^*$ and $\mu \in \Re^{n+1}(X_1, X_{n+1})$. Therefore, $f_n \dots f_1 \in \Re^{n+1}(X_1, X_{n+1})$. We assume that $f_n \dots f_1 \neq 0$. Then, modulo the mesh relations, $f_n \dots f_1$ is equal to a path of irreducible morphisms $g_n \dots g_{r+1} g_r \dots g_1$, with $g_r, \dots, g_1$ in a co-ray of Γ and $g_n, \dots, g_{r+1}$ in a ray. Since $g_r, \dots, g_1$ belong to a co-ray, their right degree is ∞, by [9, Proposition 4.11].

Since $g_n \dots g_{r+1} g_r \dots g_1 \in \Re^{n+1}(X_1, X_{n+1})$ and $d_r(g_1) = \infty$, we deduce that $g_n \dots g_{r+1} g_r \dots g_2 \in \Re^n$. Since $g_2, \dots, g_r$ also have infinite right degree we can iterate this argument and conclude that $g_n \dots g_{r+1} \in \Re^{n+1-r}$, contradicting the fact that the morphisms $g_n, \dots, g_{r+1}$ belong to a sectional path, since they belong to the same ray (see [10, Appendix]). Then $f_n \dots f_1 = 0$, proving the implication. ∎

Now, we are in a position to prove one of the main results of this section, extending the characterization given in [6, Proposition 4.3].

THEOREM 2.6. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a stable tube or a component of Γ_A of type ZA_∞. Then the following conditions are equivalent:*

- (a) *There are an integer $n > 1$ and irreducible morphisms $h_1, \dots, h_n$ between modules in Γ such that $h_n \dots h_1 \neq 0$ and $h_n \dots h_1 \in \Re^{n+1}$.*
- (b) *There are an integer $t \geq 1$ and a simple regular module W in Γ such that $\Re(W, \tau^{-t} W) \neq 0$.*
- (c) *There are an integer $m > 1$ and irreducible morphisms $f_1, \dots, f_m$ over a left (right) almost sectional path starting (ending) at a simple regular module W such that $0 \neq f_m \dots f_1 \in \Re^{m+1}$.*

Proof. Suppose that statement (a) holds. By Lemma 2.5, it follows that there are irreducible morphisms $f_i : Y_i \to Y_{i+1}$ with composite $f_n \dots f_1 = 0$. Then (b) follows from Lemma 2.4.

We get the converse by considering $n = l + 1$ in [6, Proposition 4.3].

Finally, (b) implies (c) by [6, Proposition 4.3] and clearly (c) implies (a). ∎

Applying the above theorem we get the following corollary for homogeneous tubes.

COROLLARY 2.7. *Let A be a finite-dimensional algebra over an algebraically closed field. Let Γ be a homogeneous tube and W the simple regular module in Γ. The following conditions are equivalent:*

- (a) *There are an integer $n > 1$ and irreducible morphisms $h_1, \dots, h_n$ between modules in Γ such that $h_n \dots h_1 \neq 0$ and $h_n \dots h_1 \in \Re^{n+1}$.*
- (b) $\Re(W, W) \neq 0$.

Next, we will prove that a composite of irreducible morphisms $h_1, \ldots, h_n$ between modules in components of type ZA_∞ or in stable tubes is in $\Re^{n+1}$ if and only if $h_n \ldots h_1 \in \Re^\infty$. We start with a useful technical lemma.

LEMMA 2.8. *Let Γ be a stable tube or a component of type ZA_∞. Suppose there is a zero path in the mesh $\eta : X \to Y$ of length n. Then:*

(a) *Each path in the mesh from X to Y of length greater than or equal to n is zero.*
(b) *Each path of irreducible morphisms from X to Y of length n is in $\Re^{n+1}$.*

Proof. (a) is an easy consequence of the fact that paths in the mesh from X to Y of the same length coincide in $\operatorname{mod} A$.

(b) Let η be the zero path $f_n \ldots f_1$, where $f_i : X_{i-1} \to X_i$ satisfy the mesh relations, let $f'_i : X_{i-1} \to X_i$ be irreducible morphisms for $i = 1, \ldots, n$, and set $X = X_0$ and $Y = X_n$. Since k is an algebraically closed field, we have $f'_i = \alpha_i f_i + \mu_i$ with $\alpha_i \in k^*$ and $\mu_i \in \Re^2(X_{i-1}, X_i)$. Hence

$$f'_n \ldots f'_1 = \alpha_n \ldots \alpha_1 f_n \ldots f_1 + \sum_{i=1}^{t} \theta_i$$

where $\theta_i : X \to Y$ is a composite of n morphisms in $\{f_1, \ldots, f_n, \mu_1, \ldots \mu_n\}$ with at least one of them in $\{\mu_1, \ldots \mu_n\}$. Therefore, each $\theta_i \in \Re^{n+1}(X, Y)$ and we conclude that $f'_n \ldots f'_1 \in \Re^{n+1}(X, Y)$. ∎

THEOREM 2.9. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ be a component of Γ_A of type ZA_∞ or a stable tube. If there is a zero path in the mesh from X to Y then any path of irreducible morphisms from X to Y of length n is in $\Re^\infty(X, Y)$.*

Proof. Consider irreducible morphisms $f_i : X_{i-1} \to X_i$ and $f'^{i} : X_{i-1} \to X_i$ between indecomposable modules with $i = 1, \ldots, n$, $X = X_0$, $Y = X_n$. Assume that $f_n \ldots f_1$ is a zero path in the mesh, and that $f'_n \ldots f'_1 \in \Re^{n+k}(X_0, X_n) \setminus \Re^{n+k+1}(X_0, X_n)$ for some $k \geq 0$. Then, by Lemma 2.8(b), we know that $k \geq 1$. Therefore, by [3, V, Proposition 7.4] there is a path $\delta : X_0 \to X_n$ of $n + k$ irreducible morphisms whose composite does not belong to $\Re^{n+k+1}(X_0, X_n)$. Then there is also a path in the mesh $\eta : X_0 \to X_n$ of length $n + k$ and by Lemma 2.8(a) we know that $\eta = 0$. Applying now Lemma 2.8(b) to the path η we conclude that $\delta \in \Re^{n+k+1}(X_0, X_n)$, a contradiction to our assumption. ∎

The following corollaries are immediate consequences of the above theorem.

COROLLARY 2.10. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a component of Γ_A of type ZA_∞ or a stable* *tube. Let $h_i : X_i \to X_{i+1}$ be n irreducible morphisms with $X_i \in \Gamma$ for $i=1,\dots,n$. Then $0 \neq h_n \dots h_1 \in \Re^{n+1}(X_1, X_{n+1})$ if and only if $0 \neq h_n \dots h_1 \in \Re^{\infty}(X_1, X_{n+1})$.*

Proof. Assume that there are n irreducible morphisms $h_i : X_i \to X_{i+1}$ such that $0 \neq h_n \dots h_1 \in \Re^{n+1}(X_1, X_{n+1})$ and let $f_i : X_i \to X_{i+1}$ be irreducible morphisms satisfying the mesh relations. Then $f_n \dots f_1 = 0$, and the result follows from the above theorem.

The converse is clear. ∎

Corollary 2.11. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a component of Γ_A of type ZA_∞ or a stable tube. Given an integer $n > 1$, there are no irreducible morphisms $f_1, \dots, f_n$ between modules in Γ such that $f_n \dots f_1 \in \Re^{n+1} \setminus \Re^{n+2}$.*

Corollary 2.12. *Let A be a finite-dimensional algebra over an algebraically closed field and Γ a standard stable tube. Then the composite $f_n \dots f_1$ of n irreducible morphisms between modules in Γ belongs to $\Re^{n+1}$ if and only if $f_n \dots f_1 = 0$.*

In [6], the composite of n irreducible morphisms in the border of a wing was studied, characterizing when such a composite belongs to the $n+1$-th power of the radical. In this section, we are going to apply these results in the particular case when the wing lies in a component of type ZA_∞ or a stable tube.

We recall the definition of a wing in a connected component Γ of Γ_A (see [13], p. 127). A *wing* is a configuration of almost split sequences of the form

$$\begin{tikzcd}[column sep=small]
W_{1,1} \arrow[dr] & \cdots & W_{2,2} \arrow[dr] & & \cdots & & W_{n-1,n-1} \arrow[dr] & & W_{n,n} \\
& W_{1,2} \arrow[ur] \arrow[dr] & & W_{2,3} & & W_{3,n-1} \arrow[dr] & & W_{n-1,n} \arrow[ur] & \\
& & W_{1,3} \arrow[ur] & & W_{2,n-1} \arrow[ur] \arrow[dr] & & W_{3,n} & & \\
& & & W_{1,n-1} \arrow[ur] \arrow[dr] & & W_{2,n} \arrow[ur] & & & \\
& & & & W_{1,n} \arrow[ur] & & & &
\end{tikzcd}$$

where $\alpha(W_{i,i}) = 1$ for $i = 2, \dots, n$ and $\alpha'(W_{1,1}) = 1$.

The paths

$$f : W_{1,1} \xrightarrow{f_{1,1}} W_{1,2} \xrightarrow{f_{1,2}} W_{1,3} \to \cdots \to W_{1,n-1} \xrightarrow{f_{1,n-1}} W_{1,n}$$

and

$$g : W_{1,n} \xrightarrow{g_{1,n}} W_{2,n} \xrightarrow{g_{2,n}} W_{3,n} \to \cdots \to W_{n-1,n} \xrightarrow{g_{n-1,n}} W_{n,n}$$

corresponding respectively to the sectional path starting at $W_{1,1}$ and ending at $W_{1,n}$, and to the sectional path starting at $W_{1,n}$ and ending at $W_{n,n}$, are called the *borders* of the wing.

PROPOSITION 2.13. *Let A be a finite-dimensional algebra over an algebraically closed field. Consider a wing in a component Γ of type ZA_∞ or in a stable tube. Let*

$$f : W_{1,1} \xrightarrow{f_{1,1}} W_{1,2} \xrightarrow{f_{1,2}} W_{1,3} \to \cdots \to W_{1,n-1} \xrightarrow{f_{1,n-1}} W_{1,n}$$

and

$$g : W_{1,n} \xrightarrow{g_{1,n}} W_{2,n} \xrightarrow{g_{2,n}} W_{3,n} \to \cdots \to W_{n-1,n} \xrightarrow{g_{n-1,n}} W_{n,n}$$

be the borders of the wing, corresponding to the ray starting at $W_{1,1}$ and to the co-ray ending at $W_{n,n}$, respectively. The following conditions are equivalent:

(a) *There exist irreducible morphisms $h_{1,j} : W_{1,j} \to W_{1,j+1}$ and $h'_{j,n} : W_{j,n} \to W_{j+1,n}$, for $j = 1, \ldots, n-1$, such that*
$$0 \neq h'_{n-1,n} \ldots h'_{1,n} h_{1,n-1} \ldots h_{1,1} \in \Re^{2n-1}(W_{1,1}, W_{n,n}).$$
(b) $\Re(W_{1,1}, W_{n,n}) \neq 0$.
(c) *There exists a non-zero morphism $\varphi \in \Re^2(W_{1,n}, W_{1,n})$ such that $g\varphi f \neq 0$ and $g\varphi f \in \Re^\infty(W_{1,1}, W_{n,n})$.*

Proof. (a)⇒(b) is trivial.

(b)⇒(c). By [6, Proposition 4.2], we know that there exists a non-zero morphism $\varphi \in \Re^2(W_{1,n}, W_{1,n})$ such that $g\varphi f \neq 0$. Since $gf = 0$ and $\varphi \in \Re^2(W_{1,n}, W_{1,n})$, Lemma 2.8(b) yields $0 \neq g\varphi f \in \Re^\infty(W_{1,1}, W_{n,n})$.

Finally we prove (c)⇒(a). Define the irreducible morphisms $h_{1,j} : W_{1,j} \to W_{1,j+1}$ and $h'_{i,n} : W_{i,n} \to W_{i+1,n}$ as follows: $h_{1,j} = f_{1,j}$ for $j = 1, \ldots, n-2$, $h_{1,n-1} = \varphi f_{1,n-1}$ and $h'_{j,n} = g_{j,n}$ for $j = 1, \ldots, n-1$. Then their composite is non-zero and belongs to $\Re^\infty(W_{1,1}, W_{n,n})$. ■

3. String algebras. In this section, we will apply the results proven in Section 2 to study when the composite of n irreducible morphisms between band modules over a string algebra is a non-zero morphism in $\Re^{n+1}$. First, we recall some necessary notions.

A finite-dimensional algebra $A = (kQ_A)/I_A$ is called a *string algebra* provided:

(S1) Any vertex of Q_A is the starting point of at most two arrows.
(S1′) Any vertex of Q_A is the ending point of at most two arrows.
(S2) Given an arrow β of Q_A, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I_A$.
(S2′) Given an arrow β of Q_A, there is at most one arrow γ with $e(\beta) = s(\gamma)$ and $\gamma\beta \notin I_A$.
(S3) The ideal I_A is generated by a set of paths of Q_A.

3.1. Band modules. Let $A = (kQ_A)/I_A$ be a string algebra. Given a walk ω in Q_A, when we write $w = c_n \dots c_1$ we will always assume that c_i is an arrow or the inverse of an arrow, for $i = 1, \dots, n$.

We say that w is a *reduced walk* if it is a trivial path or if $w = c_n \dots c_1$ with $c_{i+1} \neq c_i^{-1}$ for $i = 1, \dots, n-1$.

A *string* in A is either a trivial path ϵ_v, $v \in Q_0$, or a reduced walk $C = c_n \dots c_1$ of length $n \geq 1$ such that no subwalk $c_{i+t} \cdots c_{i+1} c_i$ or its inverse belongs to I_A.

Let W' be the set of all strings C such that for each positive integer m, C^m is a reduced string and C cannot be written as a power of a string of shorter length. The strings in W' are called *reduced cycles*. We observe that W' is empty if there are no cycles in Q_A.

Let I, J be finite sets, and X_i, Y_j be k-vector spaces for $i \in I$, $j \in J$. Let $h_{ij} : X_i \to Y_j$ be morphisms for (i, j) in a subset D of $I \times J$. Consider the morphism $F = (f_{ij}) : \bigoplus_{i \in I} X_i \to \bigoplus_{j \in J} Y_j$ defined as follows:

$$f_{ij} = \begin{cases} h_{ij} & \text{if } (i, j) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We will say that F is the *direct sum* of the morphisms h_{ij}.

Let $V = k^d$ and let $\varphi \in \operatorname{End} V$ be given by the Jordan block $J_{d,\lambda}$, where $\lambda \in k^*$. To a fixed string $C = c_n \dots c_1 \in W'$ and the morphism φ, we can assign an indecomposable module $M(C, \varphi)$, called a *band module*, as follows:

For $i = 1, \dots, n$ we define $V(i) = V$. For $i = 1, \dots, n-1$, let f_{c_i} be the identity map from $V(i)$ to $V(i+1)$ if c_i is an arrow, and the identity map from $V(i+1)$ to $V(i)$ otherwise. Let f_{c_n} be the k-linear map that sends $x \in V(n)$ to $\varphi(x) \in V(1)$ or $x \in V(1)$ to $\varphi^{-1}(x) \in V(n)$ according as c_n is an arrow or the inverse of an arrow, respectively.

The band module $M(C, \varphi)$ is defined as follows: Let v be a vertex in Q_A. If v is a vertex of one of the arrows c_i or c_i^{-1} involved in C, let $M(C, \varphi)_v$ be the direct sum of vector spaces $V(i)$ such that $s(c_i) = v$. Otherwise $M(C, \varphi)_v = 0$.

For an arrow α in Q_A, if α appears in C then we define $M(C, \varphi)_\alpha$ as the direct sum of the maps f_{c_i} such that $c_i = \alpha$ or $c_i = \alpha^{-1}$; in any other case we define $M(C, \varphi)_\alpha$ to be the zero map.

In this way we obtain a representation $M(C, \varphi)$ of Q_A that satisfies the relations in I_A. We refer to [4] for more details on this construction.

We recall from [4] that almost all the components of the Auslander–Reiten quiver Γ_A of a string algebra A, except for a finite number of them, are of type ZA_∞^∞ or $ZA_\infty/\langle\tau\rangle$. Moreover, all the components of Γ_A having band modules are homogeneous tubes and those without border and having string modules are regular components of type ZA_∞^∞. In particular, band modules are in homogeneous tubes. Also the simple regular modules correspond to

$V = k$ with φ the left multiplication by λ. In this case we denote $M(C, \varphi) = M(C, \lambda)$.

Given a string $C = c_n \dots c_1$ in W' consider a substring δ of C in one of the three situations illustrated below:

$$- \cdots - \xrightarrow{\alpha} \delta \xleftarrow{\beta} - \cdots -, \qquad \delta \xleftarrow{\beta} - \cdots - \xrightarrow{\alpha} \quad \text{or} \quad \xleftarrow{\beta} - \cdots - \xrightarrow{\alpha} \delta,$$

where α and β are arrows, that is, such that $\beta^{-1}\delta\alpha$ is a substring of one of the strings $c_n \dots c_1$, $c_1 c_n \dots c_2$ or $c_{n-1} \dots c_1 c_2$.

The string module $M(\delta)$ associated to a string δ has been defined in [4], considering the spaces $V(i) = k$ and the morphisms f_{c_i} as the vertices and arrows corresponding to the support of δ. Next, we are going to prove that in this situation $M(\delta)$ is a submodule of the band module $M(C, \lambda)$, and dually that $M(\delta)$ is a quotient of the band module $M(C, \lambda)$ if the direction of the arrows α, β in the above description of δ is reversed.

LEMMA 3.1. *Given a string $C = c_n \dots c_1$ in W' and a substring δ of C, let $C' = c_1 c_n \dots c_2$ and $C'' = c_{n-1} \dots c_1 c_2$. Then:*

(a) *If $\beta^{-1}\delta\alpha$ is a substring of one of the strings C, C' or C'' then $M(\delta)$ is a submodule of $M(C, \lambda)$.*
(b) *If $\beta\delta\alpha^{-1}$ is a substring of one of the strings C, C' or C'' then $M(\delta)$ is a quotient of $M(C, \lambda)$.*

In particular, if δ is the trivial string e_i then the simple S_i belongs to the socle of $M(C, \lambda)$ in case (a) *and to its top in case* (b).

Proof. We only prove (a) since (b) follows by duality. Let $C = c_n \dots c_1$ and $\delta = c_{r+s} \dots c_r$ with $r \geq 1$, $r + s \leq n$. Let $1 \leq i \leq n$ and $\lambda \in k^*$. Let $V(i) = k$ and f_{c_i} be the vector spaces and the morphisms that define the band module $M(C, \lambda)$, respectively. For $1 \leq i \leq n + 1$, we write

$$W(i) = \begin{cases} k & \text{if } r \leq i \leq r + s + 1, \\ 0 & \text{otherwise} \end{cases} \qquad g_{c_i} = \begin{cases} \mathrm{Id}_k & \text{if } r \leq i \leq r + s, \\ 0 & \text{otherwise.} \end{cases}$$

We will define linear transformations $\varphi_j : W(j) \to V(j)$ for $j = 1, \dots, n$, and $\varphi_{n+1} : W(n + 1) \to V(1)$ such that the diagrams

$$\begin{array}{ccc} V(i) & \xrightarrow{\ f_{c_i}\ } & V(i+1) \\ {\scriptstyle \varphi_i} \uparrow & & \uparrow {\scriptstyle \varphi_{i+1}} \\ W(i) & \xrightarrow{\ g_{c_i}\ } & W(i+1) \end{array} \quad \text{and} \quad \begin{array}{ccc} V(n) & \xrightarrow{\ f_{c_n}\ } & V(1) \\ {\scriptstyle \varphi_n} \uparrow & & \uparrow {\scriptstyle \varphi_{n+1}} \\ W(n) & \xrightarrow{\ g_{c_n}\ } & W(n+1) \end{array}$$

commute, for $i = i, \dots, n - 1$, where the horizontal lines stand for $\to$ or $\leftarrow$ according to the direction of the corresponding morphisms. Then an appropriate direct sum of the morphisms φ_j induces a morphism of representations $M(\delta) \to M(C, \lambda)$, which is injective if all the φ_j are. To define the morphisms φ_j we consider three cases.

Case I: $\beta^{-1}\delta\alpha$ is a substring of C, that is, $1 < r$ and $r + s < n$. Then $\alpha = c_{r-1}$ and $\beta^{-1} = c_{r+s+1}$. Let $1 \leq i \leq n$. We define $\varphi_i : W(i) \to V(i)$ as the identity from k to k if $r \leq i \leq r + s + 1$ and as the zero morphism otherwise.

Let $\varphi_{n+1} : W(n+1) \to V(1)$ be the zero morphism. If $W(i) = W(i+1) = k$ or if $W(i) = W(i+1) = 0$, then the first diagram commutes. The other cases correspond to $i = r - 1$, $i = r + s + 1$. In the first case, $c_i = \alpha$ and both g_{c_i} and φ_i have domain $W(r-1) = 0$. In the second case $c_i = \beta^{-1}$, and then g_{c_i} and φ_{i+1} have domain $W(r + s + 2) = 0$. In any case, both diagrams commute.

Case II: $r + s = n$. Define $\varphi_i : W(i) \to V(i)$ as the identity from k to k if $r \leq i \leq r + s = n$ and as the zero map for $i < r$. Let $\varphi_{n+1} : W(n+1) = k \to V(1) = k$ be multiplication by λ.

By the definition of φ_{n+1} the second diagram commutes, and it is not hard to prove that the first one also commutes.

Case III: $r = 1$. This is analogous to Case I. ∎

Since band modules over a string algebra are in homogeneous tubes, we may apply to them the results of the previous section. Thus in view of Corollary 2.7 to characterize the existence of a non-zero composite of n irreducible morphisms between indecomposable band modules in $\Re^{n+1}$ we just study the existence of a simple regular band module X such that $\Re(X, X) \neq 0$.

Let γ be a cycle of Q_A, not necessarily oriented. We will also denote by γ one of the two walks obtained by going over the cycle γ once, conveniently chosen. We say that $\alpha_s \ldots \alpha_2\alpha_1$ is a *subpath* of γ if it is a path of Q_A whose arrows are arrows of γ.

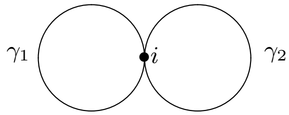
Lemma 3.2. *Let A be a connected string algebra and $X = M(C, \lambda)$ a simple regular band module of a homogeneous tube. If $\Re(X, X) \neq 0$ then Q_A has at least two cycles γ_1 and γ_2, not necessarily oriented, such that the string C goes over both cycles.*

Proof. We observe that since there exists a band module, Q_A has a cycle (not necessarily oriented). Assume that Q_A has a unique cycle. Since $C \in W'$ the string goes over the cycle only once, and therefore the multiplicity of each composition factor of $X = M(C, \lambda)$ is one. Hence $\Re(X, X) = 0$. ∎

Thus, we only need to consider string algebras whose ordinary quiver has at least two cycles γ_1 and γ_2, not necessarily oriented. We start by studying the case when these cycles share a unique vertex, i.

Lemma 3.3. *Let A be a string algebra, where Q_A consists of two cycles γ_1 and γ_2, not necessarily oriented, sharing a unique vertex, i. That is, Q_A*

is of the form



The following conditions are equivalent:

(a) *There is a string C such that $X = M(C,\lambda)$ satisfies $\Re(X,X) \neq 0$ for any $\lambda \neq 0$.*

(b) *One of the following conditions is satisfied:*

(b_1) *At the vertex i of Q_A we have the following situation:*

$$\begin{tikzcd} {} \arrow[dr, "\alpha_n"] & & {} \\ & \bullet i \arrow[ur, "\beta_1"] \arrow[dl, "\alpha_1"] & \\ {} & & {} \arrow[ul, "\beta_r"] \end{tikzcd}$$

where α_1, α_n are arrows of γ_1 and β_1, β_r are arrows of γ_2 such that:

(i) *if $n > 2$, the subpaths of γ_1 are non-zero, except those containing $\alpha_1\alpha_n$,*

(ii) *if $r > 2$, the subpaths of γ_2 are non-zero, except those containing $\beta_1\beta_r$.*

(b_2) *At the vertex i of Q_A we have the following situation:*

$$\begin{tikzcd} {} \arrow[dr, "\alpha_n"] & & {} \\ & \bullet i \arrow[ur, "\beta_1"] \arrow[dr, "\beta_r"] & \\ {} \arrow[ur, "\alpha_1"] & & {} \end{tikzcd}$$

with $I_A = \langle \beta_r\alpha_1, \beta_1\alpha_n \rangle$ or $I_A = \langle \beta_1\alpha_1, \beta_r\alpha_n \rangle$.

Proof. Assume that (a) holds and (b) does not hold. First, suppose that at the vertex i of Q_A we have the following orientation of arrows:

$$\begin{tikzcd} {} \arrow[dr, "\alpha_n"] & & {} \\ & \bullet i \arrow[ur, "\beta_1"] \arrow[dr, "\beta_r"] & \\ {} \arrow[ur, "\alpha_1"] & & {} \end{tikzcd}$$

Then at least three of the relations $\beta_r\alpha_1 = 0$, $\beta_r\alpha_n = 0$, $\beta_1\alpha_n = 0$ or $\beta_1\alpha_1 = 0$ are satisfied because A is a string algebra and (b_2) does not hold. Since (a) holds, there is a band module $X = M(C,\lambda)$ such that $\Re(X,X) \neq 0$, and then by Lemma 3.2 the string C must go over both cycles γ_1 and γ_2. Since the subpaths of C are non-zero, no path of I_A is a subpath of C, since otherwise C^2 would not be a string. So I_A is generated by exactly three of the paths $\beta_i\alpha_j$.

If the relations in (b_1) and (b_2) are not those stated above, then it is not possible to find strings going over both cycles, and by Lemma 3.2 we get $\Re(X,X)=0$, contradicting (a). Thus (b) holds.

Now, we prove the converse. We are going to prove that if (b_1) or (b_2) holds then there is a string $C \in W'$ such that the band module $X = M(C,\lambda)$ satisfies $\Re(X,X) \neq 0$.

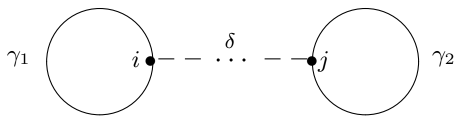
Case I: Assume that (b_1) holds and set $\gamma_1 = \alpha_n\gamma_1'$ and $\gamma_2^{-1} = \gamma_2'\beta_r^{-1}$. Consider the string $C = \gamma_2^{-1}\gamma_1$. Then $C = \gamma_2'\beta_r^{-1}i\alpha_n\gamma_1'$, where i represents the trivial string at the vertex i. By Lemma 3.1 applied to the trivial string i, the simple S_i is in the socle of $X = M(C,\lambda)$.

Now, we write $\gamma_1 = \gamma_1''\alpha_1$ and $\gamma_2^{-1} = \beta_1^{-1}\gamma_2''$. Since $C^2 = \gamma_2^{-1}\gamma_1\gamma_2^{-1}\gamma_1 = \gamma_2^{-1}\gamma_1''\alpha_1 i\beta_1^{-1}\gamma_2''\gamma_1$, we see that S_i is in the top of $M(C,\lambda)$, by applying the same lemma to i. Then S_i is a summand of $\operatorname{top} X$ and of $\operatorname{socle} X$, therefore $\Re(X,X) \neq 0$.

Case II: Suppose that (b_2) holds and that $I_A = \langle \beta_r\alpha_1, \beta_1\alpha_n\rangle$. If we consider the string $C = \gamma_2^{-1}\gamma_2^{-1}\gamma_1\gamma_1$ with $\gamma_1 = \alpha_n\gamma_1'\alpha_1^{-1}$ and $\gamma_2^{-1} = \beta_1^{-1}\gamma_2'\beta_r$, then $C = \beta_1^{-1}\gamma_2'\beta_r i\beta_1^{-1}\gamma_2'\beta_r\alpha_n\gamma_1'\alpha_1^{-1}i\alpha_n\gamma_1'\alpha_1^{-1}$. By Lemma 3.1 the simple S_i is in the socle and in the top of $X = M(C,\lambda)$.

Now, assume that (b_2) holds and that $I_A = \langle \beta_1\alpha_1, \beta_r\alpha_n\rangle$. If we consider the string $C = \gamma_2\gamma_2\gamma_1\gamma_1$, with $\gamma_1 = \alpha_n\gamma_1'\alpha_1^{-1}$ and $\gamma_2 = \beta_r^{-1}\gamma_2'\beta_1$, then $C = \beta_r^{-1}\gamma_2'\beta_1 i\beta_r^{-1}\gamma_2'\beta_1\alpha_n\gamma_1'\alpha_1^{-1}i\alpha_n\gamma_1'\alpha_1^{-1}$ and by Lemma 3.1 the simple S_i is in the socle and in the top of $X = M(C,\lambda)$. Thus, in either case $\Re(X,X) \neq 0$, as desired. ∎

Now, we are going to study algebras such that their quivers have two disjoint cycles, not necessarily oriented, joined by a quiver of type A_n:



More precisely:

Lemma 3.4. *Let A be a string algebra such that Q_A is given by two cycles γ_1 and γ_2, not necessarily oriented, joined by a quiver δ of type A_n having a unique vertex i in common with γ_1 and a unique vertex j in γ_2. Then the following conditions are equivalent:*

(a) *There is a string C such that any band module $X = M(C,\lambda)$ with $\lambda \neq 0$ satisfies $\Re(X,X) \neq 0$.*

(b) *In Q_A we have the following situation:*

$$\begin{tikzcd}[column sep=small]
{} \arrow[dr, "\alpha_n"] & & & & {} \\
& i \arrow[dl, "\alpha_1"'] \arrow[r, dash, "\varepsilon_1"] & \overset{\delta}{\cdots} \arrow[r, dash, "\varepsilon_s"] & j \arrow[ur, "\beta_1"] & \\
{} & & & & {} \arrow[ul, "\beta_r"']
\end{tikzcd}$$

with $I_A = \langle \alpha_1\alpha_n, \beta_1\beta_r \rangle$, where $\alpha_1, \alpha_n \in \gamma_1$ and $\beta_1, \beta_r \in \gamma_2$ are arrows, $\alpha_1 = \alpha_n$ if γ_1 is a loop and $\beta_1 = \beta_r$ if γ_2 is a loop.

Proof. (a)⇒(b). Assume that (a) holds and that (b) does not hold. First, suppose that we have the following orientations in γ_1:

$$(*)\quad \begin{tikzcd}[column sep=small]
{} \arrow[dr, "\alpha_n"] & & \\
& i \arrow[r, dash, "\varepsilon_1"] & {} \\
{} \arrow[ur, "\alpha_1"] & &
\end{tikzcd}
\quad \text{or} \quad (**)\quad
\begin{tikzcd}[column sep=small]
{} & & \\
& i \arrow[ul, "\alpha_n"'] \arrow[dl, "\alpha_1"'] \arrow[r, dash, "\varepsilon_1"] & {} \\
{} & &
\end{tikzcd}$$

We analyze only the case $(*)$, since $(**)$ is dual. Since A is a string algebra, $\varepsilon_1\alpha_n = 0$ or $\varepsilon_1\alpha_1 = 0$. In any case, there is no string $C \in W'$. In fact, by Lemma 3.2 the string C must go over both cycles and then C^2 is not a string.

In the cases where the orientation of γ_2 is

$$\begin{tikzcd}[column sep=small]
& & {} \arrow[dl, "\beta_1"'] \\
{} \arrow[r, dash, "\varepsilon_s"] & j & \\
& & {} \arrow[ul, "\beta_r"]
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}[column sep=small]
& & {} \\
{} \arrow[r, dash, "\varepsilon_s"] & j \arrow[ur, "\beta_1"] \arrow[dr, "\beta_r"'] & \\
& & {}
\end{tikzcd}$$

a similar analysis shows there are no strings in W' going over both cycles.

Assuming that the orientation is as described in (b), and that the ideal I_A is not generated by $\alpha_1\alpha_n$ and $\beta_r\beta_1$, we cannot find a string going over both cycles; this contradicts Lemma 3.2, proving (b).

(b)⇒(a). Suppose that Q_A and I_A are as stated in (b). We are going to prove that there is a string $C \in W'$ such that the associated band module X contains a proper submodule X' isomorphic to a quotient of X. Then we will have non-zero morphisms $X \to X'$ and $X' \to X$ (the projection and inclusion, respectively) such that their composite $X \to X' \to X$ is a non-zero morphism in $\Re(X, X)$. Let δ be the walk starting at i. Write $\gamma_1 = \gamma_1'\alpha_1$, $\gamma_2 = \gamma_2'\beta_1$ and also $\gamma_1 = \alpha_n\gamma_1''$, $\gamma_2 = \beta_r\gamma_2''$ and consider the string $C = \delta^{-1}\gamma_2^{-1}\delta\gamma_1$. Then $C = \delta^{-1}\gamma_2''^{-1}\beta_r^{-1}\delta\alpha_n\gamma_1''$. Applying Lemma 3.1 we conclude that $M(\delta)$ is a submodule of $X = M(C, \lambda)$.

On the other hand, if we consider the string $C = \delta^{-1}\beta_1^{-1}\gamma_2'^{-1}\delta\gamma_1'\alpha_1$, then applying Lemma 3.1, we deduce that $M(\delta) = M(\delta^{-1})$ is a quotient of $X = M(C, \lambda)$, as desired. ■

Now, we consider string algebras such that their quivers Q_A consist of two cycles γ_1 and γ_2, not necessarily oriented, such that their intersection is a quiver of type A_n.

LEMMA 3.5. *Let A be a string algebra such that Q_A consists of two cycles γ_1 and γ_2, not necessarily oriented, intersecting in a quiver δ of type A_n. The following conditions are equivalent:*

(a) *There is a string C such that the band module $X = M(C, \lambda)$ satisfies $\Re(X, X) \neq 0$ for each $\lambda \in k^*$.*

(b) *Either Q_A has the shape*

$$(1)\qquad \begin{array}{ccc} & i & \\ \overset{\alpha_1}{\swarrow} & | & \overset{\beta_1}{\nwarrow} \\ \gamma_1\vdots & \vdots\,\delta & \vdots\,\gamma_2 \\ \overset{\alpha_n}{\searrow} & | & \overset{\beta_r}{\nearrow} \\ & j & \end{array}$$

and $I_A = \langle \alpha_1\beta_1, \beta_r\alpha_n \rangle$, or Q_A has the shape

$$(2)\qquad \begin{array}{ccc} & i & \\ \overset{\alpha_1}{\swarrow} & | & \overset{\beta_1}{\nwarrow} \\ \gamma_1\vdots & \vdots\,\delta & \vdots\,\gamma_2 \\ \overset{\alpha_n}{\nwarrow} & | & \overset{\beta_r}{\swarrow} \\ & j & \end{array}$$

and $I_A = \langle \alpha_1\beta_1, \alpha_n\beta_r \rangle$.

Proof. Assume that (a) holds but (b) does not. First, suppose that we have the following orientation of arrows at the vertex i:

$$\begin{array}{ccc} & i & \\ \overset{\alpha_1}{\nearrow} & \downarrow \delta_s & \overset{\beta_1}{\nwarrow} \end{array}$$

Since A is a string algebra, we see that $\delta_s\alpha_1 = 0$ or $\delta_s\beta_1 = 0$. In any case, there is no string in $C \in W'$, since $C^2 = 0$ or C^2 is not a reduced string. The case

$$\begin{array}{ccc} & i & \\ \overset{\alpha_1}{\swarrow} & \uparrow \delta_s & \overset{\beta_1}{\searrow} \end{array}$$

is dual.

In case Q_A is as stated in (b), but the corresponding relations are not, it is not hard to see that it is not possible to go over both cycles, getting the implication.

(b)⇒(a). Assume that (1) holds. Consider the string $C = \gamma_1^{-1}\gamma_2$. Let X be the band module associated with C. We write $\gamma_1 = \alpha_1^{-1}\gamma_1'\delta$ and $\gamma_2 = \delta^{-1}\beta_r^{-1}\gamma_2'$.

Then $C = \gamma_1^{-1}\gamma_2 = \delta^{-1}\gamma_1'^{-1}\alpha_1\delta^{-1}\beta_r^{-1}\gamma_2'$ and applying Lemma 3.1 we find that $M(\delta) = M(\delta^{-1})$ is a quotient of X.

If we write $\gamma_1 = \gamma_1''\alpha_n^{-1}\delta$ and $\gamma_2 = \delta^{-1}\gamma_2''\beta_1^{-1}$, then we get

$$C = \delta^{-1}\alpha_n\gamma_1''^{-1}\delta^{-1}\gamma_2''\beta_1^{-1}.$$

Applying Lemma 3.1 shows that $M(\delta)$ is a submodule of X. Since it is also a quotient of X, we see that $\Re(X, X) \neq 0$.

Assume that (2) holds. With a similar analysis and considering the string $C = \gamma_1^{-1}\gamma_2\gamma_2\gamma_1^{-1}$ we deduce that $M(\delta)$ is both a submodule and a quotient of X, proving the lemma. ∎

Summarizing the above results we can state the following proposition:

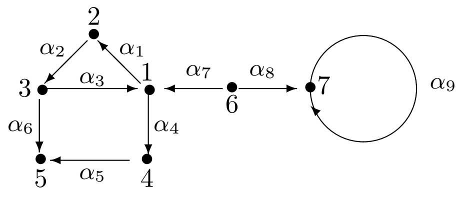
PROPOSITION 3.6. *Let $A = kQ_A/I_A$ be a connected string algebra such that Q_A contains a subquiver, not necessarily full and convex, of the type described in Lemma* 3.3, 3.4 *or* 3.5. *Then, for each $\lambda \in k^*$, there is a simple regular band module X_λ in a homogeneous tube such that $\Re(X_\lambda, X_\lambda) \neq 0$ and $X_\lambda \neq X_\mu$ for $\lambda \neq \mu$.*

Proof. Assume that Q_A contains a subquiver Q' of one of the types described in Lemmas 3.3–3.5. By the lemmas there is a string C of $A' = kQ'/I_A \cap kQ'$ such that the band module $X_\lambda = M(C, \lambda)$ satisfies $\Re(X_\lambda, X_\lambda) \neq 0$ for each $\lambda \in k^*$.

Then C is also a string of A and $M(C, \lambda)$ is a band A-module satisfying the required conditions. ∎

The converse of the above proposition does not hold, as we show in our next example:

EXAMPLE 3.7. Consider the string algebra given by the quiver



with the relations $\alpha_1\alpha_7 = 0$, $\alpha_6\alpha_2 = 0$, $\alpha_4\alpha_7 = 0$, $\alpha_4\alpha_3 = 0$ and $\alpha_9\alpha_9 = 0$. Then no subquiver of Q_A containing exactly two cycles satisfies the conditions stated in Lemma 3.3, 3.4 or 3.5. However, if X is the band module associated to the string $C = \alpha_4^{-1}\alpha_5^{-1}\alpha_6\alpha_3^{-1}\alpha_7\alpha_8^{-1}\alpha_9\alpha_8\alpha_7^{-1}\alpha_3\alpha_2\alpha_1$, then the simple S_1 is a direct summand of both the top and the socle of X. Therefore, $\Re(X,X) \neq 0$.

Combining the last proposition and Theorem 2.6 we can state the main result of this section.

THEOREM 3.8. *Let $A = kQ_A/I_A$ be a connected string algebra. Then the following conditions are equivalent:*

(a) *There are n irreducible morphisms $h_i : X_i \to X_{i+1}$ with X_i band modules for $i = 1,\dots,n$ such that $h_n \dots h_1 \neq 0$ and $h_n \dots h_1 \in \Re^{n+1}(X_1, X_{n+1})$.*
(b) *There is a simple regular band module X_1 in a homogeneous tube such that $\Re(X_1, X_1) \neq 0$.*

Moreover, when Q_A contains at most two cycles, the above conditions are equivalent to:

(c) *For each $\lambda \in k^*$, there is a simple regular band module X_λ in a homogeneous tube such that $\Re(X_\lambda, X_\lambda) \neq 0$ and $X_\lambda \neq X_\mu$ for $\lambda \neq \mu$.*
(d) *The quiver Q_A contains a subquiver, not necessarily full and convex, of the type described in Lemma 3.3, 3.4 or 3.5.*

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REFERENCES

[1] E. R. Alvares, C. Chaio and S. Trepode, *Auslander–Reiten components with sectional bypasses*, Comm. Algebra 37 (2009), 2213–2224.

[2] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, Vol. 1, London Math. Soc. Student Texts 65, Cambridge Univ. Press, 2006.

[3] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[4] M. C. R. Butler and C. M. Ringel, *Auslander–Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra 15 (1987), 145–179.

[5] C. Chaio, F. U. Coelho and S. Trepode, *On the composite of two irreducible morphisms in radical cube*, J. Algebra 312 (2007), 650–667.

[6] —, —, —, *On the composite of irreducible morphisms in almost sectional paths*, J. Pure Appl. Algebra 212 (2008), 244–261.

[7] —, —, —, *On the composite of three irreducible morphisms in the fourth power of the radical*, Comm. Algebra, to appear.

[8] C. Chaio, P. Le Meur and S. Trepode, *Degrees of irreducible morphisms and finite-representation type*, J. London Math. Soc. (2011), to appear.
[9] C. Chaio, M. I. Platzeck and S. Trepode, *On the degree of irreducible morphisms*, J. Algebra 281 (2004), 200–224.
[10] K. Igusa and G. Todorov, *A characterization of finite Auslander–Reiten quivers*, ibid. 89 (1984), 148–177.
[11] S. Liu, *Degrees of irreducible maps and the shapes of Auslander–Reiten quivers*, J. London Math. Soc. (2) 45 (1992), 32–54.
[12] —, *Shapes of connected components of the Auslander–Reiten quivers of Artin algebras*, in: CMS Conf. Proc. 19, Amer. Math. Soc., 1996, 109–137.
[13] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

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