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EQUIVARIANT K-THEORY OF FLAG VARIETIES REVISITED AND RELATED RESULTS

BY

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Abstract. We obtain several several results on the multiplicative structure constants of the T-equivariant Grothendieck ring $K_T(G/B)$ of the flag variety G/B. We do this by lifting the classes of the structure sheaves of Schubert varieties in $K_T(G/B)$ to $R(T)\otimes R(T)$, where $R(T)$ denotes the representation ring of the torus T. We further apply our results to describe the multiplicative structure constants of $K(X)_{\mathbb{Q}}$ where X denotes the wonderful compactification of the adjoint group of G, in terms of the structure constants of Schubert varieties in the Grothendieck ring of G/B.

1. Introduction. Let G be a semisimple simply connected algebraic group over an algebraically closed field k. Let B be a Borel subgroup and $T\subset B$ be a maximal torus.

In this article we construct explicit lifts of the classes of the structure sheaves of the Schubert basis in $K_T(G/B)$ to the ring $R(T)\otimes_{\mathbb{Z}}R(T)$. For this, we apply techniques similar to those developed by Marlin [16], by exploiting the properties of Demazure operators ([7]).

Using these lifts we also give new methods to describe the multiplicative structure of $K_T(G/B)$. More precisely, in §2 and §3, we give closed formulas for multiplicative structure constants and also recover some known results on these constants in this setting.

This was inspired by the results of Hiller [11, Chapter IV] who constructs a basis for $\mathrm{Sym}(X^*(T))$ as a $\mathrm{Sym}(X^*(T))^W$-module by lifting the fundamental classes of the Schubert varieties to the cohomology ring $H^*(G/B)$. He further uses this basis to develop an algebraic approach to Schubert calculus in $H^*(G/B)$.

Let X denote the wonderful compactification of the semisimple adjoint group $G_{\mathrm{ad}}=G/Z(G)$. Recall that in the main result of [20], the images in $K(G/B)$ under c_K of the Steinberg basis of $R(T)$ as an $R(G)$-module are used to describe the multiplicative structure of the Grothendieck ring $K(X)$ of X. Moreover, in that paper the multiplicative structure constants of $K(X)$

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as a $K(G/B)$-algebra involve the images of the structure constants of the Steinberg basis, which do not have known direct geometric or representation-theoretic interpretations (see [20, Theorems 3.8 and 3.12]).

In §4 we show that the above constructed lifts of the Schubert basis in $K(G/B)$ to $R(T)$ form a basis of $R(T)_{\mathfrak{p}}$ over $R(G)_{\mathfrak{p}}$, where $\mathfrak{p}$ denotes the kernel of the augmentation map $R(G) \to \mathbb{Z}$. These are a new set of bases for $R(T)_{\mathfrak{p}}$ as an $R(G)_{\mathfrak{p}}$-module different from the basis obtained by localization from that defined by Steinberg in [18]. We then reformulate the results [20, §3] using these bases instead of the Steinberg bases. Using this reformulation, in the main result of this paper (Theorem 4.4) we prove that $K(X)_{\mathbb{Q}}$ is a free $K(G/B)$-module generated by the classes of the structure sheaves of Schubert varieties in all $K(G/P)_{\mathbb{Q}}$, where $P \supseteq B$ is a parabolic subgroup. In particular, we express the multiplicative structure constants of $K(X)_{\mathbb{Q}}$ as a $K(G/B)$-algebra in terms of the structure constants of the classes of the structure sheaves of Schubert varieties in the Grothendieck ring of flag varieties.

Thus, although we seem to lose some information by going to rational coefficients, we do obtain better interpretations of the basis and the multiplicative structure of the $K(X)_{\mathbb{Q}}$ by relating it to the Schubert calculus in the Grothendieck ring of flag varieties.

1.1. Notations and conventions. As in the introduction, let G be a simply connected semisimple algebraic group over an algebraically closed field k, let B be a Borel subgroup of G and let $T \subseteq B$ be a maximal torus. Let $\Lambda = X^*(T)$ denote the weight lattice. Let Φ denote the root system and Δ the set of simple roots relative to B. Let $W = N(T)/T$ be the Weyl group of the root system Φ. Let $B^- = w_0 B w_0$ be the opposite Borel subgroup to B where w_0 is the unique maximal element of the Bruhat order on W. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Let ω_α denote the fundamental weight corresponding to the simple root $\alpha \in \Delta$.

For $w \in W$, let X_w denote the Schubert variety which is the closure of the Schubert cell BwB/B in G/B and let X^w denote the opposite Schubert variety which is the closure of the opposite Schubert cell B^-wB/B in G/B. Thus we have $X^w = w_0 X_{w_0 w}$.

Let $v, w \in W$. Recall that the *Bruhat order* is the order on W defined by: $v \preceq w$ if and only if $X_v \subseteq X_w$. Further, $X^v \cap X_w$ is nonempty if and only if $v \preceq w$; then $X^v \cap X_w$ is a variety called the *Richardson variety* and denoted by X_w^v. Moreover, X_w^v has two kinds of boundaries, namely $(\partial X_w)^v := (\partial X_w) \cap X^v$ and $(\partial X^v)_w := (\partial X^v) \cap X_w$. Here $\partial X_w = \bigcup_{w' \prec w} X_{w'}$ is the boundary of the Schubert variety X_w and $\partial X^v = \bigcup_{v \prec v'} X^{v'}$ is the boundary of the opposite Schubert variety X^v. Thus we have $(\partial X_w)^v = \bigcup_{w' \prec w} X_{w'}^v$ and $(\partial X^v)_w = \bigcup_{v \prec v'} X_w^{v'}$ (see [4, Prop. 1.3.2 and §4.2]).

For X any smooth G-variety, let $K_G(X)$ denote the Grothendieck ring of G-equivariant coherent sheaves (or equivalently, vector bundles) on X. We have the canonical forgetful homomorphism $K_G(X) \to K_T(X)$. In particular, $R(G) := K_G(\mathrm{pt})$ is the Grothendieck ring of k-representations of G. Since G is simply connected, we can identify $R(G) = \mathbb{Z}[\Lambda]^W$ via restriction to T. Furthermore, the structure morphism $X \to k$ induces a canonical $R(G)$-module structure on $K_G(X)$. Also, $K(X)$ denotes the Grothendieck ring of coherent sheaves on X and we have the canonical forgetful homomorphism $K_G(X) \to K(X)$.

For $w \in W$, let $[\mathcal{O}_{X_w}]_T$ (resp. $[\mathcal{O}_{X^w}]_T$) denote the class of the structure sheaf of the Schubert variety (resp. opposite Schubert variety) in $K_T(G/B)$. Further, note that we have the identification $[\mathcal{O}_{X^w}]_T = w_0 \cdot [\mathcal{O}_{X_{w_0 w}}]_T$ in $K_T(G/B)$. Recall from [13] that the Schubert classes $\{[\mathcal{O}_{X^w}]_T\}_{w\in W}$ form a basis of $K_T(G/B)$ as an $R(T)$-module.

For $\lambda \in \Lambda$, let $\mathcal{L}(\lambda) := (\mathbb{C}_\lambda \times G)/B$, where B acts diagonally, and the B-action on the one-dimensional vector space $\mathbb{C}_\lambda$ is given by the surjection $B \to T$ followed by λ. Then $\mathcal{L}(\lambda)$ is a T-linearized line bundle on G/B associated to λ. Let $\mathcal{L}^\lambda$ denote the class of $\mathcal{L}(\lambda)$ in $K_T(G/B)$. Further, we shall denote by e^λ the class of the trivial bundle in $K_T(G/B)$ with T-action given by λ.

Let $c_K^T : \mathbb{Z}[\Lambda] = R(T) \to K_T(G/B)$ denote the characteristic homomorphism which sends $e^\lambda \in R(T)$ to $\mathcal{L}^\lambda \in K_T(G/B)$.

Let $*$ denote the canonical involution in $K_T(G/B)$ defined by duality of a T-vector bundle. This is compatible with the involution in $R(T)$ defined by $e^\lambda \mapsto e^{-\lambda}$. In particular, $*c_K^T(e^\lambda) = [(\mathbb{C}_{-\lambda} \times G)/B] \in K_T(G/B)$.

If $Y \supset Z$ are closed T-stable subvarieties of a T-variety X, then $\mathcal{O}_Y(-Z)$ denotes the ideal sheaf of Z in Y. Thus, viewed as an element of $K_T(X)$, $[\mathcal{O}_Y(-Z)] = [\mathcal{O}_Y] - [\mathcal{O}_Z]$. Moreover, if $\mathcal{F}$ is a T-equivariant coherent sheaf on Y then $\mathcal{F}(-Z)$ will denote $\mathcal{F} \otimes \mathcal{O}_Y(-Z)$.

Recall that Demazure has defined the operators L_w, $w \in W$, on $\mathbb{Z}[\Lambda]$ with the following properties:

$$\begin{aligned} L_w L_{w'} &= L_{ww'} && \text{if } l(ww') = l(w) + l(w'), \\ L_s L_s &= L_s && \text{if } l(s) = 1, \text{ i.e. } s = s_\alpha \text{ for some } \alpha \in \Delta, \\ L_{s_\alpha}(f) &= \frac{f - s_\alpha(f)}{1 - e^\alpha} && \text{for } \alpha \in \Delta \text{ and } f \in R(T) \end{aligned} \tag{1.1}$$

(see [7, Theorem 2, pp. 86–87]). In particular, L_{s_α} (and hence L_w) is an $R(T)^W$-linear operator on $R(T)$. Also, for a simple reflection $s \in W$ we have

$$L_s \cdot L_w = \begin{cases} L_{sw} & \text{if } l(sw) = l(w) + 1, \\ L_w & \text{if } l(sw) = l(w) - 1. \end{cases} \tag{1.2}$$

Moreover, for any $w' \in W$, there exists a unique $v(w,w') \in W$ such that

$$L_{w'} L_{w^{-1}w_0} = L_{v(w,w')} \tag{1.3}$$

(see [7, §5.6]).

For each proper T-variety Y and each T-equivariant coherent sheaf $\mathcal{F}$ on Y, we define

$$\chi^T(Y,[\mathcal{F}]) = \pi_*([\mathcal{F}]) \tag{1.4}$$

where $\pi : Y \to \mathrm{pt}$ is the unique map to a point. Observe that

$$\chi^T(Y,[\mathcal{F}]) = \sum_k (-1)^k \operatorname{Char}(H^k(Y,\mathcal{F})) \in R(T) \tag{1.5}$$

where $\operatorname{Char}(H^k(Y,\mathcal{F})) \in R(T)$ is the character of the finite-dimensional T-module

$$H^k(Y,\mathcal{F}) = H^k(G/B, \mathcal{O}_Y \otimes \mathcal{F}).$$

Further, $\chi^T : K_T(Y) \to R(T)$ is an $R(T)$-linear map. In particular, let $\mathcal{F}$ be a T-equivariant coherent sheaf on G/B. Then we define the equivariant Euler–Poincaré characteristic as

$$\chi^T(X_w, c_K^T(e^\lambda)) = e^\rho \cdot L_w(e^{\lambda-\rho}). \tag{1.6}$$

Moreover, if $\epsilon : R(T) \to \mathbb{Z}$ denotes the canonical augmentation, then

$$\chi(X_w, \mathcal{L}(\lambda)) = \epsilon L_w(e^{\lambda-\rho}) \tag{1.7}$$

where $\chi(\cdot\,,\cdot)$ denotes the ordinary Euler–Poincaré characteristic (see [7, Theorem 2(b) and Cor. 1, pp. 86–87]).

In [13], Kostant and Kumar define an $R(T)$-module basis $(\tau^w)_{w\in W}$ for $K_T(G/B)$ which satisfies

$$\chi^T(X_{v^{-1}}, *\tau^w) = \delta_{v,w} \tag{1.8}$$

(see [13, p. 591, Prop. 3.39]). Let

$$\partial X^w := \bigsqcup_{v\in W,\, v>w} B^- vB/B, \tag{1.9}$$

$$\xi^w := [\mathcal{O}_{X^w}(-\partial X^w)]_T. \tag{1.10}$$

Recall from [9, Prop. 2.1] that $\{\xi^w\}_{w\in W}$ is an $R(T)$-basis for $K_T(G/B)$ dual to the Schubert basis $\{[\mathcal{O}_{X_w}]_T\}_{w\in W}$ under the pairing

$$\langle u, v\rangle := \chi^T(G/B, u\cdot v) \quad \text{for all } u,v \in K_T(G/B). \tag{1.11}$$

Further, it is shown in [9, Prop. 2.2] that

$$*\tau^w = \xi^{w^{-1}} \tag{1.12}$$

where τ^w is the Kostant–Kumar basis.

We further have the following relation between the Graham–Kumar basis and the opposite Schubert basis of $K_T(G/B)$ (see [9]):

$$[\mathcal{O}_{X^w}]_T = \sum_{w \preceq w'} \xi^{w'}. \tag{1.13}$$

For $I \subseteq \Delta$, let W_I be the subgroup of W generated by $\{s_\alpha : \alpha \in I\}$. Further, let W^I denote the minimal length coset representatives of W/W_I. Let $P = P_I \supset B$ denote the corresponding standard parabolic subgroup. In particular, for $w \in W^I$, we have the Schubert variety X_w^P (resp. the opposite Schubert variety X_P^w) which is the closure of the Bruhat cell BwP/P (resp. opposite Bruhat cell B^-wP/P) in the partial flag variety G/P.

It is well known that $\{[\mathcal{O}_{X_w^P}]_T\}_{w\in W^I}$ is an $R(T)$-basis of $K_T(G/P)$, and so is $\{[\mathcal{O}_{X_P^w}]_T\}_{w\in W^I}$. Further, in [9, §2], Graham and Kumar define the elements

$$\xi_P^v = [\mathcal{O}_{X_P^v}(-\partial X_P^v)]_T \tag{1.14}$$

which form an $R(T)$-basis $\{\xi_P^v\}_{v\in W^I}$ for $K_T(G/P)$ dual to the Schubert basis $\{[\mathcal{O}_{X_w^P}]_T\}_{w\in W^I}$ under the pairing

$$\langle [\mathcal{O}_{X_w^P}]_T, \xi_P^v\rangle = \chi^T(G/P, \mathcal{O}_{X_w^P\cap X_P^v}(-X_w^P \cap \partial X_P^v)). \tag{1.15}$$

Note that (1.15) is a generalization of (1.11) to G/P.

Let $K(G/B)$ denote the Grothendieck ring of coherent sheaves on G/B. Further,

$$c_K : \mathbb{Z}[\Lambda] = R(T) = K_G(G/B) \to K(G/B)$$

denote the characteristic homomorphism. Recall that in [7], Demazure has established the existence of a basis $(a_w)_{w\in W}$ for the $\mathbb{Z}$-module $K(G/B)$ such that

$$c_K(e^\lambda) = \sum_{w\in W} \chi(X_w, \mathcal{L}(\lambda)) a_w, \tag{1.16}$$

where $\chi(\cdot\,,\cdot)$ denotes the Euler–Poincaré characteristic.

Recall that we have the following relation between the Demazure basis and Schubert basis (see [4, Prop. 4.3.2]):

$$[\mathcal{O}_{X^w}] = \sum_{w \preceq w'} a_{w'}. \tag{1.17}$$

Let

$$f : K_T(G/B) \to K(G/B) \tag{1.18}$$

denote the forgetful homomorphism. Then we have

$$f([\mathcal{O}_{X^w}]_T) = [\mathcal{O}_{X^w}])$$

and $f(\xi^w) = a_w$ (see [13, Prop. 3.39]).

2. Equivariant K-theory of flag varieties

2.1. Lifting of Schubert basis to $R(T) \otimes R(T)$. In this section we construct explicit lifts of classes of structure sheaves of Schubert varieties in $K_T(G/B)$ to the ring $R(T) \otimes R(T)$. These will be used later to describe the multiplicative structure of $K_T(G/B)$.

We mention here that tensor products are considered over $\mathbb{Z}$ unless otherwise specified.

LEMMA 2.1. *The canonical homomorphism*

$$\Psi : R(T) \otimes R(T) \to K_T(G/B)$$

that sends an element $\sum_{i=1}^{n} a_i \otimes b_i$ in $R(T) \otimes R(T)$ to $\sum_{i=1}^{n} a_i \cdot c_K^T(b_i)$ in $K_T(G/B)$ is surjective with kernel the ideal

$$\mathcal{I} = \langle c \otimes 1 - 1 \otimes c : c \in R(T)^W \rangle$$

in $R(T) \otimes R(T)$.

Proof. We recall from [17, Prop. 4.1] that the map

$$R(T) \otimes_{R(G)} K_G(G/B) = R(T) \otimes_{R(T)^W} R(T) \to K_T(G/B) \tag{2.1}$$

defined via $a \otimes b \mapsto a \cdot c_K^T(b)$ is an isomorphism. Moreover, by definition of $R(T) \otimes_{R(T)^W} R(T)$, there is a canonical surjective homomorphism $\psi : R(T) \otimes R(T) \to R(T) \otimes_{R(T)^W} R(T)$ with kernel precisely $\mathcal{I}$. Now, Ψ is the homomorphism obtained by composing ψ with the isomorphism given by (2.1). It follows that Ψ is surjective with kernel $\mathcal{I}$. (See also [10, Theorem 1.2]). ■

DEFINITION 2.2. By defining $\mathbb{L}_w(a \otimes b) := a \otimes L_w(b)$ and extending it by linearity, we can define the Demazure operator $\mathbb{L}_w$ on $R(T) \otimes R(T)$ as an $R(T) \otimes 1$-linear operator.

We now prove a preliminary lemma which will be applied in the main proposition.

LEMMA 2.3. *Let $v(w, w')$ be as in* (1.3)*. Then*

$$v(w, w') = w_0 \ \Leftrightarrow \ w \preceq w'. \tag{2.2}$$

Proof. Let $l(w) = r$ and $w_r := w^{-1} w_0$. Let $w' = s'_1 \cdots s'_k$ be a reduced expression for w'. Hence

$$L_{w'} \cdot L_{w^{-1} w_0} = L_{w'} \cdot L_{w_r} = L_{s'_1} \cdots L_{s'_k} \cdot L_{w_r}. \tag{2.3}$$

Now, by (1.2) we see that

$$L_{s'_1} \cdots L_{s'_k} \cdot L_{w_r} = L_{s'_{i_1} \cdots s'_{i_m} w_r} \tag{2.4}$$

for some subsequence $(s'_{i_1}, \ldots, s'_{i_m})$ of $(s'_1, \ldots, s'_k)$. Now, (1.3), (2.3) and (2.4) imply that

$$v(w, w') = s'_{i_1} \cdots s'_{i_m} w_r. \tag{2.5}$$

Since $w_0 = w \cdot w_r$, it follows that $v(w, w') = w_0$ implies

$$w = s'_{i_1} \cdots s'_{i_m}. \tag{2.6}$$

Hence $w \preceq w'$ (see [4, Cor. 2.2.2]).

For the converse, we need to show:

Claim. *If $w \preceq w'$ then $L_{w'} L_{w^{-1}w_0} = L_{w_0}$.*

Proof of Claim. Note that when $l(w) = 0$ then $w = 1$. Thus by (1.2) we have

$$L_{w'} L_{w^{-1}w_0} = L_{w'} L_{w_0} = L_{w_0}. \tag{2.7}$$

Also, when $l(w') - l(w) = 0$ then $w = w'$. Again by (1.2) we have

$$L_{w'} L_{w^{-1}w_0} = L_w L_{w_r} = L_{w_0}. \tag{2.8}$$

We shall now prove the claim by induction on $l(w)$ and $l(w') - l(w)$.

Let $s'_1 \cdots s'_k$ be a reduced expression of w' and let $w = s'_{i_1} \cdots s'_{i_m}$. Now, we can write

$$L_{w'} L_{w^{-1}w_0} = L_v L_{s'_k} L_{w^{-1}w_0} \tag{2.9}$$

where $v = s'_1 \cdots s'_{k-1}$.

Case (i). If $l(s'_k w_r) = l(w_r) - 1$ then by (1.2) we have

$$L_v L_{s'_k} L_{w^{-1}w_0} = L_v L_{w_r} = L_v L_{w^{-1}w_0}. \tag{2.10}$$

Moreover, $l(s'_k w_r) = l(w_r) - 1$ is equivalent to $l(w s'_k) = l(w) + 1$. This further implies that $i_m \leq k - 1$. Hence $w \preceq v$. Now, since $l(v) - l(w) \lneq l(w') - l(w)$, the claim follows by induction on $l(w') - l(w)$.

Case (ii). If $l(s'_k w_r) = l(w_r) + 1$ then again by (1.2) we have

$$L_v L_{s'_k} L_{w^{-1}w_0} = L_v L_{s'_k w_r} = L_v L_{(w s'_k)^{-1} w_0}. \tag{2.11}$$

Note that $l(s'_k w_r) = l(w_r) + 1$ implies $l(w s'_k) = l(w) - 1$. Since $w \preceq w'$, this further implies that $w s'_k \preceq w'$ (see [12, Proposition, p. 119]). Moreover, since $l(w s'_k) \lneq l(w)$, we further see that $w s'_k \preceq v$. The claim now follows by induction on $l(w)$. ∎

Proposition 2.4. *In $R(T) \otimes R(T)$ there exists an element u_0 such that*

$$\Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})) = [\mathcal{O}_{X^w}]_T. \tag{2.12}$$

Indeed, we may take $u_0 = v_0(1 \otimes e^{-\rho})$ where v_0 is such that $\Psi(v_0) = [\mathcal{O}_{X^{w_0}}]_T$.

Proof. By (1.8) and (1.12), we have the following identity in $K_T(G/B)$:

$$c_K^T(e^\lambda) = \sum_{w\in W} \chi^T(X_w, \mathcal{L}(\lambda))\xi^w. \tag{2.13}$$

Moreover, combining (1.6) and (2.13) yields

$$c_K^T(e^\lambda) = \sum_{w\in W} e^\rho \cdot L_w(e^{\lambda-\rho})\xi^w. \tag{2.14}$$

By (1.13), it follows in particular that $\xi^{w_0} = [\mathcal{O}_{X^{w_0}}]_T = w_0 \cdot [\mathcal{O}_{X_1}]_T$.

Now, since Ψ is surjective by Lemma 2.1, there exists an element v_0 such that $\Psi(v_0) = \xi^{w_0}$. More precisely, if

$$v_0 = \sum_{i=1}^{n} a_i \otimes b_i, \tag{2.15}$$

then using (2.14) we have

$$\Psi(v_0) = \sum_{i=1}^{n} a_i \cdot c_K^T(b_i) = \sum_{i=1}^{n}\sum_{w\in W} a_i \cdot e^\rho \cdot L_w(b_i \cdot e^{-\rho})\xi^w. \tag{2.16}$$

Hence

$$\Psi(v_0) = \sum_{w\in W}\sum_{i=1}^{n} a_i \cdot e^\rho \cdot L_w(b_i \cdot e^{-\rho})\xi^w = \xi^{w_0}. \tag{2.17}$$

Therefore

$$\sum_{i=1}^{n} a_i \cdot e^\rho \cdot L_w(b_i \cdot e^{-\rho}) = \delta_{w,w_0}. \tag{2.18}$$

Let

$$u_0 := v_0 \cdot (1 \otimes e^{-\rho}). \tag{2.19}$$

Claim. *u_0 is the required element in $R(T) \otimes R(T)$ that satisfies* (2.12).

Proof of Claim. Note that if v_0 is as in (2.15) then

$$u_0 = \sum_{i=1}^{n} a_i \otimes e^{-\rho} \cdot b_i. \tag{2.20}$$

Now, by (2.20) and Def. 2.2 it follows that

$$\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) = \sum_{i=1}^{n} a_i \otimes e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot b_i). \tag{2.21}$$

Hence by (2.14) and (2.21) we have

$$\Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = \sum_{w'\in W}\sum_{i=1}^{n} e^\rho \cdot a_i \cdot L_{w'} L_{w^{-1}w_0}(b_i \cdot e^{-\rho})\xi^{w'}. \tag{2.22}$$

Now the claim follows by (1.13), (2.18), (2.22) and Lemma 2.3. ∎

LEMMA 2.5. *If* $w \in W^I$ *and* $r \in R(T)$, *then*

$$e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r) \in R(T)^{W_I}. \tag{2.23}$$

Proof. We first note that

$$s_j(e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = e^{\rho-\alpha_j} \cdot s_j(L_{w^{-1}w_0}(e^{-\rho} \cdot r)). \tag{2.24}$$

Thus for $j \in I$, the condition

$$s_j(e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r) \tag{2.25}$$

is equivalent to

$$s_j(L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = e^{\alpha_j} \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r). \tag{2.26}$$

Further, note that

$$s_j(L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = L_{w^{-1}w_0}(e^{-\rho} \cdot r) - (1-e^{\alpha_j}) \cdot L_{s_j} L_{w^{-1}w_0}(e^{-\rho} \cdot r). \tag{2.27}$$

Let $w_1 := w^{-1}w_0$. Then

$$l(s_j w_1) = l(s_j w^{-1} w_0) = l(w_0) - l(s_j w^{-1}). \tag{2.28}$$

Now, if $w \in W^I$, then for every $j \in I$ we have $l(ws_j) = l(w) + 1$, which is equivalent to $l(s_j w^{-1}) = l(w^{-1}) + 1$. By (2.28) this implies that

$$\begin{aligned} l(s_j w_1) &= l(w_0) - (l(w^{-1}) + 1) = l(w_0) - l(w^{-1}) - 1 \\ &= l(w^{-1}w_0) - 1 = l(w_1) - 1. \end{aligned} \tag{2.29}$$

This further implies by (1.2) that

$$L_{s_j} L_{w^{-1}w_0} = L_{s_j} L_{w_1} = L_{w_1}. \tag{2.30}$$

Now by substituting (2.30) in (2.27), we see that when $w \in W^I$, the condition (2.26) and hence (2.25) hold for all $j \in I$. This proves that if $w \in W^I$ then $e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r) \in R(T)^{W_I}$. ∎

PROPOSITION 2.6. *Let* u_0 *be as in Proposition* 2.4. *Then*

$$\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) \in R(T) \otimes R(T)^{W_I} \quad \text{if } w \in W^I.$$

Proof. This follows immediately from (2.21) and Lemma 2.5. ∎

NOTATION 2.7. In the following sections we let $u_0 = \sum_{i=1}^n a_i \otimes e^{-\rho} \cdot b_i \in R(T) \otimes R(T)$ be as in Prop. 2.4. (See §5 about this choice.)

2.2. Structure constants of Schubert basis in $K_T(G/B)$. In this section we determine a closed formula for the multiplicative structure constants of the basis $\{[\mathcal{O}_{X^w}]_T\}_{w \in W}$ in $K_T(G/B)$ in terms of the above elements a_i, b_i. We remark here that in [9], these structure constants as well as those of the dual basis have been studied in detail with regard to the positivity conjectures, viz., of [9, Conjectures 3.1 and 3.10]. The author is currently working to find more direct interconnections between the results in this section and those in [9].

Lemma 2.8. *For $x, y, z \in W$, let*

$$C^z_{x,y} := \sum_{w \preceq z} (-1)^{l(z)-l(w)} \cdot \sum_{1 \le i,j \le n} a_i \cdot a_j \cdot e^{\rho} \cdot L_w(L_{x^{-1}w_0}(b_i \cdot e^{-\rho}) \cdot L_{y^{-1}w_0}(b_j \cdot e^{-\rho}) \cdot e^{\rho}) \tag{2.31}$$

where $v_0 = \sum_{i=1}^n a_i \otimes b_i \in R(T) \otimes R(T)$ is such that $\Psi(v_0) = [\mathcal{O}_{X^{w_0}}]_T$. Then in $K_T(G/B)$ we have

$$[\mathcal{O}_{X^x}]_T [\mathcal{O}_{X^y}]_T = \sum_{z \in W} C^z_{x,y} [\mathcal{O}_{X^z}]_T \quad \text{for } x, y \in W. \tag{2.32}$$

Proof. Recall from [9, Lemma 4.2] that the basis $\{\xi^v\}_{v \in W}$ can be expressed in terms of the Schubert basis $\{[\mathcal{O}_{X^v}]_T\}_{v \in W}$ in $K_T(G/B)$ as follows:

$$\xi^v = \sum_{v \preceq w} (-1)^{l(w)-l(v)} [\mathcal{O}_{X^w}]_T. \tag{2.33}$$

Note that (2.33) is equivalent to (1.13) via Möbius inversion (see [4, Remark 4.3.3]). Now, using Lemma 2.1 and substituting (2.33) in (2.14) we get

$$\Psi(a \otimes b) = \sum_{w \in W} \sum_{v \in W,\, v \preceq w} (-1)^{l(w)-l(v)} a \cdot e^{\rho} \cdot L_v(b \cdot e^{-\rho}) [\mathcal{O}_{X^w}]_T \tag{2.34}$$

for $a \otimes b \in R(T) \otimes R(T)$.

Moreover, by (2.21),

$$\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho}) \cdot \mathbb{L}_{y^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho}) = \sum_{1 \le i,j \le n} a_i \cdot a_j \otimes e^{2\rho} \cdot L_{x^{-1}w_0}(e^{-\rho} \cdot b_i) \cdot L_{y^{-1}w_0}(e^{-\rho} \cdot b_j). \tag{2.35}$$

Further, by Prop. 2.4,

$$\Psi(\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho}) \cdot \mathbb{L}_{y^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})) = [\mathcal{O}_{X^x}]_T [\mathcal{O}_{X^y}]_T \tag{2.36}$$

for $x, y \in W$. Then by (2.36) and (2.34) we get (2.32), where $C^z_{x,y}$ is as in (2.31). ∎

2.3. A Chevalley formula in $K_T(G/B)$. The following lemma gives a "Chevalley formula" in $K_T(G/B)$, which determines the coefficients when the product $[\mathcal{L}^T(\lambda)]_T [\mathcal{O}_{X^x}]_T$ is expressed in terms of the Schubert basis $\{[\mathcal{O}_{X^v}]_T : v \in W\}$.

Lemma 2.9. *For $\lambda \in X^*(T)$ and $x, y \in W$ let*

$$Q^{\lambda}_{x,y} := \sum_{w \in W,\, w \preceq y} (-1)^{l(y)-l(w)} \sum_{i=1}^{n} e^{\rho} \cdot a_i \cdot L_w(e^{\lambda} \cdot L_{x^{-1}w_0}(b_i \cdot e^{-\rho})) \tag{2.37}$$

where $v_0 = \sum_{i=1}^n a_i \otimes b_i \in R(T) \otimes R(T)$ *is such that* $\Psi(v_0) = [\mathcal{O}_{X^{w_0}}]_T$. *Then in* $K_T(G/B)$ *we have*

$$[\mathcal{L}^T(\lambda)]_T \cdot [\mathcal{O}_{X^x}]_T = \sum_{y \in W} Q^\lambda_{x,y} [\mathcal{O}_{X^y}]_T. \tag{2.38}$$

Proof. Note that

$$\Psi(1 \otimes e^\lambda) = c^T_K(e^\lambda) = [\mathcal{L}^T(\lambda)]_T. \tag{2.39}$$

By (2.39) and Prop. 2.4 it follows that

$$\Psi(\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) \cdot (1 \otimes e^\lambda)) = [\mathcal{O}_{X^x}]_T \cdot [\mathcal{L}^T(\lambda)]_T \tag{2.40}$$

for $x \in W$ and $\lambda \in X^*(T)$. Thus we see that (2.38) follows immediately from (2.21), (2.34) and (2.40), where $Q^\lambda_{x,y}$ is given by (2.37). ∎

2.3.1. *Comparison with known Chevalley formulas*

LEMMA 2.10. *Let* $x, y \in W$ *and* $w = w_0 x$, $v = w_0 y$. *We then have the following interpretation of* (2.37)*:*

$$Q^\lambda_{x,y} = w_0 \cdot \chi^T(X^v_w, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)) \tag{2.41}$$

whenever $x \preceq y$, *and* $Q^\lambda_{x,y} = 0$ *otherwise* (*see* [3, *Lemma* 1]). *In particular, when* $w_0(\lambda) \in X^*(T)$ *is dominant we have*

$$Q^\lambda_{x,y} = w_0 \cdot \operatorname{Char} H^0(X^v_w, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)) \tag{2.42}$$

whenever $x \preceq y$, *and* $Q^\lambda_{x,y} = 0$ *otherwise.*

Proof. Let $\xi_y := [\mathcal{O}_{X_y}(-\partial X_y)]$. Then ξ_y is dual to $[\mathcal{O}_{X^x}]$ under the pairing (1.11). Now, (2.38) implies that

$$Q^\lambda_{x,y} = \langle [\mathcal{L}^T(\lambda)]_T \cdot [\mathcal{O}_{X^x}]_T, \xi_y \rangle = \chi^T(X^x_y, [\mathcal{L}^T(\lambda)]_T(-\partial X_y)^x] \tag{2.43}$$

whenever $x \preceq y$, and $Q^\lambda_{x,y} = 0$ otherwise. The second equality above follows because the intersections $X_y \cap X^x$ and $X^x \cap \partial X_y$ are transversal (see [4, Lemma 4.1.2]).

If $w = w_0 x$ and $v = w_0 y$, we can write (2.43) as

$$\begin{aligned} Q^\lambda_{x,y} &= \chi^T(X^{w_0 w}_{w_0 v}, \mathcal{L}^T(\lambda)(-(\partial X_{w_0 v})^{w_0 w})) \\ &= \chi^T(w_0 \cdot X^v_w, \mathcal{L}^T(\lambda)(-w_0 \cdot (\partial X^v)_w)) \\ &= w_0 \cdot \chi^T(X^v_w, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)) \end{aligned} \tag{2.44}$$

whenever $v \preceq w$, and $Q^\lambda_{x,y} = 0$ otherwise. (Note that $x \preceq y$ is equivalent to $v \preceq w$.) In particular, when $w_0(\lambda) \in X^*(T)$ is dominant, by [5, Prop.1, p. 9] it follows that

$$\chi^T(X^v_w, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)) = \operatorname{Char} H^0(X^v_w, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)).$$

Hence the lemma. ∎

REMARK 2.11. Let $w = w_0 x$ and $v = w_0 y$. Then (2.38) can be rewritten as

$$[\mathcal{L}^T(w_0\lambda)]_T \cdot [\mathcal{O}_{X_w}]_T = \sum_{v \preceq w} w_0(Q^{\lambda}_{x,y})[\mathcal{O}_{X_v}]_T \tag{2.45}$$

where $Q^{\lambda}_{x,y}$ is as in (2.44). Hence substituting (2.44) in (2.45), we derive the "Chevalley formula" as in [15] and [14]. In particular, note that $w_0(Q^{w_0\lambda}_{w_0 w, w_0 v})$ is the same as $C^{\lambda}_{w,v}$ of [15] where it is interpreted as $\sum e^{-\pi(1)}$ where the sum runs over all L-S paths π of shape λ ending in v and starting with an element smaller than or equal to w. Here we briefly recall that an L-S path π of shape λ on $X(\tau)$ is a pair of sequences $\pi = (\underline{\tau}, \underline{a})$ of Weyl group elements and rational numbers, where $\underline{\tau}$ is of the form $\underline{\tau} = (\tau_1, \dots, \tau_r)$ such that $\tau \geq \tau_1$ and $\tau_1 \geq \cdots \geq \tau_r$ in the Bruhat order on W. We call $\tau_1 = i(\pi)$ the *initial* element of π and $\tau_r = e(\pi)$ the *end* element of π (see [15, §3]).

REMARK 2.12. We refer the reader to [15] and [14] for more details on the representation-theoretic interpretation of the Chevalley formula using Standard Monomial Theory. The reader is also referred to [10] for the Chevalley formula in $K_T(G/B)$ given in terms of the combinatorics of the Littelmann path model, using the affine nil Hecke algebra. See also [21] for recent results on the Chevalley formula in equivariant K-theory of flag varieties using the Bott–Samelson resolution.

2.4. Structure constants of the Schubert basis in $K_T(G/P)$. In this section we determine a closed formula for the multiplicative structure constants of the Schubert basis $\{[\mathcal{O}_{X_P^w}]_T\}_{w\in W^I}$ of $K_T(G/P)$, again in terms of a_i, b_i.

Let μ^I be the Möbius function of the induced Bruhat ordering on W^I. Then (see [8, Theorem 1.2])

$$\mu^I(v,w) = \begin{cases} (-1)^{l(v)+l(w)} & \text{if } [v,w]\cap W^I = [v,w], \\ 0 & \text{otherwise,} \end{cases} \tag{2.46}$$

where for $v \preceq w$, $[v,w] := \{u \in W : v \preceq u \preceq w\}$.

LEMMA 2.13. *For $x, y, z \in W^I$, let*

$$D^z_{x,y} := \sum_{w\in W^I, w\preceq z} \mu^I(w,z) \cdot \sum_{1\leq i,j\leq n} e^{\rho}\cdot a_i \cdot a_j \cdot L_w(L_{x^{-1}w_0}(b_i\cdot e^{-\rho})\cdot L_{y^{-1}w_0}(b_j\cdot e^{-\rho})\cdot e^{\rho}). \tag{2.47}$$

Then in $K_T(G/P)$ we have

$$[\mathcal{O}_{X_P^x}]_T[\mathcal{O}_{X_P^y}]_T = \sum_{z\in W^I} D^z_{x,y}[\mathcal{O}_{X_P^z}]_T \quad \textit{for } x, y \in W^I. \tag{2.48}$$

Proof. Let $\pi : G/B \to G/P$ be the canonical projection. Then

$$\pi^*([\mathcal{O}_{X_P^w}]_T) = [\mathcal{O}_{X^w}]_T \quad \text{for } w \in W^I, \tag{2.49}$$

where $\pi^* : K_T(G/P) \to K_T(G/B)$ is the induced morphism.

For any $v \in W^I$ we also have (see [9, Lemma 3.4])

$$\pi^* \xi_P^v = \sum_{u \in W_I} \xi^{vu}. \tag{2.50}$$

Furthermore, for $w, v \in W^I$, we shall identify the elements ξ_P^v and $[\mathcal{O}_P^w]_T$ in $K_T(G/P)$ with their images in $K_T(G/B)$ under the injective morphism π^*.

Further, it follows from (1.1) that for $r \in R(T)^{W_I}$ and $\alpha \in I$,

$$L_{s_\alpha}(r \cdot e^{-\rho}) = \frac{r \cdot e^{-\rho} - r \cdot e^{-\rho+\alpha}}{1 - e^{\alpha}} = r \cdot e^{-\rho}. \tag{2.51}$$

This implies that for any $(w', v) \in W_I \times W^I$ we have

$$L_{vw'}(r \cdot e^{-\rho}) = L_v L_{w'}(r \cdot e^{-\rho}) = L_v(r \cdot e^{-\rho}). \tag{2.52}$$

Now, by (2.14), (2.50) and (2.51) it follows that for any $r \in R(T)^{W_I}$,

$$c_K^T(r) = \sum_{v \in W^I} e^{\rho} \cdot L_v(r \cdot e^{-\rho}) \cdot \xi_P^v. \tag{2.53}$$

Further, by Prop. 2.4 and (2.49) we have

$$\Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 - e^{\rho})) = [\mathcal{O}_{X_P^w}]_T. \tag{2.54}$$

By Lemma 2.5,

$$L_{w^{-1}w_0}(e^{-\rho} \cdot b_i) \cdot e^{\rho} \in R(T)^{W_I} \quad \text{for } 1 \leq i \leq n$$

whenever $w \in W^I$.

Now, from (2.22), (2.50) and substituting $L_{w^{-1}w_0}(e^{-\rho}) \cdot b_i \cdot e^{\rho}$ for r in (2.52) we get

$$\begin{aligned} [\mathcal{O}_{X_P^w}]_T &= \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})) \\ &= \sum_{v \in W^I} \sum_{i=1}^{n} e^{\rho} \cdot a_i \cdot L_v(L_{w^{-1}w_0}(e^{-\rho} \cdot b_i)) \xi_P^v. \end{aligned} \tag{2.55}$$

Let $v(w, w') \in W$ be such that $L_{v(w,w')} = L_v \cdot L_{w^{-1}w_0}$. Then by (2.18),

$$\sum_{i=1}^{n} e^{\rho} \cdot a_i \cdot L_{v(w,w')}(e^{-\rho} \cdot b_i) = \delta_{v(w,w'),w_0}. \tag{2.56}$$

Further, by Lemma 2.3, (2.55) can be rewritten as

$$[\mathcal{O}_{X_P^w}]_T = \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})) = \sum_{v \in W^I,\, w \preceq v} \xi_P^v. \tag{2.57}$$

Further, via Möbius inversion, (2.57) is equivalent to

$$\xi_P^v = \sum_{w \in W^I} \mu^I(v,w)[\mathcal{O}_{X_P^w}]_T \tag{2.58}$$

where $\mu^I(v,w)$ is as defined in (2.46).

Now, substituting (2.58) in (2.53) and using Lemma 2.1 we get

$$\Psi(t \otimes r) = \sum_{w \in W^I} \mu^I(v,w) \cdot e^{\rho} \cdot t \cdot L_v(r \cdot e^{-\rho})[\mathcal{O}_{X_P^w}]_T \tag{2.59}$$

for $t \otimes r \in R(T) \otimes R(T)^{W_I}$. Since by Props. 2.4 and 2.6 we have

$$[\mathcal{O}_{X_P^w}] = \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})),$$

(2.59) implies that

$$\Psi(\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho}) \cdot \mathbb{L}_{y^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})) = [\mathcal{O}_{X_P^x}]_T[\mathcal{O}_{X_P^y}]_T \tag{2.60}$$

for $x, y \in W^I$.

Thus by (2.35), (2.59) and (2.60) we get (2.48), where $D_{x,y}^z$ is as in (2.47). Hence the lemma. ∎

Remark 2.14. Note that Lemma 2.13 is a generalization of Lemma 2.8 to partial flag varieties.

3. Analogous results in ordinary K-theory. In this section we construct explicit lifts of structure sheaves of the Schubert varieties in $K(G/B)$ to $1 \otimes R(T)$. Indeed, the forgetful homomorphism $f : R(T) \otimes_{R(T)^W} R(T) = K_T(G/B) \to K(G/B)$ lifts to a map $\widetilde{f} : R(T) \otimes R(T) \to R(T)$ given by $e^{\lambda} \otimes e^{\mu} \mapsto e^{\mu}$.

Let v_0 be as in (2.15) and $u_0 = (1 \otimes e^{-\rho}) \cdot v_0$. Further, let $v_0' := \widetilde{f}(v_0)$ and $u_0' := \widetilde{f}(u_0)$. Then

$$v_0' = \sum_{i=}^{n} \epsilon(a_i) \cdot b_i, \tag{3.1}$$

$$u_0' = \sum_{i=1}^{n} \epsilon(a_i) \cdot e^{-\rho} \cdot b_i. \tag{3.2}$$

The following proposition describes explicit lifts of $[\mathcal{O}_{X^w}] \in K(G/B)$ to $R(T)$.

Proposition 3.1. *Let $v_0', u_0' \in R(T)$ be as in* (3.1) *and* (3.2) *respectively. Then $c_K(v_0') = [\mathcal{O}_{X^{w_0}}]$ and*

$$c_K(L_{w^{-1}w_0}(u_0') \cdot e^{\rho}) = [\mathcal{O}_{X^w}]. \tag{3.3}$$

Proof. Recall that $f([\mathcal{O}_{X^w}]_T) = [\mathcal{O}_{X^w}]$ (see (1.18)). Moreover, by Prop. 2.4, $\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})$ lifts $[\mathcal{O}_{X^w}]_T$ to $R(T) \otimes R(T)$. Now, (2.21) implies that

$$\widetilde{f}(\mathbb{L}_{w^{-1}w_0}(u_0)) = L_{w^{-1}w_0}(u_0').$$

Thus by a simple diagram chase it follows that the image $L_{w^{-1}w_0}(u_0') \cdot e^{\rho}$ of $\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^{\rho})$ under $\widetilde{f}$ lifts $[\mathcal{O}_{X^w}]$ to $R(T)$. ∎

PROPOSITION 3.2. *We have* $L_{w^{-1}w_0}(u_0') \cdot e^{\rho} \in R(T)^{W_I}$ *if* $w \in W^I$.

Proof. Apply Lemma 2.5 for $r = v_0'$. ∎

We now state the results in the ordinary K-ring analogous to Lemmas 2.8, 2.9, 2.13. Since the proofs follow the same lines as the proofs of the above mentioned lemmas, we avoid the repetition here.

LEMMA 3.3. *For* $x, y, z \in W$, *let*

$$c_{x,y}^z := \sum_{w \preceq z} (-1)^{l(z)-l(w)} \epsilon L_w(L_{x^{-1}w_0}(u_0') \cdot L_{y^{-1}w_0}(u_0') \cdot e^{\rho}). \tag{3.4}$$

Then in $K(G/B)$ *we have*

$$[\mathcal{O}_{X^x}][\mathcal{O}_{X^y}] = \sum_{z \in W} c_{x,y}^z [\mathcal{O}_{X^z}] \quad \text{for } x, y \in W. \tag{3.5}$$

LEMMA 3.4. *For* $\lambda \in X^*(T)$ *and* $x, y \in W$ *let*

$$q_{x,y}^{\lambda} := \sum_{w \preceq y} (-1)^{l(y)-l(w)} \epsilon L_w(e^{\lambda} \cdot L_{x^{-1}w_0}(u_0')). \tag{3.6}$$

Then in $K(G/B)$ *we have*

$$[\mathcal{L}(\lambda)][\mathcal{O}_{X^x}] = \sum_{y \in W} q_{x,y}^{\lambda} [\mathcal{O}_{X^y}]. \tag{3.7}$$

LEMMA 3.5. *For* $x, y, z \in W^I$, *let*

$$d_{x,y}^z := \sum_{w \preceq z} \mu^I(z, w) \epsilon L_w(L_{x^{-1}w_0}(u_0') \cdot L_{y^{-1}w_0}(u_0') \cdot e^{\rho}). \tag{3.8}$$

Then in $K(G/P)$ *we have*

$$[\mathcal{O}_{X_P^x}][\mathcal{O}_{X_P^y}] = \sum_{z \in W^I} d_{x,y}^z [\mathcal{O}_{X_P^z}] \quad \text{for } x, y \in W^I. \tag{3.9}$$

4. K-ring of the wonderful compactification

4.1. Some preliminaries. Let $\alpha_1, \ldots, \alpha_r$ be an ordering of the set Δ of simple roots and $\omega_1, \ldots, \omega_r$ denote the corresponding fundamental weights for the root system of (G, T). Since G is simply connected, the fundamental weights form a basis for $X^*(T)$ and hence for every $\lambda \in \Lambda$, $e^{\lambda} \in R(T)$ is a Laurent monomial in the elements e^{ω_i}, $1 \le i \le r$.

In [18, Theorem 2.2] Steinberg has defined a basis

$$\{f_v : v \in W^I\} \tag{4.1}$$

of $R(T)^{W_I}$ as a free $R(G)$-module of rank $|W^I|$. We recall here this definition: For $v \in W^I$ let

$$p_v := \prod_{v^{-1}\alpha_i<0} e^{\omega_i} \in R(T).$$

Then

$$f_v := \sum_{x\in W_I(v)\backslash W_I} x^{-1}v^{-1}p_v$$

where $W_I(v)$ denotes the stabilizer of $v^{-1}p_v$ in W_I.

Let $c_K : R(T)^{W_I} \to K(G/P_I)$ denote the restriction of the characteristic homomorphism (see [16, §8]). Let $I(G) := \{a - \epsilon(a) : a \in R(G)\}$ denote the augmentation ideal. Then it is known that c_K is a surjective ring homomorphism and

$$\ker(c_K) = I(G) \cdot R(T)^{W_I}. \tag{4.2}$$

Let $r_v := L_{v^{-1}w_0}(u_0)\cdot e^{\rho} \in R(T)^{W_I}$ for $v \in W^I$ for every $I \subset \Delta$. Then by Lemma 3.2, $c_K(r_v) = [\mathcal{O}_{X^v_{P_I}}]$ for every $v \in W^I$. Let $\lambda_I := c_K(\prod_{\alpha\in I}(1-e^{-\alpha}))$ for $I \subseteq \Delta$.

Further, recall that $R(T)^{W_I} = \mathbb{Z}[\Lambda]^{W_I}$ and $R(G) = R(T)^W = \mathbb{Z}[\Lambda]^W$.

Note that $\mathfrak{p} := I(G)$ is a prime ideal in $R(G)$ and let $R(G)_{\mathfrak{p}}$ denote the corresponding localization. Observe that the augmentation extends to

$$R(G)_{\mathfrak{p}} \to \mathbb{Q} \tag{4.3}$$

with kernel the maximal ideal $\mathfrak{p} \cdot R(G)_{\mathfrak{p}}$. Further, the characteristic homomorphism extends to

$$c_K : R(T)_{\mathfrak{p}}^{W_I} \twoheadrightarrow K(G/P_I)_{\mathbb{Q}} \tag{4.4}$$

with kernel $\mathfrak{p} \cdot R(T)_{\mathfrak{p}}^{W_I}$.

LEMMA 4.1. *The elements $\{r_v : v \in W^I\}$ form a basis of $R(T)_{\mathfrak{p}}^{W_I}$ as an $R(G)_{\mathfrak{p}}$-module.*

Proof. Recall that $R(T)_{\mathfrak{p}}^{W_I}$ is a finitely generated $R(G)_{\mathfrak{p}}$-module. Moreover, $\{c_K(r_v) = [\mathcal{O}_{X^v}] : v \in W^I\}$ is a basis of $K(G/P_I)_{\mathbb{Q}}$ as an $R(G)_{\mathfrak{p}}/\mathfrak{p}\cdot R(G)_{\mathfrak{p}} \simeq \mathbb{Q}$-vector space. Now, by the Nakayama lemma (see [2, Prop. 2.8]), $\{r_v : v \in W^I\}$ spans $R(T)_{\mathfrak{p}}^{W_I}$ as an $R(G)_{\mathfrak{p}}$-module. Furthermore, since $R(T)_{\mathfrak{p}}^{W_I}$ is free over $R(G)_{\mathfrak{p}}$ of rank $|W^I|$, it follows that $\{r_v : v \in W^I\}$ is a basis of $R(T)_{\mathfrak{p}}^{W_I}$ as an $R(G)_{\mathfrak{p}}$-module. ∎

We now fix some notations (see also [20, p. 378]).

Note that $J \subseteq I$ implies that $W^{\Delta\setminus J} \subseteq W^{\Delta\setminus I}$. Let

$$C^I := W^{\Delta\setminus I} \setminus \bigcup_{J\subsetneq I} W^{\Delta\setminus J}, \tag{4.5}$$

$$R(T)_I := \bigoplus_{v\in C^I} R(T)_{\mathfrak{p}}^W \cdot r_v, \tag{4.6}$$

where $R(T)_{\mathfrak{p}}^W = R(G)_{\mathfrak{p}}$.

Lemma 4.2. *We have the following direct sum decompositions of $R(T)^W$-modules:*

$$R(T)_{\mathfrak{p}}^{W_{\Delta\setminus I}} = \bigoplus_{J\subseteq I} R(T)_J, \tag{4.7}$$

$$R(T)_{\mathfrak{p}}^{W_{\Delta\setminus I}} = \Big(\sum_{J\subsetneq I} R(T)_{\mathfrak{p}}^{W_{\Delta\setminus J}}\Big) \oplus R(T)_I, \tag{4.8}$$

for $I \subseteq \Delta$.

Proof. By (4.5), $W^{\Delta\setminus I} = \bigsqcup_{J\subseteq I} C^J$. Hence Lemma 4.1 implies that

$$R(T)_{\mathfrak{p}}^{W_{\Delta\setminus I}} = \bigoplus_{J\subseteq I} \bigoplus_{v\in C^J} R(T)_{\mathfrak{p}}^W \cdot r_v. \tag{4.9}$$

Using (4.6), the proof is now exactly as that of [20, Lemma 1.10]. ∎

In $R(T)_{\mathfrak{p}}$ we have

$$r_v \cdot r_{v'} = \sum_{J\subseteq I\cup I'} \sum_{w\in C^J} a_{v,v'}^w \cdot r_w \tag{4.10}$$

for certain elements $a_{v,v'}^w \in R(G)_{\mathfrak{p}} = R(T)_{\mathfrak{p}}^W$, for all $v \in C^I$, $v' \in C^{I'}$ and $w \in C^J$, $J \subseteq I \cup I'$.

Finally, let

$$K(G/B)_{\mathbb{Q},I} := \bigoplus_{v\in C^I} \mathbb{Q}\cdot[\mathcal{O}_{X^v}].$$

Then

$$K(G/B)_{\mathbb{Q}} = \bigoplus_{I\subseteq\Delta} K(G/B)_I.$$

4.2. Main Theorem. Let $X := \overline{G_{\mathrm{ad}}}$ denote the wonderful compactification of the semisimple adjoint group $G_{\mathrm{ad}} = G/Z(G)$, where $Z(G)$ denotes the center of G, constructed by De Concini and Procesi [6].

Note that $K_{G\times G}(X)$ is an $\mathcal{R} := R(G)\otimes R(G)$-module. Let

$$\mathcal{S} := R(G)\otimes R(G)_{\mathfrak{p}}.$$

Then the forgetful homomorphism extends to

$$f : K_{G\times G}(X)\otimes_{\mathcal{R}} \mathcal{S} \to K(X)_{\mathbb{Q}}. \tag{4.11}$$

THEOREM 4.3. *The ring $K_{G\times G}(X)\otimes_{\mathcal{R}}\mathcal{S}$ has the following direct sum decomposition as an $\mathcal{S}$-module:*

$$K_{G\times G}(X)\otimes_{\mathcal{R}}\mathcal{S}=\bigoplus_{I\subseteq\Delta}\prod_{\alpha\in I}(1-e^{\alpha(u)})\cdot R(T)\otimes R(T)_I. \tag{4.12}$$

Further, the above direct sum is a free $R(T)\otimes R(G)_{\mathfrak{p}}$-module of rank $|W|$ with basis

$$\Big\{\prod_{\alpha\in I}(1-e^{\alpha(u)})\otimes r_v : v\in C^I \text{ and } I\subseteq\Delta\Big\},$$

where C^I is as defined in (4.5) *and $\{r_v\}$ is as defined above. Moreover, we can identify the component $R(T)\otimes 1\subseteq R(T)\otimes R(G)_{\mathfrak{p}}$ in the above direct sum with the subring of $K_{G\times G}(X)$ generated by $\mathrm{Pic}^{G\times G}(X)$.* (*We refer to* [19] *for a similar description of the equivariant cohomology ring of the wonderful compactification.*)

Proof. Recall from [20, Lemma 3.2] the inclusions

$$R(T)\otimes R(G)\subseteq K_{G\times G}(X)\subseteq R(T)\otimes R(T) \tag{4.13}$$

where $K_{G\times G}(X)$ consists of all elements $f(u,v)\in R(T)\otimes R(T)$ that satisfy

$$(1,s_\alpha)f(u,v)\equiv f(u,v)\ (\mathrm{mod}\ (1-e^{\alpha(u)}))\quad\text{for every }\alpha\in\Delta. \tag{4.14}$$

Now, since $1\otimes R(G)_{\mathfrak{p}}$ is a flat $1\otimes R(G)$-module, we see that $\mathcal{S}$ is flat as an $\mathcal{R}$-module. This implies from (4.13) that

$$R(T)\otimes R(G)\otimes_{\mathcal{R}}\mathcal{S}\subseteq K_{G\times G}(X)\otimes_{\mathcal{R}}\mathcal{S}\subseteq R(T)\otimes R(T)\otimes_{\mathcal{R}}\mathcal{S}.$$

Further, $K_{G\times G}(X)\otimes_{\mathcal{R}}\mathcal{S}$ consists of all elements

$$f(u,v)\in R(T)\otimes R(T)\otimes_{\mathcal{R}}\mathcal{S}$$

that satisfy

$$(1,s_\alpha)f(u,v)\equiv f(u,v)\ (\mathrm{mod}\ (1-e^{\alpha(u)}))\quad\text{for every }\alpha\in\Delta.$$

Here we use the fact that if $f(u,v)\in\mathcal{S}$, then $(1,s_\alpha)f(u,v)=f(u,v)$ for every $\alpha\in\Delta$. The theorem now follows by using Lemma 4.2 above, and replacing the Steinberg basis $\{f_v\}_{v\in W^I}$ by the canonical lifting of the Schubert basis $\{r_v\}_{v\in W^I}$, in the proof of Theorem 3.8 of [20]. ∎

THEOREM 4.4. *The subring of $K(X)$ generated by the classes of line bundles is isomorphic to $K(G/B)$. Moreover, $K(X)$ is a free module of rank $|W|$ over $K(G/B)$. More explicitly, let*

$$\gamma_v:=1\otimes[\mathcal{O}_{X^v}]\in K(G/B)\otimes K(G/B)_{\mathbb{Q},I}$$

for $v\in C^I$ for every $I\subseteq\Delta$. Then

$$K(X)_{\mathbb{Q}}\simeq\bigoplus_{v\in W}K(G/B)\cdot\gamma_v.$$

Further, the above isomorphism is a ring isomorphism, where the product of any two basis elements γ_v *and* $\gamma_{v'}$ *is defined as follows:*

$$\gamma_v \cdot \gamma_{v'} := \sum_{J \subseteq I \cup I'} \sum_{w \in C^J} (\lambda_{I \cap I'} \cdot \lambda_{(I \cup I') \setminus J} \cdot c^w_{v,v'}) \cdot \gamma_w$$

where $c^w_{v,v'} \in \mathbb{Z}$ *are as defined in* (3.4).

Proof. Since $c_K(r_v) = [\mathcal{O}_{X^v}]$ for $v \in W^I$ and $I \subseteq \Delta$, the image under c_K of the element $a^w_{v,v'} \in R(G)_{\mathfrak{p}}$ defined in (4.10) is nothing but the structure constant $c^w_{v,v'} \in \mathbb{Z}$ defined in (3.4). The proof now follows exactly that of [20, Theorem 3.12, p. 403]. ∎

Remark 4.5. Note that Theorem 4.4 is a restatement of [20, Theorem 3.12], obtained by replacing the Steinberg basis $\{f_v\}_{v \in W^I}$ by the lift of the Schubert basis $\{r_v\}_{v \in W^I}$. In [20, Theorem 3.12], the multiplicative structure of the ordinary K-ring of the wonderful group compactifications was described in terms of the structure constants of the image of the Steinberg basis $\{f_v\}_{v \in W^I}$ under c_K. These structure constants do not have any known relations to geometry or representation theory.

Whereas now we see that the multiplicative structure constants of the basis $\gamma_v = 1 \otimes [\mathcal{O}_{X^v}]$ of $K(X)_{\mathbb{Q}}$ as $K(G/B)$-module are determined explicitly in terms of the multiplicative structure constants of the Schubert basis $c_K(r_v)$ described above in Prop. 3.3. These structure constants have been described in §2 and §3 above, and are also known to have nice geometric and representation-theoretic interpretations (see for example [3] and [15]).

5. Appendix

5.1. An explicit lift of $[\mathcal{O}_{X^{w_0}}]_T$ to $R(T) \otimes R(T)$. Let $\{e^{p_w}\}_{w \in W}$ be the basis defined by Steinberg of $R(T)$ as an $R(T)^W$-module, where

$$p_w = w\Big(\sum_{\alpha \in \Delta,\, w(\alpha) < 0} \omega_\alpha\Big)$$

for $w \in W$ (see [18]). Then by [16, lemme 4 and prop. 3] we see that the matrix $M = (L_{w'}(e^{p_w}))_{w,w' \in W}$ with entries in $R(T)$ is invertible. Thus there exists a unique vector $(a_w)_{w \in W}$ such that

$$\sum_{w \in W} a_w \cdot L_{w'}(e^{p_w}) = e^{-\rho} \cdot \delta_{w',w_0} \tag{5.1}$$

for every $w' \in W$. Now, defining $b_w := e^{\rho + p_w}$, we see that the element

$$v_0 = \sum_{w \in W} a_w \otimes b_w$$

in $R(T)\otimes R(T)$ satisfies (2.18). Thus we have a canonical choice of an element

$$u_0 = v_0 \cdot (1 \otimes e^{-\rho}) = \sum_{w\in W} a_w \otimes e^{p_w} \tag{5.2}$$

in $R(T) \otimes R(T)$ which satisfies Prop. 2.4. We now illustrate the computation of $u_0 = \sum_{w\in W} a_w \otimes b_w$ in $R(T) \otimes R(T)$ for the case when G is of type A_2.

EXAMPLE 5.1. When G is of type A_2, we have $\Delta = \{\alpha, \beta\}$ and ω_α and ω_β are the fundamental weights dual to α and β respectively. Further, $W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha\}$, where s_α and s_β are the simple reflections corresponding to α and β satisfying the braid relation $(s_\alpha s_\beta)^3 = 1$. Moreover, $\rho = \omega_\alpha + \omega_\beta$. In this case the Steinberg basis elements are

$$\begin{aligned} e^{p_1} &= 1, \\ e^{p_{s_\alpha}} &= e^{\omega_\beta - \omega_\alpha}, \\ e^{p_{s_\beta}} &= e^{\omega_\alpha - \omega_\beta}, \\ e^{p_{s_\alpha s_\beta}} &= e^{-\omega_\alpha}, \\ e^{p_{s_\beta s_\alpha}} &= e^{-\omega_\beta}, \\ e^{p_{s_\alpha s_\beta s_\alpha}} &= e^{-\omega_\alpha - \omega_\beta}. \end{aligned} \tag{5.3}$$

Furthermore, the matrix $(L_{w'}(e^{p_w}))$ is

$$\begin{pmatrix} 1 & e^{\omega_\beta-\omega_\alpha} & e^{\omega_\alpha-\omega_\beta} & e^{-\omega_\alpha} & e^{-\omega_\beta} & e^{-\omega_\alpha-\omega_\beta} \\ 0 & e^{\omega_\beta-\omega_\alpha} & -e^{-\omega_\alpha} & e^{-\omega_\alpha} & 0 & e^{-\omega_\alpha-\omega_\beta} \\ 0 & -e^{-\omega_\beta} & e^{\omega_\alpha-\omega_\beta} & 0 & e^{-\omega_\beta} & e^{-\omega_\alpha-\omega_\beta} \\ 0 & 0 & -e^{-\omega_\alpha} & 0 & 0 & e^{-\omega_\alpha-\omega_\beta} \\ 0 & -e^{-\omega_\beta} & 0 & 0 & 0 & e^{-\omega_\alpha-\omega_\beta} \\ 0 & 0 & 0 & 0 & 0 & e^{-\omega_\alpha-\omega_\beta} \end{pmatrix} \tag{5.4}$$

where the rows correspond respectively to $L_{w'}(e^{p_w})$ for $w' \in W$ for the above ordering. We can now solve the system (5.1) to get

$$\begin{aligned} a_1 &= -e^{-\omega_\alpha-\omega_\beta}, \\ a_{s_\alpha} &= e^{-\omega_\alpha}, \\ a_{s_\beta} &= e^{-\omega_\beta}, \\ a_{s_\alpha s_\beta} &= -e^{-\omega_\alpha+\omega_\beta}, \\ a_{s_\beta s_\alpha} &= -e^{-\omega_\beta+\omega_\alpha}, \\ a_{s_\alpha s_\beta s_\alpha} &= 1. \end{aligned} \tag{5.5}$$

Hence from (5.2) we get

$$u_0 = 1 \otimes e^{-\omega_\alpha-\omega_\beta} - e^{-\omega_\alpha-\omega_\beta} \otimes 1 + e^{-\omega_\alpha} \otimes e^{\omega_\beta-\omega_\alpha} - e^{\omega_\beta-\omega_\alpha} \otimes e^{-\omega_\alpha} + e^{-\omega_\beta} \otimes e^{\omega_\alpha-\omega_\beta} - e^{\omega_\alpha-\omega_\beta} \otimes e^{-\omega_\beta}. \tag{5.6}$$

5.2. Examples for computations of structure constants. We now illustrate the computations in Lemmas 2.8 and 2.9 respectively by the following examples, when G is of type A_2. We follow the notations of Example 5.1.

EXAMPLE 5.2. Let $u_0 = \sum_{w\in W} a_w \otimes e^{p_w} \in R(T)\otimes R(T)$ be the lift of $[\mathcal{O}_{X^{w_0}}]_T$ as in (5.6). Further, let $x = s_\alpha$, $y = s_\alpha s_\beta$, and

$$t_{x,y}^{w,w'} := L_{s_\beta s_\alpha}(b_w \cdot e^{-\rho}) \cdot L_{s_\alpha}(b_{w'} \cdot e^{-\rho}) \cdot e^{\rho} = L_{s_\beta s_\alpha}(e^{p_w}) \cdot L_{s_\alpha}(e^{p_{w'}}) \cdot e^{\rho}.$$

Then by (2.31), the multiplicative structure constants of $[\mathcal{O}_x]_T \cdot [\mathcal{O}_y]_T$ in $K_T(G/B)$ are obtained recursively as follows:

$$\begin{aligned}
C_{x,y}^{1} &= \sum_{w,w'} a_w \cdot a_{w'} \cdot e^{\rho} \cdot t_{x,y}^{w,w'},\\
C_{x,y}^{s_\alpha} &= \sum_{w,w'} a_w \cdot a_{w'} \cdot e^{\rho} \cdot L_{s_\alpha}(t_{x,y}^{w,w'}) - C_{x,y}^{1},\\
C_{x,y}^{s_\beta} &= \sum_{w,w'} a_w \cdot a_{w'} \cdot e^{\rho} \cdot L_{s_\beta}(t_{x,y}^{w,w'}) - C_{x,y}^{1},\\
C_{x,y}^{s_\alpha s_\beta} &= \sum_{w,w'} a_w \cdot a_{w'} \cdot e^{\rho} \cdot L_{s_\alpha s_\beta}(t_{x,y}^{w,w'}) - C_{x,y}^{s_\alpha} - C_{x,y}^{s_\beta} - C_{x,y}^{1},\\
C_{x,y}^{s_\beta s_\alpha} &= \sum_{w,w'} a_w \cdot a_{w'} \cdot e^{\rho} \cdot L_{s_\beta s_\alpha}(t_{x,y}^{w,w'}) - C_{x,y}^{s_\alpha} - C_{x,y}^{s_\beta} - C_{x,y}^{1},\\
C_{x,y}^{s_\alpha s_\beta s_\alpha} &= \sum_{w,w'} a_w \cdot a_{w'} \cdot e^{\rho} \cdot L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{w,w'}) - C_{x,y}^{s_\alpha s_\beta} - C_{x,y}^{s_\beta s_\alpha}\\
&\quad - C_{x,y}^{s_\alpha} - C_{x,y}^{s_\beta} - C_{x,y}^{1}.
\end{aligned} \tag{5.7}$$

Now, from (5.4) it follows that $t_{x,y}^{w,w'} = 0$ when $w = 1, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha$ or $w' = 1, s_\beta s_\alpha$. Further, we have

$$\begin{aligned}
t_{x,y}^{s_\alpha,s_\alpha} &= -e^{\omega_\beta}, & t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha} &= e^{\omega_\beta-\omega_\alpha},\\
t_{x,y}^{s_\alpha,s_\beta} &= 1, & t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\beta} &= -e^{-\omega_\alpha},\\
t_{x,y}^{s_\alpha,s_\alpha s_\beta} &= -1, & t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta} &= e^{-\omega_\alpha},\\
t_{x,y}^{s_\alpha,s_\alpha s_\beta s_\alpha} &= -e^{-\omega_\beta}, & t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta s_\alpha} &= e^{-\omega_\alpha-\omega_\beta};
\end{aligned} \tag{5.8}$$

$$(5.9)\quad \begin{aligned} L_{s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha}) &= 0, & L_{s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha}) &= e^{\omega_\beta-\omega_\alpha},\\ L_{s_\alpha}(t_{x,y}^{s_\alpha,s_\beta}) &= 0, & L_{s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\beta}) &= -e^{-\omega_\alpha},\\ L_{s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha s_\beta}) &= 0, & L_{s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta}) &= e^{-\omega_\alpha},\\ L_{s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha s_\beta s_\alpha}) &= 0, & L_{s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta s_\alpha}) &= e^{-\omega_\alpha-\omega_\beta}; \end{aligned}$$

$$(5.10)\quad \begin{aligned} L_{s_\beta}(t_{x,y}^{s_\alpha,s_\alpha}) &= e^{-\omega_\beta+\omega_\alpha}, & L_{s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha}) &= -e^{-\omega_\beta},\\ L_{s_\beta}(t_{x,y}^{s_\alpha,s_\beta}) &= 0, & L_{s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\beta}) &= 0,\\ L_{s_\beta}(t_{x,y}^{s_\alpha,s_\alpha s_\beta}) &= 0, & L_{s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta}) &= 0,\\ L_{s_\beta}(t_{x,y}^{s_\alpha,s_\alpha s_\beta s_\alpha}) &= -e^{-\omega_\beta}, & L_{s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta s_\alpha}) &= e^{-\omega_\alpha-\omega_\beta}; \end{aligned}$$

$$(5.11)\quad \begin{aligned} L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha,s_\alpha}) &= -e^{-\omega_\alpha}, & L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha}) &= 0,\\ L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha,s_\beta}) &= 0, & L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\beta}) &= 0,\\ L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha,s_\alpha s_\beta}) &= 0, & L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta}) &= 0,\\ L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha,s_\alpha s_\beta s_\alpha}) &= 0, & L_{s_\alpha s_\beta}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta s_\alpha}) &= e^{-\omega_\alpha-\omega_\beta}; \end{aligned}$$

$$(5.12)\quad \begin{aligned} L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha}) &= 0, & L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha}) &= -e^{-\omega_\beta},\\ L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\beta}) &= 0, & L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\beta}) &= 0,\\ L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha s_\beta}) &= 0, & L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta}) &= 0,\\ L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha s_\beta s_\alpha}) &= 0, & L_{s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta s_\alpha}) &= e^{-\omega_\alpha-\omega_\beta}; \end{aligned}$$

$$(5.13)\quad \begin{aligned} L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha}) &= 0, & L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha}) &= 0,\\ L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\beta}) &= 0, & L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\beta}) &= 0,\\ L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha s_\beta}) &= 0, & L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta}) &= 0,\\ L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha,s_\alpha s_\beta s_\alpha}) &= 0, & L_{s_\alpha s_\beta s_\alpha}(t_{x,y}^{s_\alpha s_\beta s_\alpha,s_\alpha s_\beta s_\alpha}) &= e^{-\omega_\alpha-\omega_\beta}; \end{aligned}$$

$$(5.14)\quad \begin{aligned} e^\rho\cdot a_{s_\alpha}\cdot a_{s_\alpha} &= e^{\omega_\beta-\omega_\alpha}, & e^\rho\cdot a_{s_\alpha s_\beta s_\alpha}\cdot a_{s_\alpha} &= e^{\omega_\beta},\\ e^\rho\cdot a_{s_\alpha}\cdot a_{s_\beta} &= 1, & e^\rho\cdot a_{s_\alpha s_\beta s_\alpha}\cdot a_{s_\beta} &= e^{\omega_\alpha},\\ e^\rho\cdot a_{s_\alpha}\cdot a_{s_\alpha s_\beta} &= -e^{2\omega_\beta-\omega_\alpha}, & e^\rho\cdot a_{s_\alpha s_\beta s_\alpha}\cdot a_{s_\alpha s_\beta} &= -e^{2\omega_\beta},\\ e^\rho\cdot a_{s_\alpha}\cdot a_{s_\alpha s_\beta s_\alpha} &= e^{\omega_\beta}, & e^\rho\cdot a_{s_\alpha s_\beta s_\alpha}\cdot a_{s_\alpha s_\beta s_\alpha} &= e^{\rho}. \end{aligned}$$

Now, substituting (5.8) to (5.14) in (5.7) we get

$$(5.15)\quad \begin{aligned} C_{x,y}^{1} &= 0, & C_{x,y}^{s_\alpha s_\beta} &= 1-e^{-\alpha},\\ C_{x,y}^{s_\alpha} &= 0, & C_{x,y}^{s_\beta s_\alpha} &= 0,\\ C_{x,y}^{s_\beta} &= 0, & C_{x,y}^{s_\alpha s_\beta s_\alpha} &= 1-(1-e^{-\alpha}) = e^{-\alpha}. \end{aligned}$$

REMARK 5.3. In particular, note that

$$(-1)^{l(x)+l(y)+l(s_\alpha s_\beta)} \cdot C_{x,y}^{s_\alpha s_\beta} = e^{-\alpha} - 1,$$
$$(-1)^{l(x)+l(y)+l(s_\alpha s_\beta s_\alpha)} \cdot C_{x,y}^{s_\alpha s_\beta s_\alpha} = (e^{-\alpha} - 1) + 1.$$

This verifies [9, Conjecture 3.10] (conjecture of Griffeth–Ram) for this example. We refer the reader to [10, §5] for the computation of multiplicative structure constants in rank 2 cases using different methods, and a proof of the positivity conjecture in [1].

EXAMPLE 5.4. Let $x = w_0 = s_\alpha s_\beta s_\alpha$ and $\lambda = \rho$. Let u_0 be the lift of $[\mathcal{O}_{X^{w_0}}]_T$ as in (5.6). In particular, we note that $e^\lambda \cdot L_{x^{-1}w_0}(b_w \cdot e^{-\rho}) = b_w$. Then the Chevalley structure constants of $[\mathcal{O}_{X^x}]_T \cdot [\mathcal{L}^T(\rho)]_T$ are obtained recursively as follows:

$$\begin{aligned}
Q^\rho_{x,1} &= \sum_{w\in W} e^\rho \cdot a_w \cdot b_w,\\
Q^\rho_{x,s_\alpha} &= \sum_{w\in W} e^\rho \cdot a_w \cdot L_{s_\alpha}(b_w) - Q^\rho_{x,1},\\
Q^\rho_{x,s_\beta} &= \sum_{w\in W} e^\rho \cdot a_w \cdot L_{s_\beta}(b_w) - Q^\rho_{x,1},\\
Q^\rho_{x,s_\alpha s_\beta} &= \sum_{w\in W} e^\rho \cdot a_w \cdot L_{s_\alpha s_\beta}(b_w) - Q^\rho_{x,s_\alpha} - Q^\rho_{x,s_\beta} - Q^\rho_{x,1},\\
Q^\rho_{x,s_\beta s_\alpha} &= \sum_{w\in W} e^\rho \cdot a_w \cdot L_{s_\beta s_\alpha}(b_w) - Q^\rho_{x,s_\alpha} - Q^\rho_{x,s_\beta} - Q^\rho_{x,1},\\
Q^\rho_{x,s_\alpha s_\beta s_\alpha} &= \sum_{w\in W} e^\rho \cdot a_w \cdot L_{s_\alpha s_\beta s_\alpha}(b_w) - Q^\rho_{x,s_\alpha s_\beta} - Q^\rho_{x,s_\beta s_\alpha} - Q^\rho_{x,s_\alpha}\\
&\quad - Q^\rho_{x,s_\beta} - Q^\rho_{x,1}.
\end{aligned} \tag{5.16}$$

Now, since $b_w = e^{\rho+p_w}$, from (5.3) it follows that

$$\begin{aligned}
b_1 &= e^\rho, & b_{s_\alpha s_\beta} &= e^{\omega_\beta},\\
b_{s_\alpha} &= e^{2\omega_\beta}, & b_{s_\beta s_\alpha} &= e^{\omega_\alpha},\\
b_{s_\beta} &= e^{2\omega_\alpha}, & b_{s_\alpha s_\beta s_\alpha} &= 1.
\end{aligned} \tag{5.17}$$

Hence by substituting (5.5) and (5.17) in (5.16) we get

$$\begin{aligned}
Q^\rho_{x,1} &= 0, & Q^\rho_{x,s_\alpha s_\beta} &= 0,\\
Q^\rho_{x,s_\alpha} &= 0, & Q^\rho_{x,s_\beta s_\alpha} &= 0,\\
Q^\rho_{x,s_\beta} &= 0, & Q^\rho_{x,s_\alpha s_\beta s_\alpha} &= e^{-2\rho}.
\end{aligned}$$

REMARK 5.5. It was recently brought to the notice of the author that some results in this article coincide with or follow from results in the paper by Kostant and Kumar [13]. We wish to mention that these results have been independently proved during the course of this work using slightly different techniques. We give here the exact cross references for the benefit of the reader. Our Lemma 2.1 is essentially [13, Theorem 4.4]. Also, Prop. 2.5 follows essentially from [13, Lemma 4.12], and Lemma 2.9 is the analogue of [13, Prop. 2.25] where the structure constants are determined with respect to the dual of the structure sheaf basis.

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