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SOME GENERALIZATION OF STEINHAUS' LATTICE POINTS PROBLEM

BY

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Abstract. Steinhaus' lattice points problem addresses the question of whether it is possible to cover exactly n lattice points on the plane with an open ball for every fixed nonnegative integer n. This paper includes a theorem which can be used to solve the general problem of covering elements of so-called quasi-finite sets in Hilbert spaces. Some applications of this theorem are considered.

1. Introduction. Hugo Steinhaus, Polish mathematician and populariser of mathematics, published in [4] a collection of interesting problems in elementary mathematics (see also [3]). Several of these problems are strictly connected with geometry, and some relate to lattice points on the plane. The inspiration for this paper is the following question: for a fixed nonnegative integer n, is it always possible to find a circle on the plane such that exactly n of the lattice points are within this circle?

It turns out that this question has a positive answer. An elementary solution by H. Steinhaus, which involves some basic arguments from number theory and analytic geometry, can be found in [3] or [4]. We solve this problem in a more general setting.

Definition 1.1. Let X be a metric space. A set $A \subset X$ is called *quasi-finite* if A is countable and the intersection of A with any open ball B in X is finite.

In particular every finite subset of a metric space is quasi-finite. Also the set of lattice points in the Euclidean space $\mathbb{R}^n$ is an example of a quasi-finite set (the case $n = 2$ is considered in Corollary 3.2).

Now we present the main result of this paper:

Main Theorem 1.2. *Let A be a quasi-finite set in a Hilbert space X over a field $\mathbb{K}$ $(= \mathbb{R}, \mathbb{C})$. Then the set Y_A of all points $y \in X$ such that for every nonnegative integer n there exists an open ball centered at y containing exactly n elements of A, is dense in X.*

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We prove the Main Theorem in the next section.

The last section of this paper includes some applications. Firstly, we show an alternative solution for the classical version of Steinhaus' problem. Secondly, we present some criterion for Hilbert spaces, which can be used in some cases instead of the parallelogram identity.

REMARK 1.3. More complicated lattice points problems of H. Steinhaus are an inspiration for intensive research. For example, in [2] the authors consider the problem of existence of a set which has exactly one point in common with every subset of the plane which is congruent to the set of all lattice points. Also, in [1] some generalizations of solutions of Steinhaus' lattice points problem are considered.

2. The proof of the Main Theorem. Enumerate $A = \{x_1, x_2, \ldots\}$ and for $i \neq j$ let

$$A_j^i = \{x \in X : \|x - x_i\| = \|x - x_j\|\}.$$

We will show that A_j^i is closed and nowhere dense in X for $i \neq j$.

Fix positive integers i, j such that $i \neq j$. To deduce that A_j^i is closed, take $\phi\colon X \to \mathbb{K}$ given by the formula

$$\phi(x) = \|x - x_i\| - \|x - x_j\|.$$

From the continuity of the norm ϕ is a continuous map. So, A_j^i is closed since $\phi^{-1}(\{0\}) = A_j^i$.

To show that A_j^i has an empty interior we define a map $\psi\colon X \to X$ by

$$\psi(x) = x + \frac{x_i + x_j}{2}$$

and denote $w = (x_i - x_j)/2$. Clearly $w \neq 0$. If

$$B = \{x \in X : \|w - x\| = \|w + x\|\},$$

then $\psi(B) = A_j^i$. Since the function ψ is a translation it is sufficient to show that B has an empty interior. Take an arbitrary $x \in B$. Then

$$\begin{gathered}
\langle w - x \,|\, w - x\rangle = \langle w + x \,|\, w + x\rangle, \\
\|w\|^2 - \langle w \,|\, x\rangle - \langle x \,|\, w\rangle + \|x\|^2 = \|w\|^2 + \langle w \,|\, x\rangle + \langle x \,|\, w\rangle + \|x\|^2, \\
\langle w \,|\, x\rangle = -\langle x \,|\, w\rangle, \\
\overline{\langle x \,|\, w\rangle} + \langle x \,|\, w\rangle = 0.
\end{gathered}$$

From the above equalities we find that $B = \{x \in X : \mathfrak{Re}\langle x \,|\, w\rangle = 0\}$. It is easy to notice that B is a linear subspace of X. If a ball is contained in B, then $B = X$. In particular $\langle w \,|\, w\rangle = 0$, a contradiction. Consequently, B and hence A_j^i has an empty interior.

It follows from the Baire Category Theorem that the set $\bigcup_{i\neq j} A^i_j$ has an empty interior. If we denote its complement by Y_A, then Y_A is a dense subset of X.

Fix an arbitrary nonnegative integer n and $y \in Y_A$. As A is quasi-finite, we obtain $A \subset \bigcup_{k=1}^{\infty} B(y,k)$, where $B(y,k)$ is the open ball of radius k centered at y. Hence there exists some k_0 such that $B(y,k_0)$ has at least n elements common with A. We denote these elements by $y_1,\dots,y_N$ for some $N \geq n$ and write d_i for $\|y-y_i\|$. Without loss of generality, we can assume

$$d_1 < \cdots < d_N.$$

If we additionally define $d_0 := 0$ and $d_{N+1} := k_0$, then we can choose $r > 0$ satisfying $d_n < r < d_{n+1}$. Finally, the open ball of radius r centered at y contains exactly n points of A, which was to be demonstrated.

3. Some applications. Firstly, we recall the formal definition of a lattice point.

Definition 3.1. A *lattice point* is an element of a finite-dimensional linear space over $\mathbb{R}$ or $\mathbb{C}$ with a fixed basis all of whose coefficients are integers.

Now we use the Main Theorem to solve the classical version of Steinhaus' problem (the case of the Euclidean space $\mathbb{R}^n$ is analogous).

Corollary 3.2 (Steinhaus' classical lattice points problem). *For every $k \in \mathbb{N}$ there is an open ball on the plane which contains exactly k lattice points.*

Proof. By the Main Theorem it is sufficient to show that the set L of lattice points on the plane is quasi-finite. It is obvious that L is countable. If $r < k$ for some positive integer k, then every ball of radius r contains at most $4k^2$ lattice points. ∎

We proceed to further applications. The Main Theorem is useful to prove that some Banach spaces are not unitary: if it is possible to find a quasi-finite subset A and a positive integer m such that no open ball contains a collection of m elements of A, then the space cannot be unitary. This method is used in the following corollaries.

Corollary 3.3. *The space $\mathbb{R}^n$ with the norm $\|x\| = \max_{1\leq i\leq n}|x_i|$, where $x = (x_1,\dots,x_n)$, is not unitary for $n > 1$.*

Proof. Let L denote the set of lattice points in $\mathbb{R}^n$. Any open interval of fixed length includes either $k-1$ or k integers for some $k \in \mathbb{N}$. Hence every n-dimensional cube contains

$$k^l(k-1)^{n-l} \tag{3.1}$$

lattice points for some $k, l \in \mathbb{N}$. No prime $p > 2$ can be written in the form (3.1). This means that the set Y_L of points with the property mentioned in the Main Theorem is empty in that case. ∎

REMARK 3.4. A similar proof can be used to obtain the analogous result for $\mathbb{R}^n$ $(n > 1)$ with the norm $\|(x_1, \ldots, x_n)\| = \sum_{i=1}^{n} |x_i|$.

The method can also be applied to the space of all continuous functions:

COROLLARY 3.5. *The space* $C[0,1]$ *of all continuous functions on* $[0,1]$ *with the norm* $\|f\| = \max_{t\in[0,1]} |f(t)|$ *is not unitary.*

Proof. Assume that $C[0,1]$ is a Hilbert space. Then the subspace

$$Y = \{ax + b : a, b \in \mathbb{R}\}$$

is also a Hilbert space. Consider now the set $A = \{ax + b : a, b \in \mathbb{Z}\}$. It is easy to see that A is quasi-finite in Y. Every $f \in A$ is uniquely determined by the condition $f(0), f(1) \in \mathbb{Z}$. If B is an arbitrary ball in Y of radius $r > 0$ centered at g, then the intervals $(g(0) - r, g(0) + r)$ and $(g(1) - r, g(1) + r)$ contain k or $k + 1$ integers for some $k \in \mathbb{N}$. If $f(0)$ and $f(1)$ have to be integers, then f can be chosen in k^2 or $k(k+1)$ different ways. Obviously, if we choose f in one of these ways, then f is also an element of B. Hence an arbitrary ball in Y can contain only k^2 or $k(k + 1)$ elements of A for some $k \in \mathbb{N}$. ∎

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