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THE LOCALISATION OF PRIMES IN ARITHMETIC PROGRESSIONS OF IRRATIONAL MODULUS

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Abstract. A new method for counting primes in a Beatty sequence is proposed, and it is shown that an asymptotic formula can be obtained for the number of such primes in a short interval.

1. Opening. The distribution of primes in arithmetic progressions is an intensively investigated and important subject in the theory of numbers. More recently and perhaps triggered by work of Long [8] on certain Hamiltonian systems, there has been a flurry of papers on primes near the general linear sequence $\xi n+\eta$ in which ξ, η are given real numbers, $\xi > 1$, and n varies over the natural numbers. When ξ and η are integers, $\xi n+\eta$ is an arithmetic progression of modulus ξ. In all other cases, $\xi n + \eta$ is not usually an integer. Thus, we shall be concerned with primes p of the form $p = [\xi n + \eta]$. When ξ is a rational number, the values taken by $[\xi n + \eta]$ are members of certain traditional arithmetic progressions, so that the distribution of their prime values is still part of the classical theory. Therefore, we assume henceforth that ξ is a positive irrational. Although certainly better known as a *Beatty sequence*, we refer to the values of $[\xi n + \eta]$ as an arithmetic progression of modulus ξ.

A prime number theorem for these generalised progressions has been provided by Ribenboim [9]. His work was recently extended by Banks and Shparlinski [1]. We refer to [1] and [2] for further background on the problem. In short, the main argument in [1] transfers the original problem into one in diophantine approximation, and then solves the latter by a familiar Fourier technique attributed to Vinogradov. Alternatively, one might observe that $p = [\xi n + \eta]$ is equivalent to the diophantine inequality

$$0 \le \xi n + \eta - p < 1, \tag{1}$$

which may be treated as a binary additive problem. Since $\xi n+\eta$ is a periodic sequence, this is easily done by straightforward application of the Davenport–Heilbronn Fourier transform method and an intersection estimate that the

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authors obtained in collaboration with Wooley [4]. This approach is very flexible, and applies to a wide variety of questions. The sole purpose of the present short note is to popularise its use, and to illustrate its performance by counting the solutions of (1) when p is constrained to a short interval. Thus, let $\pi(x;\xi,\eta)$ be the number of primes $p \le x$ that are representable as $p = [\xi n + \eta]$ with $n \in \mathbb{N}$. We shall conclude as follows.

THEOREM. *Suppose that $\xi > 1$ is an irrational number. Let $\theta > 5/8$. Then, whenever $x^\theta \le y \le x$, one has*

$$\pi(x+y;\xi,\eta) - \pi(x;\xi,\eta) = \frac{y}{\xi \log x}(1+o(1)) \quad (x \to \infty).$$

For comparison, when ξ is rational, one can evalutate the number of primes in question in the wider range $\theta > 7/12$. This follows from Huxley and Iwaniec [7], for example.

If $\xi < 1$, then all large natural numbers occur as values of $[\xi n + \eta]$, so counting primes in this set of values is the same as counting primes. However, if one is prepared to count primes in the sequence $[\xi n + \eta]$ with multiplicity, then our method yields the Theorem for $\xi > 0$. The hypothesis that $\xi > 1$ makes these values distinct, but is not needed otherwise. For further comments on possible extensions, we refer to the closing section of this paper.

2. Initial transformation. Some notation is required before the attack can be launched. For any positive real number τ, define the measure $d_\tau\alpha$ on $\mathbb{R}$ that relates to Lebesgue measure $d\alpha$ through

$$d_\tau\alpha = \tau\left(\frac{\sin \pi\tau\alpha}{\pi\tau\alpha}\right)^2 d\alpha. \tag{2}$$

Its Fourier transform is

$$\int_{-\infty}^{\infty} e(-\alpha\beta)\, d_\tau\alpha = \max(0, 1 - |\beta|/\tau) \tag{3}$$

where, as usual, $e(\alpha) = \exp(2\pi i\alpha)$. Let Λ be von Mangoldt's function, and consider the exponential sums

$$f(\alpha) = \sum_{x<m\le x+y} \Lambda(m)e(\alpha m), \qquad g(\alpha) = \sum_{x<\xi n\le x+y} e(\alpha\xi n). \tag{4}$$

For $\zeta \in \mathbb{R}$ and x, y, τ, ξ as before, we shall study the integral

$$I_\tau = I_\tau(x,y;\xi,\zeta) = \int_{-\infty}^{\infty} f(\alpha)g(-\alpha)e(-\alpha\zeta)\, d_\tau\alpha \tag{5}$$

by means of the Davenport–Heilbronn method, and prove the following.

PROPOSITION. *Let* $\zeta \in \mathbb{R}$. *Then, uniformly for* x, y *as in the Theorem, and* $1/3 \le \tau \le 2$, *one has*

$$I_\tau = \xi^{-1}\tau y + o(y) \quad \text{as } x \to \infty.$$

In the remainder of this section, we briefly indicate why this implies the Theorem. The argument is largely standard, and very similar to the work in Sections 2.1–2.2 of [4]. We are therefore very brief. Let

$$W(\alpha) = \begin{cases} 1 & \text{for } -1/2 < \alpha \le 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Whenever $0 < \delta < 1/3$, define the functions

$$W^+(\alpha) = \begin{cases} 1 & \text{when } |\alpha| < 1/2, \\ 1 - \dfrac{|\alpha| - 1/2}{\delta} & \text{when } 1/2 \le |\alpha| < 1/2 + \delta, \\ 0 & \text{when } |\alpha| \ge 1/2 + \delta, \end{cases}$$

$$W^-(\alpha) = \begin{cases} 1 & \text{when } |\alpha| < 1/2 - \delta, \\ 1 - \dfrac{|\alpha| - 1/2 + \delta}{\delta} & \text{when } 1/2 - \delta \le |\alpha| < 1/2, \\ 0 & \text{when } |\alpha| \ge 1/2. \end{cases}$$

Then $W^-(\alpha) \le W(\alpha) \le W^+(\alpha)$ for all $\alpha \in \mathbb{R}$. Now consider

$$P(x, y) = \sum_{\substack{x < m \le x+y \\ x < \xi n \le x+y}} \Lambda(m) W\left(m - \xi n - \eta + \frac{1}{2}\right). \tag{6}$$

Define $P^+(x, y)$, $P^-(x, y)$ similarly, but with W replaced by W^+, W^-, respectively, so that

$$P^-(x, y) \le P(x, y) \le P^+(x, y). \tag{7}$$

From (3), one readily obtains the identities

$$W^\pm(\alpha) = \left(1 \pm \frac{1}{2\delta}\right) \int_{-\infty}^{\infty} e(-\alpha\beta)\, d_{1/2\pm\delta}\beta \mp \frac{1}{2\delta} \int_{-\infty}^{\infty} e(-\alpha\beta)\, d_{1/2}\beta,$$

whence, by (4) and (5),

$$P^\pm(x, y) = \left(1 \pm \frac{1}{2\delta}\right) I_{1/2\pm\delta}\left(x, y; \xi, \eta - \frac{1}{2}\right) \mp \frac{1}{2\delta} I_{1/2}\left(x, y; \xi, \eta - \frac{1}{2}\right).$$

The Proposition ensures the existence of a function $R(y)$ with $R(y) \to 0$ as $y \to \infty$ and such that $|I_\tau - \xi^{-1}\tau y| \le yR(y)$. It follows that

$$P^\pm(x, y) = \frac{y}{\xi}(1 \pm \delta) + O(y\delta^{-1}R(y)).$$

We may choose $\delta = \sqrt{R(y)}$. Then, invoking (7), we infer the asymptotic formula

$$P(x,y)=\frac{y}{\xi}+o(y). \tag{8}$$

An inspection of (6) now reveals that $P(x,y)$ counts solutions of $0\le \xi n+\eta-m<1$, and hence of $m=[\xi n+\eta]$, with weight $\Lambda(m)$, in the range $x<m\le x+y$, $x<n\xi\le x+y$. Since $y\ge x^{5/8}$, an elementary argument suffices to show that prime powers $m=p^k$ with $k\ge 2$ make a negligible contribution to (8), and the weight $\log p$ from solutions with $m=p$ may be removed by partial summation. Thus, (8) implies the Theorem.

3. Amplifiers. In this section, we establish two estimates by standard applications of the Hardy–Littlewood method. In the proof of the Proposition, they will serve as amplifiers, in the sense of Wooley [10] and Brüdern, Kawada and Wooley [4].

Throughout this section, x,y,ξ are as in the Theorem. When $1\le Q\le \frac{1}{2}\sqrt{y}$, let $\mathfrak{N}(Q)$ denote the disjoint union of all intervals $|q\alpha-a|\le Q/y$ with $1\le a\le q<Q$ and $(a,q)=1$. Let $\mathfrak{n}(Q)=[Q/y,1+Q/y]\setminus\mathfrak{N}(Q)$.

LEMMA 1. *Let* $1\le Q\le(\log x)^9$. *Then*

$$\int\limits_{\mathfrak{n}(Q)}|f(\alpha)|^3\,d\alpha\ll y^2Q^{-1}.$$

Proof. By Parseval's identity and the Brun–Titchmarsh inequality,

$$\int\limits_0^1|f(\alpha)|^2\,d\alpha=\sum_{x<m\le x+y}\Lambda(m)^2\ll y\log x.$$

We now apply Theorems 2 and 3 of Zhan [11]. These assert that there exists a constant B such that

$$\sup_{\alpha\in\mathfrak{n}((\log x)^B)}|f(\alpha)|\ll y(\log x)^{-10}. \tag{9}$$

Note that it is here that the condition $\theta>5/8$ in the lower bound for y is needed. The rest of the argument works in the wider range $\theta>7/12$. On combining the previous two bounds, we see that

$$\int\limits_{\mathfrak{n}((\log x)^B)}|f(\alpha)|^3\,d\alpha\ll y^2(\log x)^{-9}. \tag{10}$$

It will be convenient to write $Q_0=(\log x)^B$. Also, in the following argument, $\varphi(q)$ is Euler's totient function, and $\mu(q)$ is Möbius' function. Then, uniformly for $x\le u\le x+y$, $1\le a\le q\le Q_0$ and $(a,q)=1$, one has

$$\sum_{\substack{x\le m\le u\\ m\equiv a \bmod q}}\Lambda(m)=\frac{u-x}{\varphi(q)}+O(y(\log x)^{-4B}).$$

This familiar result is a consequence of Huxley's density estimate [6], and indeed immediate from the work of Huxley and Iwaniec [7]. Now, sorting m into residue classes, one finds that in the same ranges one has

$$\sum_{x\le m\le u} \Lambda(m)e\left(\frac{am}{q}\right) = \frac{\mu(q)}{\varphi(q)}(u-x) + O(qy(\log x)^{-4B}).$$

Hence, on summing by parts, one readily establishes that

$$f\left(\frac{a}{q}+\beta\right) = \frac{\mu(q)}{\varphi(q)}\int_x^{x+y} e(\beta\gamma)\,d\gamma + O(y(\log x)^{-3B}(1+y|\beta|)) \tag{11}$$

uniformly for $\beta\in\mathbb{R}$ and a, q as before. The integral in (11) can be integrated by parts, which gives the upper bound $O(y(1+y|\beta|)^{-1})$ for it. In particular, when $\alpha\in\mathfrak{N}(Q_0)$, (11) implies that

$$\left|f\left(\frac{a}{q}+\beta\right)\right| \ll \frac{y}{\varphi(q)}(1+y|\beta|)^{-1},$$

and a routine estimation now yields the bound

$$\int_{\mathfrak{N}(Q_0)\setminus\mathfrak{N}(Q)} |f(\alpha)|^3\,d\alpha \ll y^2Q^{-1}.$$

On combining this with (10), the conclusion in Lemma 1 is immediate.

The corresponding estimate for g is much easier to obtain.

LEMMA 2. *Let* $\mathfrak{n}_\xi(Q) = \{\alpha\in\mathbb{R} : \xi\alpha\in\mathfrak{n}(Q)\}$. *Then, for* $1\le Q\le (\log x)^9$, *one has*

$$\int_{\mathfrak{n}_\xi(Q)} |g(\alpha)|^{3/2}\,d\alpha \ll y^{1/2}Q^{-1/2}.$$

Proof. Of course, much more is true, but the following simple argument suffices for our purposes. First substitute $\xi\alpha=\beta$. Then, on denoting the distance of the real number β to the nearest integer by $\|\beta\|$, the integral in question is readily seen to be

$$\ll \int_{\mathfrak{n}(Q)} \Big|\sum_{x\le\xi n\le x+y} e(\beta n)\Big|^{3/2} d\beta \ll \int_{(Q-1)/y}^{1-(Q-1)/y} \min(y^{3/2}, \|\beta\|^{-3/2})\,d\beta$$
$$\ll y^{1/2}Q^{-1/2},$$

as required.

For the application in the next section, one needs a slight variant of these estimates. Let $\mathfrak{m}(Q) = \mathfrak{n}(Q)+\mathbb{Z}$ and $\mathfrak{M}(Q) = \mathfrak{N}(Q)+\mathbb{Z}$ be the 1-periodic versions of minor and major arcs. Then, uniformly for $1/3\le\tau\le 2$, one finds from (2) that whenever $1 \le Q \le (\log x)^9$, one has

$$\int_{\mathfrak{m}(Q)} |f(\alpha)|^3 \, d_\tau\alpha \ll y^2 Q^{-1}, \tag{12}$$

$$\int_{\xi\alpha \in \mathfrak{m}(Q)} |g(\alpha)|^{3/2} \, d_\tau\alpha \ll y^{1/2} Q^{-1/2}. \tag{13}$$

4. The complementary compositum. Let $z = (\log x)^9 y^{-1}$, and split the integral I_τ in (5) into one over the *central interval* $|\alpha| \le z$ and one over the *complementary compositum* $|\alpha| > z$. Accordingly, we write

$$I_\tau = I_\tau' + I_\tau'' \tag{14}$$

in which

$$I_\tau' = \int_{-z}^{z} f(\alpha) g(-\alpha) e(-\zeta\alpha) \, d_\tau\alpha, \tag{15}$$

and I_τ'' is defined likewise, but with the integration extended over $|\alpha| > z$. In this section we show that I_τ'' is small. This is the core of the method; the rest is essentially routine. Our approach is based on our version, developed in collaboration with Wooley [4], of the Bentkus–Götze–Freeman device (see [3], [5] and [4, Section 2.5] for a thorough discussion). The new variant is an estimate purely within the theory of diophantine approximation, and makes no reference at all to the exponential sums that occur in the Fourier transform (5). We only need a very special case of our tool that we now intend to formulate. Let

$$T(R) = \max\{r \le R : \|\xi r\| < 1/r\}.$$

Since ξ is irrational, $T(R)$ is unbounded, and of course increasing. Let

$$\tilde{Q} = \min(T(\sqrt{y})^{1/3}, (\log x)^8),$$

and write

$$\mathfrak{K} = \{\alpha \in \mathbb{R} : |\alpha| \ge z,\ \alpha \in \mathfrak{M}(\tilde{Q}),\ \xi\alpha \in \mathfrak{M}(\tilde{Q})\}.$$

In Theorem 2.4 of [4] we take $\lambda_1 = \xi$, $\lambda_2 = -1$, $N \asymp y$, $Y = z$ and $Q(N) = Q_1 = Q_2 = \tilde{Q}$ to infer that

$$\int_{\mathfrak{K}} d_\tau\alpha \ll y^{-1} \tilde{Q}^2 T(\sqrt{y})^{-1} = o(y^{-1}). \tag{16}$$

The Brun–Titchmarsh inequality provides the upper bound $|f(\alpha)| \le f(0) \ll y$ whereas $g(\alpha) \ll y$ is trivial. This already suffices to conclude that

$$\int_{\mathfrak{K}} |f(\alpha) g(-\alpha)| \, d_\tau\alpha = o(y).$$

It remains to consider α with $|\alpha| \ge z$ but $\alpha \notin \mathfrak{K}$. Then $\alpha \in \mathfrak{m}(\tilde{Q})$ or $\xi\alpha \in \mathfrak{m}(\tilde{Q})$. In the first case, we use (12) with $Q = \tilde{Q}$ and (13) with $Q = 1$. Then, by Hölder's inequality,

$$\int\limits_{\mathfrak{m}(\tilde{Q})} |f(\alpha)g(-\alpha)|\, d_\tau\alpha \le \Big(\int\limits_{\mathfrak{m}(\tilde{Q})} |f(\alpha)|^3\, d_\tau\alpha\Big)^{1/3} \Big(\int\limits_{-\infty}^{\infty} |g(\alpha)|^{3/2}\, d_\tau\alpha\Big)^{2/3} \ll y\tilde{Q}^{-1/3}.$$

Reversing the roles in the previous argument also yields

$$\int\limits_{\xi\alpha\in\mathfrak{m}(\tilde{Q})} |f(\alpha)g(-\alpha)|\, d_\tau\alpha \le \Big(\int\limits_{-\infty}^{\infty} |f(\alpha)|^3\, d_\tau\alpha\Big)^{1/3} \Big(\int\limits_{\xi\alpha\in\mathfrak{m}(\tilde{Q})} |g(\alpha)|^{3/2}\, d_\tau\alpha\Big)^{2/3} \ll y\tilde{Q}^{-1/3}.$$

We may sum the last three estimates to confirm that $I''_\tau = o(y)$, and hence by (14) that

$$I_\tau = I'_\tau + o(y) \quad \text{as } y \to \infty. \tag{17}$$

5. The central interval. The evaluation of I'_τ begins with the special case of (11) where $a = q = 1$. Then, for $|\alpha| \le z$, one has

$$f(\alpha) = \int\limits_{x}^{x+y} e(\alpha\gamma)\, d\gamma + O(y(\log x)^{-10}). \tag{18}$$

Also, by (4) and Euler's summation formula, followed by an obvious substitution,

$$g(\alpha) = \frac{1}{\xi} \int\limits_{x}^{x+y} e(\alpha\gamma)\, d\gamma + O(1 + y|\alpha|). \tag{19}$$

Now let

$$h(\alpha) = \int\limits_{0}^{y} e(\alpha\gamma)\, d\gamma.$$

Then, by (18) and (19), again for $|\alpha| \le z$ one has

$$f(\alpha)g(-\alpha) = \xi^{-1}|h(\alpha)|^2 + O(y^2(\log x)^{-10}).$$

We integrate over $|\alpha| \le z$ to conclude from (15) that

$$I'_\tau = \frac{1}{\xi} \int\limits_{-z}^{z} |h(\alpha)|^2 e(-\alpha\zeta)\, d_\tau\alpha + O(y(\log y)^{-1})$$

uniformly for $1/3 \le \tau \le 2$. The upper bound $h(\alpha) \ll y(1 + y|\alpha|)^{-1}$, which in turn is readily confirmed by partial integration, is enough to complete the integral, with an error of $O(y(\log y)^{-9})$. But, by (3), one finds that

$$\int_{-\infty}^{\infty} |h(\alpha)|^2 e(-\alpha\zeta)\, d_\tau\alpha = \int_0^y\int_0^y \max(0, 1-\tau^{-1}|t_1-t_2-\zeta|)\, dt_1\, dt_2,$$

and when $\zeta \in \mathbb{R}$ is fixed and $1/3 \le \tau \le 2$, the integral on the right hand side is $\tau y + O(1)$, as one readily checks. Altogether, we find that

$$I'_\tau = \xi^{-1}\tau y + O(y(\log y)^{-1}),$$

and the Proposition follows from (17).

6. Closing. The analytic method that we have described here has certain advantages over the line of attack followed by previous writers, most notably Banks and Shparlinski [1]. In particular, our method applies to all irrational numbers ξ, and not only to irrationals "of finite type" (see [1]). On the other hand, [1] gives an asymptotic formula for $\pi(x, \xi, y)$ with an error term that is as small one can expect, given our current knowledge about zero free regions of the Riemann zeta function, but only for ξ of finite type. Although our exposition might perhaps suggest that our method is limited to weaker error terms, like $o(y)$, this is definitely not the case. Indeed, when ξ is of finite type, the bound (16) can be improved considerably, and with a slightly wider choice for the central interval, the savings in (16) become a power of y. This has been worked out in detail in Section 2.5 of [4], in particular Theorem 2.3. Thus, the results of Banks and Shparlinski are within the scope of our method as well. Banks and Shparlinski also discuss primes of the form $q[\xi n + \eta] + a$ in which q and a are given natural numbers. Our method easily extends to this set-up, and is also capable of counting primes $p \equiv a \bmod q$ that are of the form $p = [\xi n + \eta]$.

Finally, we repeat a comment that we made in Section 4. For rational values of ξ, the appropriate analogue of our Theorem is known for $\theta > 7/12$. It is therefore natural to expect the same limitation for the irrational case. However, the bottleneck here is the minor arc estimate for exponential sums over primes from a short interval. Zhan [11] still holds the record (estimate (9) above). If (9) could be established for $\theta > 7/12$, then a corresponding improvement can be made in our Theorem.

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