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THE NORM SPECTRUM IN CERTAIN CLASSES OF COMMUTATIVE BANACH ALGEBRAS

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Abstract. Let A be a commutative Banach algebra and let Σ_A be its structure space. The norm spectrum $\sigma(f)$ of the functional $f \in A^*$ is defined by $\sigma(f) = \overline{\{f \cdot a : a \in A\}} \cap \Sigma_A$, where $f \cdot a$ is the functional on A defined by $\langle f \cdot a, b\rangle = \langle f, ab\rangle$, $b \in A$. We investigate basic properties of the norm spectrum in certain classes of commutative Banach algebras and present some applications.

1. Introduction. Let A be a commutative Banach algebra and let Σ_A be its structure space. By $\widehat{a}$, we denote the Gelfand transform of an element $a \in A$. The *hull* of any ideal $I \subset A$ is defined by

$$h(I) = \{\phi \in \Sigma_A : \widehat{a}(\phi) = 0,\ \forall a \in I\}.$$

For $f \in A^*$ and $a \in A$, we define the functional $f \cdot a$ on A by

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle,\ b \in A.$$

If $f \in A^*$, then I_f will denote the (closed) ideal $\{a \in A : f \cdot a = 0\}$. For a closed subset S of Σ_A, let

$$I_S = \{a \in A : \widehat{a}(S) = \{0\}\}.$$

Then I_S is a closed ideal in A.

Recall that a closed ideal I of A is said to be *synthesizable* if $I = I_{h(I)}$. It follows from the well-known Malliavin's Theorem that not every closed ideal of $L^1(\mathbb{R})$ is synthesizable. De Vito proved in [1] that synthesizable ideals of $L^1(\mathbb{R})$ are exactly the ideals of the form

$$I_f = \{k \in L^1(\mathbb{R}) : f * k = 0\},$$

where f is an almost periodic function on $\mathbb{R}$ (if we define the duality between $f \in L^\infty(\mathbb{R})$ and $k \in L^1(\mathbb{R})$ as $\langle f, k\rangle = \int_{\mathbb{R}} f(-t)k(t)\,dt$, then $f \cdot k$ is just $f * k$). Recall also that the algebra of all almost periodic functions on $\mathbb{R}$ is identified with $\overline{\operatorname{span}}\,\Sigma_{L^1(\mathbb{R})}$.

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To study synthesizable ideals for general algebras, A. Ülger defined in [17] the *norm spectrum* $\sigma(f)$ of $f \in \overline{\operatorname{span}}\,\Sigma_A$ by

$$\sigma(f) = \overline{\{f \cdot a : a \in A\}} \cap \Sigma_A,$$

which coincides with the definition given, for instance, by Y. Katznelson [7, pp. 157–171] in the case where $A = L^1(\mathbb{R})$. As proved in [7, p. 163], $\sigma(f) \neq \emptyset$ for every nonzero almost periodic function f on $\mathbb{R}$. In [17], A. Ülger also introduced the separating ball property (SBP for short), which plays an important role in the study of synthesizable ideals. A Banach algebra A is said to have the SBP if given any two distinct elements ϕ and ψ in Σ_A, there exists an element $a \in A$ with $\|a\| \leq 1$ such that $\widehat{a}(\phi) = 1$ and $\widehat{a}(\psi) = 0$. Under the assumption that $\sigma(f) \neq \emptyset$ for all $f \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$ plus the SBP, A. Ülger [17, Theorem 5.5] gave the following generalization of De Vito's result: The ideal I is synthesizable with a separable hull iff $I = I_f$ for some $f \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$. Consequently, the following question posed by A. Ülger [17] is important: Under which hypotheses the norm spectrum of each $f \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$ is nonempty? For the Herz algebras the answer was given by Z. G. Hu in [6].

In this paper, we introduce the class of boundedly regular Ditkin algebras. These algebras do not have the SBP in general. In Section 3, we investigate some basic properties of weak* and norm spectra in boundedly regular Ditkin algebras and present some applications. Among other things it is shown that if I is a synthesizable closed ideal of a boundedly regular Ditkin algebra with w^*-separable hull, then $I = I_f$ for some $f \in l^1(\Sigma_A) \setminus \{0\}$. In Section 4, we consider the class of Banach algebras A for which there exists a continuous homomorphism $\omega : L^1(G) \to A$ with dense range, where G is a locally compact abelian group. It is shown that the norm spectrum of any $f \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$ is then nonempty.

2. Boundedly regular Ditkin algebras. In this section, we introduce the class of boundedly regular Ditkin algebras and present some examples. Throughout the paper, we will need the following notation. For a Banach space X, we denote by X^* and X^{**} the dual and the second dual of X, respectively. If $f \in X^*$ and $x \in X$, the value of f at x will be written as $\langle f, x\rangle$ or $f(x)$. By $\overline{E}$ and $\overline{E}^w$ we will denote, respectively, the norm closure and the weak closure of $E \subset X$. If $E \subset X^*$, then $\overline{E}^{w^*}$ will denote the w^*-closure of E.

Let A be a commutative Banach algebra. If A has no unit element, then the algebra formed by adjoining an identity is denoted by A_e. It is well known that $\Sigma_{A_e} = \Sigma_A \cup \{\infty\}$, the one-point compactification of Σ_A. We put

$$A_{00} := \{a \in A : \operatorname{supp} \widehat{a} \text{ is compact}\}.$$

Recall that A is said to be *Tauberian* if A_{00} is dense in A.

Now, let A be a commutative regular semisimple Banach algebra and let S be a closed subset of Σ_A. As usual, to S we associate two ideals, I_S (already defined) and J_S, where

$$J_S = \{a \in A_{00} : \operatorname{supp} \widehat{a} \cap S = \emptyset\}.$$

Notice that $J_{\{\infty\}} = A_{00}$ and $I_{\{\infty\}} = A$. Among the closed ideals of A whose hull is S, I_S is the largest one and $\overline{J_S}$ is the smallest. If $I_S = \overline{J_S}$, then S is said to be a set of *spectral synthesis* (*s-set* for short) [9, Chapter 8]. It can be seen that if I is a proper closed ideal of A and if $h(I)$ is an s-set, then I is synthesizable. But the converse is not true in general (see, for instance, [6]).

The algebra A is said to satisfy the *Ditkin condition* at $\phi \in \Sigma_A \cup \{\infty\}$ if for each $a \in I_{\{\phi\}}$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $J_{\{\phi\}}$ such that $\lim_{n\to\infty} \|aa_n - a\| = 0$ [9, p. 204]. Notice that if A satisfies Ditkin's condition at $\phi \in \Sigma_A \cup \{\infty\}$, then $\{\phi\}$ is an s-set.

Definition 2.1. We say that a commutative regular semisimple Banach algebra A is a *Ditkin algebra* if each point of Σ_A is an s-set and if in addition A satisfies Ditkin's condition at ∞ whenever A has no unit element.

The following definition is contained in [10, p. 418].

Definition 2.2. A commutative Banach algebra A is said to be *boundedly regular* if there exists a constant $C > 0$ such that for each $\phi \in \Sigma_A$ and each neighborhood U of ϕ there exists an element $a \in A$ for which $\widehat{a}(\phi) = 1$, $\operatorname{supp} \widehat{a} \subset U$, and $\|a\| \le C$.

The following examples show that many algebras of harmonic analysis are boundedly regular Ditkin algebras.

Example 2.3. (a) Let X be a locally compact Hausdorff space and let $C_0(X)$ be the Banach algebra of all continuous functions on X vanishing at infinity. As is well-known, $C_0(X)$ is a Ditkin algebra and by the Urysohn Lemma it is boundedly regular.

(b) If G is a locally compact abelian group, then $L^1(G)$ is a boundedly regular Ditkin algebra (see [9, Chapter 8] and [16, Theorem 2.6.1]).

(c) If G is a compact abelian group, then the spaces $C(G)$ and $L^p(G)$ $(1 \le p < \infty)$ with convolution multiplication and usual norms are boundedly regular Ditkin algebras.

(d) Let G be a locally compact group. For $1 < p < \infty$, we denote by $A_p(G)$ the Herz algebra of G [4]. Elements of $A_p(G)$ can be represented nonuniquely as

$$f = \sum_{n=1}^{\infty} u_n * \widetilde{v}_n$$

with $u_n \in L^p(G)$, $v_n \in L^q(G)$ $(1/p + 1/q = 1)$, $\widetilde{v}_n(g) = v_n(g^{-1})$ and

$$\|f\| = \inf \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty.$$

Here, the infimum is taken over all such representations of f. It is known that $A_p(G)$ with the above norm and pointwise multiplication is a commutative semisimple regular Banach algebra whose structure space is G (via Dirac measures). If G is amenable, then $A_p(G)$ is a boundedly regular Ditkin algebra [4, 17].

Let G be a locally compact abelian group and let $S(G)$ be a Segal algebra of G. As is known, $S(G)$ is a commutative regular semisimple Banach algebra with convolution multiplication [14, Chapter 6, §2]. The maximal ideal space of $S(G)$ can be identified with $\widehat{G}$, the dual group of G. Moreover, the Gelfand transform of $f \in S(G)$ is just $\widehat{f}(\chi)$ $(\chi \in \widehat{G})$, the Fourier transform of f. It is known [18] that every Segal algebra satisfies Ditkin's condition at each point of $\widehat{G} \cup \{\infty\}$. Consequently, every Segal algebra is a Ditkin algebra.

For example, the space $L^{1,2}(G) := L^1(G) \cap L^2(G)$ with the norm $\|f\| = \|f\|_1 + \|f\|_2$ is a Segal algebra [14, Chapter 6, §2]. We claim that $L^{1,2}(G)$ does not have the SBP. Assume that for any distinct χ_1 and χ_2 in $\widehat{G}$, there exists $f \in L^{1,2}(G)$ such that $\|f\|_1 + \|f\|_2 \leq 1$, $\widehat{f}(\chi_1) = 1$, and $\widehat{f}(\chi_2) = 0$. Since $\|f\|_1 \geq |\widehat{f}(\chi_1)| = 1$, we have $\|f\|_2 \leq 1 - \|f\|_1 \leq 0$. This is a contradiction.

We shall denote by 1_K the characteristic function of the set K. If K is a measurable subset of a locally compact group, then $|K|$ will denote its Haar measure.

Proposition 2.4. *The algebra $L^{1,2}(G)$ is a boundedly regular Ditkin algebra.*

Proof. We only need to show that $L^{1,2}(G)$ is boundedly regular. If $\widehat{G}$ is discrete, then G is compact and $L^{1,2}(G) = L^2(G)$, so there is nothing to prove. Therefore, assume that $\widehat{G}$ is not discrete. Let $\chi \in \widehat{G}$ and let U be a compact neighborhood of χ. Then there exists a compact symmetric neighborhood V of χ such that $\chi V^2 \subset U$ and $|V| \leq 1$. By Plancherel's Theorem, there exist $h, k \in L^2(G)$ such that $\widehat{h} = 1_V$ and $\widehat{k} = 1_{\chi V}$. We put

$$f(g) = |V|^{-1} h(g) k(g).$$

Then $f \in L^1(G)$ and $\widehat{f} = |V|^{-1}(h * k)$. We can see that $\operatorname{supp} \widehat{f} \subset U$ and $\widehat{f}(\chi) = 1$. Moreover,

$$\|f\|_1 \leq |V|^{-1} \|h\|_2 \|k\|_2 = |V|^{-1} \|\widehat{h}\|_2 \|\widehat{k}\|_2 = |V|^{-1} |V|^{1/2} |\chi V|^{1/2} = 1.$$

On the other hand, since $\widehat{f}$ has compact support, we have $f \in L^2(G)$ and

$$\|f\|_2 = \|\widehat{f}\|_2 \le |V|^{-1}\|\widehat{h}\|_1\|\widehat{k}\|_2 = |V|^{-1}|V|\,|\chi V|^{1/2} = |V|^{1/2} \le 1.$$

Hence, $\|f\| = \|f\|_1 + \|f\|_2 \le 2$. ∎

Another example of a Segal algebra is the following. Let $C_p(G)$ $(1 < p < \infty)$ be the set of all functions $f \in L^1(G)$ for which $\widehat{f} \in L^p(\widehat{G})$. Then $C_p(G)$ with the norm $\|f\| = \|f\|_1 + \|\widehat{f}\|_p$ is a Segal algebra [14, Chapter 6, §2]. As above, we can see that $C_p(G)$ does not have the SBP. The proof of the following proposition is similar to that of the preceding proposition.

PROPOSITION 2.5. *The algebra $C_p(G)$ is a boundedly regular Ditkin algebra.*

Let $\mathrm{BV}(\mathbb{R})$ be the space of all complex-valued functions of bounded variation on $\mathbb{R}$. We put $\mathrm{BVC}_0(\mathbb{R}) = C_0(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$. Then $\mathrm{BVC}_0(\mathbb{R})$ equipped with the pointwise multiplication and the norm $\|f\| = \|f\|_\infty + \mathrm{Var}_{\mathbb{R}}(f)$ is a commutative regular semisimple Banach algebra. Its maximal ideal space can be identified with $\mathbb{R}$. As above, it can be seen that $\mathrm{BVC}_0(\mathbb{R})$ does not have the SBP.

PROPOSITION 2.6. *The algebra* $\mathrm{BVC}_0(\mathbb{R})$ *is a boundedly regular Ditkin algebra.*

Proof. Let $a \in \mathbb{R}$ and let U be a neighborhood of a. Choose $\delta > 0$ so small that $(a - 2\delta, a + 2\delta) \subset U$. Then the triangle function defined by

$$\Delta_\delta(x) = \max(0, 1 - (x - a)/\delta)$$

is in $\mathrm{BVC}_0(\mathbb{R})$. Moreover, $\Delta_\delta(a) = 1$, $\operatorname{supp} \Delta_\delta \subset U$, and $\|\Delta_\delta\| = 3$. Hence, the algebra $BVC_0(\mathbb{R})$ is boundedly regular.

Let us show that $\mathrm{BVC}_0(\mathbb{R})$ satisfies Ditkin's condition at every $a \in \mathbb{R} \cup \{\infty\}$. First, consider the case when $a \in \mathbb{R}$. It is no restriction to assume that $a = 0$. Let $f \in \mathrm{BVC}_0(\mathbb{R})$ be such that $f(0) = 0$. For each $n \in \mathbb{N}$, let $U_n := \{x \in \mathbb{R} : |f(x)| < 1/n\}$. As in [15, A.2.5], define the function f_n by

$$f_n(x) = \begin{cases} -1/n, & f(x) \le -1/n, \\ f(x), & x \in U_n, \\ 1/n, & f(x) \ge 1/n. \end{cases}$$

It can be seen that $f_n - f$ vanishes in a neighborhood of $\{0\}$ and

$$\|f - (f - f_n)\| = \|f_n\| \le 1/n + \mathrm{Var}_{U_n}(f) \to 0 \quad (n \to \infty).$$

For each $n \in \mathbb{N}$, let e_n be the trapezium function defined by $e_n(x) = 1$ if $|x| \le n$ and $e_n(x) = 0$ if $|x| \ge n + 1$. Then $e_n \in BVC_0(\mathbb{R})$ and

$$\lim_{n\to\infty} \|fe_n - f\| = 0, \quad \forall f \in BVC_0(\mathbb{R}).$$

This shows that $\mathrm{BVC}_0(\mathbb{R})$ satisfies Ditkin's condition at ∞. ∎

Let A be a boundedly regular Banach algebra. By Definition 2.2, there exists a constant $C > 0$ such that for each $\phi \in \Sigma_A$ and each neighborhood U of ϕ there exists an element $a \in A$ for which $\widehat{a}(\phi) = 1$, $\mathrm{supp}\widehat{a} \subset U$, and $\|a\| \leq C$. It follows that $\|\phi\| \geq 1/C$ for all $\phi \in \Sigma_A$. This shows that Σ_A is a norm closed subset of A^*. Now, let $\{U_\lambda^\phi\}_{\lambda \in \Lambda}$ be a directed basic neighborhood system of $\phi \in \Sigma_A$. Then there exists a net $(a_\lambda^\phi)_{\lambda \in \Lambda}$ in A such that $\widehat{a_\lambda^\phi}(\phi) = 1$, $\mathrm{supp}\, \widehat{a_\lambda^\phi} \subset U_\lambda^\phi$, and $\|a_\lambda^\phi\| \leq C$ $(\lambda \in \Lambda)$. The net $(a_\lambda^\phi)_{\lambda \in \Lambda}$ will be called a *δ-net* at $\phi \in \Sigma_A$. Notice that if $\psi \in \Sigma_A$, then

$$\lim_\lambda \langle \psi, a_\lambda^\phi \rangle = \begin{cases} 1, & \psi = \phi, \\ 0, & \psi \neq \phi. \end{cases} \tag{2.1}$$

It follows that if F_ϕ is a w^*-limit point of the net $(a_\lambda^\phi)_{\lambda \in \Lambda}$ in A^{**}, then $\|F_\phi\| \leq C$, $F_\phi(\phi) = 1$, and $F_\phi(\psi) = 0$ for all $\psi \in \Sigma_A \setminus \{\phi\}$. Consequently, the space $(\Sigma_A, \mathrm{weak})$ is discrete. Further, if ϕ and ψ are two distinct points of Σ_A, then

$$\|\phi - \psi\| \geq \frac{1}{C} |\langle F_\phi, \phi - \psi \rangle| = \frac{1}{C}.$$

This shows that the space Σ_A is uniformly discrete. We will call F_ϕ a *δ-functional* at $\phi \in \Sigma_A$. In summary, we have the following

PROPOSITION 2.7. *If A is a boundedly regular Banach algebra, then the following assertions hold:*

(a) *Σ_A is a norm closed subset of A^*.*
(b) *The space $(\Sigma_A, \mathrm{weak})$ is discrete.*
(c) *Σ_A is uniformly discrete.*

3. The weak* and norm spectra. In this section, we investigate some basic properties of weak* and norm spectra in boundedly regular Ditkin algebras. Let A be a commutative Banach algebra. Recall that for $f \in A^*$ and $a \in A$, the functional $f \cdot a$ on A is defined by

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle, \quad b \in A.$$

Recall also that for every $f \in A^*$,

$$I_f := \{a \in A : f \cdot a = 0\}$$

is a closed ideal in A. Notice that if A satisfies Ditkin's condition at ∞, then $f \in I_f^\perp$.

The functional $f \in A^*$ is said to be (*weakly*) *almost periodic* on A if the set $\{f \cdot a : a \in A_1\}$ is relatively (weakly) compact, where A_1 is the closed unit ball of A. We will denote by $\mathrm{ap}(A)$ (resp. $\mathrm{wap}(A)$) the set of all almost periodic (resp. weakly almost periodic) functionals on A. Clearly, both $\mathrm{wap}(A)$ and $\mathrm{ap}(A)$ are norm closed A-submodules of A^*. Notice that if

$\phi \in \Sigma_A$ and $a \in A$, then $\phi \cdot a = \widehat{a}(\phi)\phi$ and therefore $\phi \in \mathrm{ap}(A)$. Thus, every $f \in \overline{\mathrm{span}}\,\Sigma_A$ is almost periodic.

As is well-known [2], on the second dual A^{**} of A there exists a Banach algebra multiplication (noncommutative, in general) extending that of A. This multiplication is constructed as follows. Let $a \in A$, $f \in A^*$, and $F, H \in A^{**}$. The elements $H \cdot f$ and $F \cdot H$ are defined by $\langle H \cdot f, a\rangle = \langle H, f \cdot a\rangle$ and $\langle F \cdot H, f\rangle = \langle F, H \cdot f\rangle$.

A *bounded approximate identity* in A is a bounded net $(a_i)_{i\in I}$ such that

$$\lim_i \|aa_i - a\| = 0, \qquad \forall a \in A.$$

Let A be a commutative Banach algebra. Recall that the *weak* spectrum* (*w^*-spectrum* for short) of $f \in A^*$ is defined by

$$\sigma_*(f) = \overline{\{f \cdot a : a \in A\}}^{w^*} \cap \Sigma_A.$$

We will need the following certainly well-known facts (see, for instance, [11]).

Proposition 3.1. *If A is a commutative Banach algebra, then the following assertions hold:*

(a) *For every $f \in A^*$, $\sigma_*(f) = h(I_f)$.*
(b) *For every $f \in A^*$ and $a \in A$,*

$$\sigma_*(f) \cap \{\phi \in \Sigma_A : \widehat{a}(\phi) \neq 0\} \subset \sigma_*(f \cdot a).$$

(c) *If the algebra A is Tauberian and if the linear span of $\{ab : a, b \in A\}$ is dense in A, then $\sigma_*(f) \neq \emptyset$ for all $f \in A^* \setminus \{0\}$.*
(d) *If the algebra A is regular and semisimple, then for every $f \in A^*$ and $a \in A$,*

$$\sigma_*(f \cdot a) \subset \sigma_*(f) \cap \operatorname{supp} \widehat{a}.$$

It follows from Proposition 3.1(c) that if A is a Ditkin algebra, then $\sigma_*(f) \neq \emptyset$ whenever $f \in A^* \setminus \{0\}$.

We shall need the following

Lemma 3.2. *Let A be a regular semisimple Banach algebra. If $(f_\lambda)_{\lambda\in\Lambda}$ is a net in A^* converging to $f \in A^*$ in the w^*-topology, then*

$$\sigma_*(f) \subset \bigcap_{\lambda\in\Lambda} \Big(\overline{\bigcup_{\mu\geq\lambda} \sigma_*(f_\mu)}^{w^*}\Big).$$

Proof. First, we claim that if $(I_\gamma)_{\gamma\in\Gamma}$ is a family of closed ideals in A, then

$$h\Big(\bigcap_{\gamma\in\Gamma} I_\gamma\Big) = \overline{\bigcup_{\gamma\in\Gamma} h(I_\gamma)}^{w^*}.$$

To see this, let $S_\gamma := h(I_\gamma)$ and $S := \overline{\bigcup_{\gamma\in\Gamma} S_\gamma}^{w^*}$. Since $\bigcap_{\gamma\in\Gamma} I_\gamma \subset I_\gamma$, we have $S_\gamma \subset h(\bigcap_{\gamma\in\Gamma} I_\gamma)$ for every $\gamma \in \Gamma$ and therefore $S \subset h(\bigcap_{\gamma\in\Gamma} I_\gamma)$. For the reverse inclusion, let $\phi \in \Sigma_A \setminus S$. Since the algebra A is regular, there exists $a \in A$ such that $\widehat{a}(\phi) \neq 0$ and $\widehat{a}$ vanishes in a neighborhood of S. Consequently, $a \in J_S$, where J_S is the smallest closed ideal in A whose hull is S. We see that $J_S \subset I_\gamma$ for every $\gamma \in \Gamma$ and so $J_S \subset \bigcap_{\gamma\in\Gamma} I_\gamma$. Hence, $a \in \bigcap_{\gamma\in\Gamma} I_\gamma$, but $\widehat{a}(\phi) \neq 0$. This means that $\phi \notin h(\bigcap_{\gamma\in\Gamma} I_\gamma)$.

Now, we claim that $\bigcap_{\mu\geq\lambda} I_{f_\mu} \subset I_f$ for every $\lambda \in \Lambda$. Indeed, if $a \in \bigcap_{\mu\geq\lambda} I_{f_\mu}$, then $f_\mu \cdot a = 0$ for all $\mu \geq \lambda$. This implies $0 = \langle f_\mu \cdot a, b\rangle = \langle f_\mu, ab\rangle$ for all $\mu \geq \lambda$ and $b \in A$. Since $f_\mu \to f$ in the w^*-topology, we have

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle = \lim_\mu \langle f_\mu, ab\rangle = 0, \qquad \forall b \in A.$$

Thus, $f \cdot a = 0$ and so $a \in I_f$. Consequently,

$$\sigma_*(f) = h(I_f) \subset h\Big(\bigcap_{\mu\geq\lambda} I_{f_\mu}\Big) = \overline{\bigcup_{\mu\geq\lambda} h(I_{f_\mu})}^{w^*} = \overline{\bigcup_{\mu\geq\lambda} \sigma_*(f_\mu)}^{w^*}, \qquad \forall\lambda \in \Lambda,$$

and so $\sigma_*(f) = \bigcap_{\lambda\in\Lambda}(\overline{\bigcup_{\mu\geq\lambda} \sigma_*(f_\mu)}^{w^*})$. ∎

Let A be a commutative Banach algebra. Recall that the *norm spectrum* [17] of $f \in A^*$ is defined by

$$\sigma(f) = \overline{\{f \cdot a : a \in A\}} \cap \Sigma_A.$$

Clearly, $\sigma(f) \subset \sigma_*(f)$. Note that $\sigma(f)$ may be empty even if f is a nonzero element of $L^\infty(\mathbb{R})$. For instance if $f \in C_0(\mathbb{R})$, then $\sigma(f) = \emptyset$. As mentioned in the introduction, $\sigma(f) \neq \emptyset$ for all $f \in \overline{\operatorname{span}}\, \Sigma_{L^1(\mathbb{R})} \setminus \{0\}$. But this is not the case for general Banach algebras.

We have the following

THEOREM 3.3. *Let A be a boundedly regular Ditkin algebra and let F_ϕ be a δ-functional at $\phi \in \Sigma_A$. Then the following assertions hold:*

(a) *For every $f \in \operatorname{wap}(A)$,*

$$\sigma(f) = \{\phi \in \Sigma_A : F_\phi \cdot f \neq 0\}.$$

(b) *If A has a bounded approximate identity, then for every $f \in \operatorname{wap}(A)$,*

$$\sigma(f) = \{\phi \in \Sigma_A : \langle F_\phi, f\rangle \neq 0\}.$$

Proof. (a) Let $f \in \operatorname{wap}(A)$ and $\phi \in \Sigma_A$. Assume that $F_\phi \cdot f \neq 0$. Let us show that $\phi \in \sigma(f)$. For a given $a \in A$, we have

$$\lim_\lambda \langle f \cdot a_\lambda^\phi, a\rangle = \lim_\lambda \langle f \cdot a, a_\lambda^\phi\rangle = \langle F_\phi, f \cdot a\rangle = \langle F_\phi \cdot f, a\rangle,$$

where $(a_\lambda^\phi)_{\lambda\in\Lambda}$ is a δ-net at $\phi \in \Sigma_A$. This shows that $f \cdot a_\lambda^\phi \to F_\phi \cdot f$ in the w^*-topology. By Proposition 3.1(d), since

$$\sigma_*(f \cdot a_\lambda^\phi) \subset \sigma_*(f) \cap \operatorname{supp} \widehat{a_\lambda^\phi},$$

we have $\sigma_*(f \cdot a_\lambda^\phi) \subset U_\lambda^\phi$. Recall that $\{U_\lambda^\phi\}_{\lambda\in\Lambda}$ is a directed basic neighborhood system of $\phi \in \Sigma_A$. Taking into account that $\bigcap_{\lambda\in\Lambda} U_\lambda^\phi = \{\phi\}$, by Lemma 3.2 we obtain $\sigma_*(F_\phi \cdot f) \subset \{\phi\}$. Since $F_\phi \cdot f \neq 0$, we have $\sigma_*(F_\phi \cdot f) = \{\phi\}$. Also since $\{\phi\}$ is an s-set, $I_{F_\phi \cdot f}$ is the unique ideal of A whose hull is $\{\phi\}$. Therefore, $I_{F_\phi\cdot f} = I_{\{\phi\}}$. Consequently, $F_\phi \cdot f \in I_{\{\phi\}}^\perp = \mathbb{C}\phi$, so that there exists $c(\phi) \in \mathbb{C} \setminus \{0\}$ such that $F_\phi \cdot f = c(\phi)\phi$. Thus, $f \cdot a_\lambda^\phi \to c(\phi)\phi$ in the w^*-topology. Since $f \in \mathrm{wap}(A)$ and the net $(a_\lambda^\phi)_{\lambda\in\Lambda}$ is bounded, the set $\{f \cdot a_\lambda^\phi : \lambda \in \Lambda\}$ is relatively weakly compact. From this, we deduce that $f \cdot a_\lambda^\phi \to c(\phi)\phi$ weakly. Thus we have

$$\phi \in \overline{\{f \cdot a : a \in A\}}^w = \overline{\{f \cdot a : a \in A\}}.$$

This shows that $\phi \in \sigma(f)$.

For the reverse inclusion, suppose that for some $\phi \in \sigma(f)$, $F_\phi \cdot f = 0$. Then, as above, $f \cdot a_\lambda^\phi \to 0$ weakly. It follows that there exists a net $(b_i^\phi)_{i\in I}$ in the convex hull of $\{a_\lambda^\phi : \lambda \in \Lambda\}$ such that $\lim_i \|f \cdot b_i^\phi\| = 0$. Clearly, $\|b_i^\phi\| \le C$ and $\widehat{b_i^\phi}(\phi) = 1$ $(i \in I)$. On the other hand, since $\phi \in \sigma(f)$, there exists a net $(c_j)_{j\in J}$ in A such that $\lim_j \|f \cdot c_j - \phi\| = 0$. Let $\varepsilon > 0$. Then $\|f \cdot c_{j_0} - \phi\| < \varepsilon$ for some $j_0 \in J$. It follows that for all $i \in I$,

$$\|f \cdot c_{j_0} \cdot b_i^\phi - \phi \cdot b_i^\phi\| < C\varepsilon.$$

Since $\widehat{b_i^\phi}(\phi) = 1$, we have $\phi \cdot \widehat{b_i^\phi}(\phi) = \widehat{b_i^\phi}(\phi)\phi = \phi$ $(i \in I)$. Thus, we obtain

$$\|f \cdot c_{j_0} \cdot b_i^\phi - \phi\| < C\varepsilon \quad (i \in I).$$

As $\lim_i \|f \cdot b_i^\phi\| = 0$, by passing to the limit in the preceding inequality we get $\|\phi\| < C\varepsilon$. This contradiction completes the proof.

(b) Let $\phi \in \sigma(f)$ and let $(a_i)_{i\in I}$ be a bounded approximate identity for A. As in the proof of (a), there exists a $c(\phi) \in \mathbb{C}\setminus\{0\}$ such that $F_\phi \cdot f = c(\phi)\phi$. It can be seen that $\lim_i \widehat{a}_i(\phi) = 1$. Notice also that $f \cdot a_i \to f$ in the w^*-topology. Since $f \in \mathrm{wap}(A)$, the set $\{f \cdot a_i : i \in I\}$ is relatively weakly compact. This clearly implies that $f \cdot a_i \to f$ weakly. Consequently,

$$\langle F_\phi, f\rangle = \lim_i \langle F_\phi, f \cdot a_i\rangle = \lim_i \langle F_\phi \cdot f, a_i\rangle = c(\phi) \lim_i \widehat{a}_i(\phi) = c(\phi) \neq 0.$$

Conversely, assume that $\phi \notin \sigma(f)$. Then, by (a), $F_\phi \cdot f = 0$ and so

$$\langle F_\phi, f\rangle = \lim_i \langle F_\phi, f \cdot a_i\rangle = \lim_i \langle F_\phi \cdot f, a_i\rangle = 0. \ \blacksquare$$

PROPOSITION 3.4. *If A is a boundedly regular Ditkin algebra, then the following assertions hold:*

(a) *If $f \in \mathrm{wap}(A)$ and $\sigma(f)$ is nonempty, then for every $a \in A$,*

$$\sigma(f \cdot a) = \{\phi \in \sigma(f) : \widehat{a}(\phi) \neq 0\}.$$

(b) *If* $f \in \operatorname{ap}(A)$, *then* $\sigma(f)$ *is countable.*

(c) *If* A *has a bounded approximate identity, then for every* $f \in \operatorname{wap}(A)$, $\sigma(f)$ *is countable. Moreover, if* $f \in \overline{\operatorname{span}}\, \Sigma_A$, $\phi \in \Sigma_A$ *and* $(a_\lambda^\phi)_{\lambda\in\Lambda}$ *is a* δ*-net at* ϕ *then* $C_\phi(f) := \lim_\lambda \langle f, a_\lambda^\phi\rangle$ *exists and* $\sigma(f) = \{\phi \in \Sigma_A : C_\phi(f) \neq 0\}$.

Proof. (a) Let $\phi \in \sigma(f\cdot a)$. We already noted in the proof of Theorem 3.3 that

$$\sigma(f) = \{\phi \in \Sigma_A : \exists c(\phi) \in \mathbb{C}\setminus\{0\},\ F_\phi \cdot f = c(\phi)\phi\}, \tag{3.1}$$

where F_ϕ is a δ-functional at $\phi \in \Sigma_A$. By Theorem 3.3, $F_\phi \cdot f \cdot a \neq 0$, which implies $F_\phi \cdot f \neq 0$ and therefore $\phi \in \sigma(f)$. Further, since $F_\phi \cdot f = c(\phi)\phi$, $c(\phi) \neq 0$, we have $0 \neq F_\phi \cdot f \cdot a = c(\phi)\widehat{a}(\phi)\phi$. It follows that $\widehat{a}(\phi) \neq 0$.

Conversely, let $\phi \in \sigma(f)$ and $a \in A$ with $\widehat{a}(\phi) \neq 0$. By Theorem 3.3, $F_\phi \cdot f = c(\phi)\phi$, $c(\phi) \neq 0$. It follows that $F_\phi \cdot f \cdot a = c(\phi)\widehat{a}(\phi)\phi \neq 0$ and therefore $\phi \in \sigma(f \cdot a)$.

(b) For a given $n \in \mathbb{N}$, we put

$$\sigma_n(f) = \{\phi \in \sigma(f) : \|F_\phi \cdot f\| \geq 1/n\}.$$

By Theorem 3.3, we can write

$$\sigma(f) = \bigcup_{n=1}^{\infty} \sigma_n(f).$$

Hence, we only need to show that each $\sigma_n(f)$ is finite. Since $f \in \operatorname{ap}(A)$, the operator $T_f : A \to A^*$ defined by $T_f a = f \cdot a$ is compact. Consequently, the operator $T_f^* : A^{**} \to A^*$, where $T_f^* F = F\cdot f$ ($F \in A^{**}$), is also compact. Since the set $\{F_\phi : \phi \in \Sigma_A\}$ is bounded, $\{F_\phi \cdot f : \phi \in \sigma(f)\}$ is a relatively compact subset of A^*. From the identity (3.1), we deduce that $\{c(\phi)\phi : \phi \in \sigma(f)\}$ is a relatively compact subset of A^*. Since the set $\{|c(\phi)|\,\|\phi\| : \phi \in \sigma(f)\}$ is bounded, there exists a constant $L > 0$ such that $|c(\phi)|\,\|\phi\| \leq L$ for all $\phi \in \sigma(f)$. Also since $\|\phi\| \geq 1/C$, we have $|c(\phi)| \leq LC$ for all $\phi \in \sigma(f)$. On the other hand, $|c(\phi)| \geq 1/n$ for all $\phi \in \sigma_n(f)$. Thus,

$$1/n \leq |c(\phi)| \leq LC, \qquad \forall \phi \in \sigma_n(f).$$

From this and from Proposition 2.7(a), we deduce that $\sigma_n(f)$ is a relatively compact subset of $(\Sigma_A, \|\cdot\|)$. By Proposition 2.7(c), since Σ_A is uniformly discrete, it follows that $\sigma_n(f)$ is a finite set.

(c) As in the proof of (b), we can see that if $f \in \operatorname{wap}(A)$, then

$$\{c(\phi)\phi : \phi \in \sigma(f)\}$$

is a relatively weakly compact subset of A^*. Moreover, $1/n \leq |c(\phi)| \leq LC$ for all $\phi \in \sigma_n(f)$. Further, since A has a bounded approximate identity, (Σ_A, weak) is weakly closed [17]. So, $\sigma_n(f)$ is a relatively compact subset of (Σ_A, weak). By Proposition 2.7(b), since (Σ_A, weak) is discrete, it follows that $\sigma_n(f)$ is a finite set.

If $f \in \overline{\text{span}}\, \Sigma_A$, then for a given $\varepsilon > 0$, there exist distinct characters $\phi_1, \dots, \phi_n$ in Σ_A and $c_1, \dots, c_n$ in $\mathbb{C} \setminus \{0\}$ such that

$$\|f - c_1\phi_1 - \dots - c_n\phi_n\| < \varepsilon.$$

This implies

$$|\langle f, a_\lambda^\phi\rangle - c_1\widehat{a_\lambda^\phi}(\phi_1) - \dots - c_n\widehat{a_\lambda^\phi}(\phi_n)| \le C\varepsilon.$$

Now, from the relations

$$|\langle f, a_\lambda^\phi\rangle - \langle f, a_\mu^\phi\rangle| \le 2C\varepsilon + |c_1\widehat{a_\lambda^\phi}(\phi_1) - c_1\widehat{a_\mu^\phi}(\phi_1)| + \dots + |c_n\widehat{a_\lambda^\phi}(\phi_n) - c_n\widehat{a_\mu^\phi}(\phi_n)|$$

and from the identity (2.1), we deduce that $C_\phi(f) := \lim_\lambda \langle f, a_\lambda^\phi\rangle$ exists. Notice that $C_\phi(f) = \langle F_\phi, f\rangle$, where F_ϕ is a δ-functional at $\phi \in \Sigma_A$. Hence, by Theorem 3.3(b), we have $\sigma(f) = \{\phi \in \Sigma_A : C_\phi(f) \ne 0\}$. ∎

Let X be a locally compact Hausdorff space and let $M(X)$ $(= C_0(X)^*)$ be the Banach space of all finite regular complex Borel measures on X. By $M_c(X)$ and $M_d(X)$, respectively, we denote the spaces of all continuous and all discrete measures in $M(X)$. Note that $\overline{\text{span}}\{\delta_x : x \in X\} = M_d(X)$, where δ_x is a Dirac measure.

If A is a commutative Banach algebra, then every $\mu \in M(\Sigma_A)$ can be considered as an element of A^* with respect to the duality

$$\langle \mu, a\rangle = \int_{\Sigma_A} \widehat{a}(\phi)\, d\mu(\phi).$$

Since $\mu \cdot a = \widehat{a}(\phi) d\mu(\phi)$ $(a \in A)$, we have $I_\mu = I_{\operatorname{supp}\mu}$. It follows that $\sigma_*(\mu) = h(I_\mu) = \overline{\operatorname{supp}\mu}^{hk}$, where hk denotes the hull-kernel topology. Consequently, if A is a regular Banach algebra, then $\sigma_*(\mu) = \operatorname{supp}\mu$.

The following result in the case when $A = A_2(G)$ (the Fourier algebra of G) was proved in [3, Theorem 2.8]. The proof is similar.

LEMMA 3.5. *Let A be a commutative Banach algebra. If $\mu \in M(\Sigma_A)$, then $\mu \in \text{wap}(A)$.*

Proof. It is enough to show that the operator $T_\mu : A \to A^*$ defined by $T_\mu a = \mu \cdot a = \widehat{a}(\phi)\, d\mu(\phi)$ $(a \in A)$ is weakly compact. It is no restriction to assume that μ is a positive measure with $\|\mu\| = 1$. Consider the linear map $S : L^2(\Sigma_A, \mu) \to A^*$ defined by $Sf = f(\phi)\, d\mu(\phi)$ $(\phi \in \Sigma_A)$. Then

$$\|Sf\|_{A^*} \le \int_{\Sigma_A} |f(\phi)|\, d\mu(\phi) \le \|f\|_2$$

and thus S is bounded. Since the space $L^2(\Sigma_A, \mu)$ is reflexive, S is weakly compact. Notice also that $T_\mu = S \circ \Gamma$, where $\Gamma : a \mapsto \widehat{a}$ is the Gelfand homomorphism. It follows that the operator T_μ is weakly compact. ∎

THEOREM 3.6. *If A is a boundedly regular Ditkin algebra, then for every $\mu \in M(\Sigma_A)$,*

$$\sigma(\mu) = \{\phi \in \Sigma_A : \mu\{\phi\} \neq 0\}.$$

Proof. By Lemma 3.5, $\mu \in \mathrm{wap}(A)$. Therefore, by Theorem 3.3 it is enough to show that $F_\phi \cdot \mu = \mu\{\phi\}\phi$, where F_ϕ is a δ-functional at $\phi \in \Sigma_A$. Let $\{U_\lambda^\phi\}_{\lambda \in \Lambda}$ be a directed basic neighborhood system of $\phi \in \Sigma_A$ and let $(a_\lambda^\phi)_{\lambda \in \Lambda}$ be the corresponding δ-net. We already noted above that $\mu \cdot a_\lambda^\phi \to F_\phi \cdot \mu$ in the w^*-topology. Hence, we only need to show that $\mu \cdot a_\lambda^\phi \to \mu\{\phi\}\phi$ in the w^*-topology. Let us show that $\mu \cdot a_\lambda^\phi \to \mu\{\phi\}\phi$ even in norm. In view of regularity of μ, for a given $\varepsilon > 0$, there exists a neighborhood U_λ^ϕ such that

$$|\mu|(U_\lambda^\phi \setminus \{\phi\}) < \varepsilon/C.$$

Consequently,

$$\begin{aligned} \|\mu \cdot a_\lambda^\phi - \mu\{\phi\}\phi\| &= \sup_{\|a\| \le 1} |\langle \mu, a_\lambda^\phi \cdot a\rangle - \mu\{\phi\}\widehat{a}(\phi)| \\ &= \sup_{\|a\| \le 1} \Big| \int_{\Sigma_A} \widehat{a_\lambda^\phi}(\psi)\widehat{a}(\psi)\, d\mu(\psi) - \mu\{\phi\}\widehat{a}(\phi)\Big| \\ &= \sup_{\|a\| \le 1} \Big| \int_{U_\lambda^\phi \setminus \{\phi\}} \widehat{a_\lambda^\phi}(\psi)\widehat{a}(\psi)\, d\mu(\psi)\Big| \le C|\mu|(U_\lambda^\phi \setminus \{\phi\}) < \varepsilon. \end{aligned}$$ ∎

As a consequence of Theorem 3.6, we have the following

COROLLARY 3.7. *If A is a boundedly regular Ditkin algebra, then the following assertions hold:*

(a) *If $\mu \in \overline{M_d(\Sigma_A)} \setminus \{0\}$, then $\sigma(\mu) \neq \emptyset$.*
(b) *If $f \in \overline{M_c(\Sigma_A)}$, then $\sigma(f) = \emptyset$.*

Proof. (a) follows from the preceding theorem.

(b) If $f \in \overline{M_c(\Sigma_A)}$, then there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of continuous measures on Σ_A such that $\mu_n \to f$ in the A^* norm. It follows that $F_\phi \cdot \mu_n \to F_\phi \cdot f$ in the A^* norm, where F_ϕ is a δ-functional at $\phi \in \Sigma_A$. Taking into account Theorem 3.3, Lemma 3.5, and Theorem 3.6, we have $F_\phi \cdot \mu_n = 0$ for all $n \in \mathbb{N}$. Hence, $F_\phi \cdot f = 0$ for all $\phi \in \Sigma_A$. On the other hand, since $\mathrm{wap}(A)$ is norm closed, by Lemma 3.5, $f \in \mathrm{wap}(A)$. Now, it follows from Theorem 3.3 that $\sigma(f) = \emptyset$. ∎

If G is a locally compact group, then $M(G)$ can be considered as a linear subset of $A_p(G)^*$. Let $\mathrm{PF}_p(G)$ denote the norm closure of $L^1(G)$

in $A_p(G)^*$. As proved in [6, Lemma 3.14], $\sigma(\mu) = \{g \in G : \mu\{g\} \neq 0\}$ for all $\mu \in \overline{\operatorname{span}}\, \Sigma_{A_p(G)} \cap M(G)$, and if G nondiscrete, then $\sigma(f) = \emptyset$ for all $f \in \overline{\operatorname{span}}\, \Sigma_{A_p(G)} \cap \mathrm{PF}_p(G)$.

However, we have the following

Corollary 3.8. *If G is amenable, then the following assertions hold:*

(a) *For every $\mu \in A_p(G)^* \cap M(G)$,*

$$\sigma(\mu) = \{g \in G : \mu\{g\} \neq 0\}.$$

(b) *If G is nondiscrete, then $\sigma(f) = \emptyset$ for all $f \in \mathrm{PF}_p(G)$.*

Now, we observe the following description of the discreteness of Σ_A in terms of norm spectra (see also [6, Theorem 3.4]).

Theorem 3.9. *Let A be a boundedly regular Ditkin algebra. Then the space Σ_A is discrete if and only if $\sigma(f) = \sigma_*(f)$ for all $f \in l^1(\Sigma_A)$.*

Proof. Assume that Σ_A is discrete. Let $f \in A^*$ and $\phi_0 \in \sigma_*(f)$. Then there exists $a \in A$ such that $\widehat{a}(\phi_0) = 1$ and $\widehat{a}(\phi) = 0$ for all $\phi \in \sigma_*(f) \setminus \{\phi_0\}$. By Proposition 3.1(d), $\sigma_*(f \cdot a) \subset \{\phi_0\}$. On the other hand, by Proposition 3.1(b), $\{\phi_0\} \subset \sigma_*(f \cdot a)$. Hence, $\sigma_*(f \cdot a) = \{\phi_0\}$. As in the proof of Theorem 3.3(a), there exists $c \neq 0$ such that $f \cdot a = c\phi_0$. This shows that $\phi_0 \in \sigma(f)$.

Conversely, suppose $\sigma(f) = \sigma_*(f)$ for all $f \in l^1(\Sigma_A)$, but Σ_A is not discrete. Then, Σ_A contains a countable nonclosed subset $\{\phi_n\}_{n\in\mathbb{N}}$ [6, Lemma 3.3]. If we put $f = \sum_{n=1}^{\infty} 2^{-n}\phi_n$, then $f \in l^1(\Sigma_A)$. Consequently, for a given $\phi \in \Sigma_A$, we can write

$$F_\phi \cdot f = \sum_{n=1}^{\infty} 2^{-n}\langle F_\phi, \phi_n\rangle \phi_n,$$

where F_ϕ is the corresponding δ-functional. Recall that $F_\phi(\phi) = 1$ and $F_\phi(\psi) = 0$ for all $\psi \in \Sigma_A \setminus \{\phi\}$. It easily follows from Theorem 3.3 that $\sigma(f) = \{\phi_n : n \in \mathbb{N}\}$. Now, let us show that

$$\sigma_*(f) = \overline{\{\phi_n : n \in \mathbb{N}\}}^{w^*}.$$

Clearly, $\overline{\{\phi_n : n \in \mathbb{N}\}}^{w^*} \subset \sigma_*(f)$. For the reverse inclusion, assume that there exists $\phi_0 \in \sigma_*(f) \setminus \overline{\{\phi_n : n \in \mathbb{N}\}}^{w^*}$. Then there exists $a \in A$ such that $\widehat{a}(\phi_0) \neq 0$ and $\widehat{a}(\phi_n) = 0$ for all $n \in \mathbb{N}$. It follows that

$$f \cdot a = \sum_{n=1}^{\infty} c_n \widehat{a}(\phi_n)\phi_n = 0.$$

Hence, $a \in I_f$ and therefore $\widehat{a}(\phi_0) = 0$. So $\sigma(f) \neq \sigma_*(f)$, a contradiction. ∎

Let A be a commutative Banach algebra. An element $a \in A$ is said to be (*weakly*) *compact* if the map $\tau_a : A \to A$ defined by $\tau_a(b) = ab$ is (weakly) compact. It is well known [2, Lemma 3] that each $a \in A$ is weakly compact if and only if A is an ideal in A^{**}. In [17, Theorem 3.1], it is proved that if A is an ideal in its second dual, then the weak and weak* topologies coincide on Σ_A. Using this, in summary we have the following

COROLLARY 3.10. *If A is a boundedly regular Ditkin algebra, then the following assertions are equivalent:*

(a) *Each $a \in A$ is compact.*
(b) *Each $a \in A$ is weakly compact.*
(c) *The algebra A is an ideal in A^{**}.*
(d) *The space Σ_A is discrete.*
(e) *$\sigma(f) = \sigma_*(f)$ for all $f \in l^1(\Sigma_A)$.*

The following theorem characterizes synthesizable ideals in boundedly regular Ditkin algebras.

THEOREM 3.11. *If A is a boundedly regular Ditkin algebra, then the following assertions hold:*

(a) *For every $f \in l^1(\Sigma_A) \setminus \{0\}$, $\sigma_*(f)$ is w^*-separable and the ideal I_f is synthesizable.*
(b) *If I is a proper synthesizable closed ideal of A and if $h(I)$ is w^*-separable, then $I = I_f$ for some $f \in l^1(\Sigma_A) \setminus \{0\}$.*

Proof. (a) Let $f = \sum_{n=1}^{\infty} c_n \phi_n$, where $\{\phi_n\}_{n\in\mathbb{N}} \subset \Sigma_A$ and $\sum_{n=1}^{\infty} |c_n| < \infty$. As in the proof of Theorem 3.9, we have $\sigma(f) = \{\phi_1, \phi_2, \dots\}$ and

$$\sigma_*(f) = \overline{\{\phi_n : n \in \mathbb{N}\}}^{w^*}.$$

Let us show that the ideal I_f is synthesizable. Let $a \in A$ be such that $\widehat{a}$ vanishes on $\sigma_*(f)$. It follows that $\widehat{a}(\phi_n) = 0$ for all $n \in \mathbb{N}$. Thus, we have

$$f \cdot a = \sum_{n=1}^{\infty} c_n \widehat{a}(\phi_n) \phi_n = 0,$$

and so $a \in I_f$.

(b) Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a w^*-dense sequence in $h(I)$ and let $f = \sum_{n=1}^{\infty} 2^{-n} \phi_n$. As in the proof of Theorem 3.9, we have

$$h(I) = \overline{\sigma(f)}^{w^*} = \sigma_*(f).$$

By (a), the ideal I_f is synthesizable and therefore $I_f = I_{\sigma_*(f)}$. On the other hand, since the ideal I is synthesizable, we can write

$$I = I_{h(I)} = I_{\sigma_*(f)} = I_f. \ \blacksquare$$

4. Nonemptiness of the norm spectrum. In this section, G will be a locally compact abelian group with the Haar measure, and $L^1(G)$ the group algebra of G. It will be convenient to consider the following pairing between the spaces $L^\infty(G)$ and $L^1(G)$: $\langle f, k\rangle = \int_G f(-g)k(g)\,dg$, where $f \in L^\infty(G)$ and $k \in L^1(G)$. We will consider the class of Banach algebras A such that there exists a continuous homomorphism $\omega : L^1(G) \to A$ with dense range. The *spectrum* of ω, denoted by $\mathrm{sp}(\omega)$, is defined as the hull of the closed ideal $I_\omega := \ker \omega$. Standard techniques in Banach algebras show that ω^* maps Σ_A homeomorphically onto $\mathrm{sp}(\omega)$. More precisely, each $\chi \in \mathrm{sp}(\omega)$ corresponds to an element $\phi_\chi \in \Sigma_A$, where $\langle \phi_\chi, \omega(k)\rangle = \widehat{k}(\chi)$, the Fourier transform of $k \in L^1(G)$. Moreover, each $\phi \in \Sigma_A$ is of this form. As shown in [12, Corollary 2], $\mathrm{ap}(A) = \overline{\mathrm{span}}\,\Sigma_A$.

The class of Banach algebras satisfying the above conditions is quite rich. In general, these algebras arise in the following way. Let X be a Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. Recall that a *representation* of G on X is a strongly continuous homomorphism T of G into $B(X)$ with $T(0) = I$. A representation T is said to be *bounded* if there exists $C > 0$ such that $\|T_g\| \le C$ $(g \in G)$. We will also consider the adjoint operators T_g^* on X^*, but we note that $g \mapsto T_g^*$ may not be a representation of G on X^* as strong continuity may fail.

If T is a bounded representation of G on X, then for every $k \in L^1(G)$ we can define $\widehat{k}(T) \in B(X)$ by $\widehat{k}(T)x = \int_G k(g)T_g x\,dg$ $(x \in X)$. The map $k \mapsto \widehat{k}(T)$ is a continuous homomorphism from $L^1(G)$ into $B(X)$. We let $L_T(G)$ denote the closure of the set $\{\widehat{k}(T) : k \in L^1(G)\}$ in the operator-norm topology. Then the algebras $L_T(G)$ satisfy the above conditions imposed on A. If $X = L^p(G)$ $(1 < p < \infty)$ and T is the regular representation on $L^p(G)$, then $L_T(G)$ coincides with $\mathrm{PF}_p(G)$. Note also that $L_T(G)$ is not semisimple in general.

Before stating the main result of this section, we shall need some preliminaries. Let G be a locally compact abelian group. Recall [13, p. 137] that a net $\{K_i\}_{i\in I}$ of compact subsets of G is called a *Følner* (or *summing*) *net* for G if the following conditions are satisfied:

(i) $|K_i| > 0$ for each $i \in I$;
(ii) $K_i \subset K_j$ if $i \le j$;
(iii) $G = \bigcup_{i\in I} \operatorname{int} K_i$;
(iv) $|(g + K_i) \vartriangle K_i|/|K_i| \to 0$ uniformly for g in any compact subset of G.

As is known [13, p. 137], there exists a Følner net for G.

In the following, let $\mathrm{AP}(G, X)$ be the space of all almost periodic functions on G with values in the Banach space X. Then $\mathrm{AP}(G, X)$ admits a unique *invariant mean* Φ,

$$\Phi(f) = \lim_i \frac{1}{|K_i|} \int_{K_i} f(g)\,dg, \qquad f \in \mathrm{AP}(G, X),$$

where $\{K_i\}_{i\in I}$ is a Følner net for G [13, p. 189]. The *Fourier–Bohr coefficients* of $f \in \mathrm{AP}(G, X)$ are defined by

$$C_\chi(f) = \lim_i \frac{1}{|K_i|} \int\limits_{K_i} \overline{\chi(g)} f(g)\, dg, \qquad \chi \in \widehat{G}.$$

The *Bohr spectrum* $\mathrm{sp_B}(f)$ of $f \in \mathrm{AP}(G, X)$ is defined by

$$\mathrm{sp_B}(f) = \{\chi \in \widehat{G} : C_\chi(f) \neq 0\}.$$

It follows from the uniqueness theorem that $\mathrm{sp_B}(f) \neq \emptyset$ whenever $f \neq 0$.

The main result of this section is the following

THEOREM 4.1. *Let A be a commutative Banach algebra. If there exists a continuous homomorphism $\omega : L^1(G) \to A$ with dense range, then $\sigma(f) \neq \emptyset$ for every $f \in \mathrm{ap}(A) \setminus \{0\}$.*

Proof. Let $f \in \mathrm{ap}(A) \setminus \{0\}$. We must show that there exists a net $(k_i)_{i\in I}$ in $L^1(G)$ and a character χ in $\mathrm{sp}(\omega)$ such that

$$\lim_i \|f \cdot \omega(k_i) - \phi_\chi\| = 0.$$

For a given $g \in G$, define the operator T_g on $\omega(L^1(G))$ by $T_g\omega(k) = \omega(k_g)$, where $k \in L^1(G)$ and $k_g(s) = k(s+g)$. Let $(e_i)_{i\in I}$ be an approximate identity for $L^1(G)$, bounded by one. Since

$$\omega((e_i)_g)\omega(k) \to \omega(k_g),$$

we have

$$\|T_g\omega(k)\| = \|\omega(k_g)\| \leq \|\omega\|\, \|\omega(k)\|.$$

On the other hand,

$$\|T_g\omega(k) - \omega(k)\| = \|\omega(k_g) - \omega(k)\| \leq \|\omega\|\, \|k_g - k\|_1 \to 0 \qquad (g \to 0).$$

Thus, since $\omega(L^1(G))$ is dense in A, the mapping $g \mapsto T_g$ can be extended to the whole A as a bounded representation which we also denote by T.

It is easy to verify that the net $(\omega(e_i))_{i\in I}$ is a bounded approximate identity for A. Now, let $\varphi \in \mathrm{ap}(A)$. Since the set $\{\varphi \cdot \omega(a_i) : i \in I\}$ is relatively compact and $\varphi \cdot \omega(a_i) \to \varphi$ in the w^*-topology, it follows that $\varphi \cdot \omega(a_i) \to \varphi$ in norm. Also, since $\mathrm{ap}(A)$ is a Banach A-module, by the Cohen–Hewitt Factorization Theorem [5, 32.22] we have

$$\mathrm{ap}(A) = \{\varphi \cdot a : \varphi \in \mathrm{ap}(A),\, a \in A\}.$$

Consequently, f can be written as $f = \varphi \cdot a$ for some $\varphi \in \mathrm{ap}(A)$ and $a \in A$. Since the set $\{T_g a : g \in G\}$ is bounded and $\varphi \in \mathrm{ap}(A)$, from the identity $T_g^* f = \varphi \cdot (T_g a)$ (which can be easily verified) we deduce that the set $\{T_g^* f : g \in G\}$ is relatively norm compact. Hence, the function $g \mapsto T_g^* f$ is in $\mathrm{AP}(G, A^*)$.

It follows from the uniqueness theorem that there exist $\chi \in \widehat{G}$ and $f_\chi \in A^* \setminus \{0\}$ such that

$$\lim_i \frac{1}{|K_i|} \int_{K_i} \overline{\chi(s)}(T_s^* f)\, ds = f_\chi, \tag{4.1}$$

where $\{K_i\}_{i\in I}$ is a Følner net for G. Consequently, we have

$$f_\chi = \lim_i \frac{1}{|K_i|} \int_{K_i} \overline{\chi(s-g)}(T_{s-g}^* f)\, ds = \chi(g) T_{-g}^* f_\chi,$$

and so

$$T_{-g}^* f_\chi = \overline{\chi(g)} f_\chi \qquad (g \in G).$$

It follows that for every $k \in L^1(G)$,

$$\int_G k(g)(T_{-g}^* f_\chi)\, dg = \widehat{k}(\chi) f_\chi.$$

On the other hand, it is easy to check that for all $k \in L^1(G)$ and $f \in A^*$,

$$\int_G k(g)(T_{-g}^* f)\, dg = f \cdot \omega(k). \tag{4.2}$$

Hence, for every $k \in L^1(G)$,

$$f_\chi \cdot \omega(k) = \widehat{k}(\chi) f_\chi.$$

We see that $\omega(k) = 0$ implies $\widehat{k}(\chi) = 0$ and therefore $\chi \in \mathrm{sp}(\omega)$. Since the set $\{\omega(k) : k \in L^1(G)\}$ is dense in A, we also have

$$f_\chi \cdot a = \langle \phi_\chi, a\rangle f_\chi, \qquad \forall a \in A.$$

It follows that if $a \in \ker \phi_\chi$, then $f_\chi \cdot a = 0$. Since

$$\langle f_\chi, a\rangle = \lim_i \langle f_\chi, a\omega(e_i)\rangle = \lim_i \langle f_\chi \cdot a, \omega(e_i)\rangle = 0,$$

we see that $\ker \phi_\chi \subset \ker f_\chi$, which implies that $f_\chi = c\phi_\chi$ for some $c \neq 0$.

For a given $i \in I$, let

$$k_i(g) := \frac{1}{c} \frac{\chi(g)}{|K_i|} 1_{K_i}(g) \qquad (g \in G).$$

Then $k_i \in L^1(G)$ and by identities (4.1) and (4.2), we get

$$f \cdot \omega(k_i) = \frac{1}{c} \frac{1}{|K_i|} \int_{K_i} \overline{\chi(g)}(T_g^* f)\, dg \to \phi_\chi \qquad \text{in norm.}$$

This shows that $\phi_\chi \in \sigma(f)$. ■

As an immediate consequence of Theorem 4.1, we have the following

COROLLARY 4.2. *The norm spectrum of any* $f \in \mathrm{ap}(\mathrm{PF}_p(G)) \setminus \{0\}$ *is nonempty.*

Notice that $\mathrm{PF}_p(G)$ is a regular semisimple Tauberian Banach algebra satisfying the SBP. The structure space of $\mathrm{PF}_p(G)$ can be identified with $\widehat{G}$. Applying Theorem 5.5 of [17], we have the following

COROLLARY 4.3. *Let G be a locally compact abelian group such that $\widehat{G}$ is second countable. Then a proper closed ideal I of $\mathrm{PF}_p(G)$ is synthesizable if and only if $I = I_f$ for some $f \in \mathrm{ap}(\mathrm{PF}_p(G))$.*

In [6, Corollary 4.7], it is shown that if G is a locally compact abelian group, then the norm spectrum of any $f \in \overline{\mathrm{span}}\, \Sigma_{A_p(G)} \setminus \{0\}$ is nonempty. This result can be derived from Theorem 4.1 as follows. We know that $L^1(\widehat{G})$ is isometric (algebra) isomorphic to $A_2(G)$ via Fourier transform $\mathcal{F}$. As the functions which are continuous on G with compact support are dense in $L^p(G)$ $(1 < p < \infty)$, $A_2(G)$ is dense in $A_p(G)$. Consequently, the map $\mathcal{F} : L^1(\widehat{G}) \to A_p(G)$ is a continuous homomorphism with dense range. We also have $\mathrm{ap}(A_p(G)) = \overline{\mathrm{span}}\, \Sigma_{A_p(G)}$ [12, Corollary 2]. Now, applying Theorem 4.1, we obtain what was claimed above.

One can deduce even more. If K is a closed subset of G, then $A_p(K)$ is the Banach algebra of restrictions to K of the functions in $A_p(G)$ with the norm

$$\|k\|_{A_p(K)} = \inf\{\|h\|_{A_p(G)} : k = h|_K,\ h \in A_p(G)\}.$$

Notice that $\pi_K \circ \mathcal{F} : L^1(\widehat{G}) \to A_p(K)$ is a continuous homomorphism with dense range, where $\pi_K : A_p(G) \to A_p(K)$ is the canonical homomorphism.

COROLLARY 4.4. *The norm spectrum of any $f \in \mathrm{ap}(A_p(K)) \setminus \{0\}$ is nonempty.*

We conclude this paper with the following

PROPOSITION 4.5. *Let A be a commutative Banach algebra such that Σ_A contains a nonempty perfect subset. If there exists a continuous homomorphism $\omega : L^1(G) \to A$ with dense range, then there exists a nonzero functional f in $\mathrm{wap}(A) \setminus \mathrm{ap}(A)$ such that $\sigma(f) = \emptyset$.*

Proof. In view of [8, p. 52, Theorem 10], there exists a nonzero continuous finite regular Borel measure μ on $\mathrm{sp}(\omega)$. By Lemma 3.5, $\mu \in \mathrm{wap}(A)$. Let us show that $\mu \notin \mathrm{ap}(A)$. First, we claim that $\omega^*\mu = \widehat{\mu}$, where

$$\widehat{\mu}(g) = \int_{\widehat{G}} \chi(g)\, d\mu(\chi)$$

is the Fourier–Stieltjes transform of μ. To see this, let $k \in L^1(G)$. Then

$$\langle \omega^*\mu, k\rangle = \langle \mu, \omega(k)\rangle = \int_{\widehat{G}} \widehat{k}(\chi)\, d\mu(\chi) = \int_{\widehat{G}} \Big(\int_G k(g)\overline{\chi(g)}\, dg\Big)\, d\mu(\chi)$$
$$= \int_G \Big(\int_{\widehat{G}} \overline{\chi(g)}\, d\mu(\chi)\Big) k(g)\, dg = \int_G \widehat{\mu}(-g)k(g)\, dg.$$

Since this is true for all $k \in L^1(G)$, we obtain $\omega^*\mu = \widehat{\mu}$.

Now, assume that $\mu \in \mathrm{ap}(A)$. By [12, Corollary 2], since

$$\mathrm{ap}(A) = \overline{\mathrm{span}}\{\phi_\chi : \chi \in \mathrm{sp}(\omega)\},$$

we have

$$\mu \in \overline{\mathrm{span}}\{\phi_\chi : \chi \in \mathrm{sp}(\omega)\}.$$

From this, we deduce that the function $\widehat{\mu}$ can be approximated in the $\|\cdot\|_\infty$-norm by linear combinations of characters in $\mathrm{sp}(\omega)$. Hence, $\widehat{\mu}$ is an almost periodic function on G. Since μ is a continuous measure, we have

$$\langle \Phi, \overline{\chi(g)}\widehat{\mu}(g)\rangle = \mu\{\chi\} = 0,$$

where Φ is the invariant mean on the space of almost periodic functions on G. This shows that all Fourier–Bohr coefficients of the function $\widehat{\mu}$ are zero. By the uniqueness theorem, $\widehat{\mu} \equiv 0$ and therefore $\mu = 0$.

It remains to show that $\sigma(\mu) = \emptyset$. Assume on the contrary that $\phi_{\chi_0} \in \sigma(\mu)$. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a directed basic neighborhood system of $\{\chi_0\}$ in $\widehat{G}$. Since the algebra $L^1(G)$ is boundedly regular, there exists a net $(k_\lambda)_{\lambda\in\Lambda}$ in $L^1(G)$ such that $\mathrm{supp}\, \widehat{k}_\lambda \subset U_\lambda$, $\widehat{k}_\lambda(\chi_0) = 1$, and $\|k_\lambda\|_1 = 1$ $(\lambda \in \Lambda)$. Let us see that $(\widehat{\mu} * k_\lambda)(g) \to 0$ uniformly on G. It is easy to check that

$$(\widehat{\mu} * k_\lambda)(g) = \int_{U_\lambda} \chi(g)\widehat{k}_\lambda(\chi)\, d\mu(\chi).$$

Since $\mu\{\chi_0\} = 0$, for a given $\varepsilon > 0$, there exists $\lambda \in \Lambda$ such that $|\mu|(U_\lambda) < \varepsilon$. Hence, we have

$$\Big| \int_{U_\lambda} \chi(g)\widehat{k}_\lambda(\chi)\, d\mu(\chi)\Big| \le |\mu|(U_\lambda) < \varepsilon.$$

Further, since $\phi_{\chi_0} \in \sigma(\mu)$, there exists a function $k \in L^1(G)$ such that

$$\|\mu \cdot \omega(k) - \phi_{\chi_0}\| < \varepsilon.$$

Taking into account the identities

$$\omega^*(f \cdot \omega(k)) = (\omega^* f) * k \quad (f \in A^*,\ k \in L^1(G))$$

and $\omega^*\mu = \widehat{\mu}$, from the preceding inequality we have

$$\|(\widehat{\mu} * k)(g) - \chi_0(g)\|_\infty < \varepsilon\|\omega\|.$$

It follows that

$$\|(\widehat{\mu} * k_\lambda * k)(g) - (\chi_0 * k_\lambda)(g)\|_\infty < \varepsilon\|\omega\|, \quad \forall \lambda \in \Lambda.$$

Since $(\widehat{\mu} * k_\lambda * k)(g) \to 0$ uniformly and $(\chi_0 * k_\lambda)(g) = \chi_0(g)$ $(\lambda \in \Lambda)$, we obtain a contradiction. ■

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