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THE HEYDE THEOREM ON $\boldsymbol{a}$-ADIC SOLENOIDS

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Abstract. We prove the following analogue of the Heyde theorem for $\boldsymbol{a}$-adic solenoids. Let ξ_1, ξ_2 be independent random variables with values in an $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$ and with distributions μ_1, μ_2. Let α_j, β_j be topological automorphisms of $\Sigma_{\boldsymbol{a}}$ such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1}$ are topological automorphisms of $\Sigma_{\boldsymbol{a}}$ too. Assuming that the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, we describe the possible distributions μ_1, μ_2.

1. Introduction. Many studies have been devoted to characterizing Gaussian distributions on the real line. Specifically, in 1970 Heyde proved the following theorem, which characterizes a Gaussian distribution by the symmetry of the conditional distribution of one linear form given another.

The Heyde Theorem (Heyde [12]; see also [13, Section 13.4.1]). *Let $\xi_1, \dots, \xi_n$, $n \ge 2$, be independent random variables, and α_j, β_j be nonzero constants such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \ne 0$ whenever $i \ne j$. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric then all the random variables ξ_j are Gaussian.*

In recent years, a great deal of attention has been devoted to generalizing the classical characterization theorems to random variables with values in locally compact Abelian groups (see e.g. [1]–[4], [6]–[8], [14], [15]; see also [5] and references therein). The articles [2]–[4], [14], [15] (see also [5, Chapter VI]) were devoted to finding group-theoretic analogues of the Heyde theorem. This article continues this research.

Let X be a second countable locally compact Abelian group, and $\mathrm{Aut}(X)$ be the group of topological automorphisms of X. Let ξ_j, $j = 1, \dots, n$, $n \ge 2$, be independent random variables with values in X and with distributions μ_j. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ be such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ whenever $i \ne j$. Define the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$.

We formulate the following problem.

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Problem 1. *Assuming that the conditional distribution of L_2 given L_1 is symmetric, describe the possible distributions μ_j.*

Problem 1 was solved for finite Abelian groups in [2], [14] and then for countable discrete Abelian groups in [4], [15]. Problem 1 for $\boldsymbol{a}$-adic solenoids was formulated in the book [5].

$\boldsymbol{a}$-adic solenoids are important examples of connected Abelian groups. We note that if X is a connected Abelian group of dimension one then X is topologically isomorphic to either the real line $\mathbb{R}$, or the circle group $\mathbb{T}$, or an $\boldsymbol{a}$-adic solenoid. Problem 1 was solved for $X = \mathbb{R}$ by Heyde. For $X = \mathbb{T}$ the problem is vacuous because there are no topological automorphisms α_j, β_j such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ whenever $i \neq j$.

In this article we solve Problem 1 for $\boldsymbol{a}$-adic solenoids (Theorem 3.1). It turns out that the answer depends on the topological automorphisms α_j, β_j. Note that it follows from [3] (see also [5, §16.2]) that under the condition that the characteristic functions of the distributions μ_j do not vanish, the symmetry of the conditional distribution of L_2 given L_1 implies that the μ_j are all Gaussian.

2. Notation and definitions. Let X be a locally compact Abelian group, $Y = X^*$ be its character group, and (x, y) be the value of a character $y \in Y$ at an element $x \in X$. Let K be a subgroup of X. Denote by

$$A(Y, K) = \{y \in Y : (x, y) = 1 \text{ for all } x \in K\}$$

the annihilator of K. If $\delta : X \to X$ is a continuous endomorphism, then the adjoint endomorphism $\widetilde{\delta} : Y \to Y$ is defined by $(x, \widetilde{\delta} y) = (\delta x, y)$ for all $x \in X,\ y \in Y$. We note that $\delta \in \mathrm{Aut}(X)$ if and only if $\widetilde{\delta} \in \mathrm{Aut}(Y)$. For each integer $n \neq 0$, let $f_n : X \to X$ be the endomorphism

$$f_n x = nx.$$

Set

$$X^{(n)} = f_n(X), \qquad X_{(n)} = \mathrm{Ker}\, f_n.$$

It is clear that the adjoint endomorphism $\widetilde{f_n} : Y \to Y$ is the mapping $\widetilde{f_n} y = ny$, i.e. $\widetilde{f_n} = f_n$. Denote by $\mathbb{R}$ the additive group of real numbers, by $\mathbb{Z}$ the additive group of integers, by $\mathbb{Q}$ the additive group of rational numbers with the discrete topology, and by $\mathbb{Z}(n)$ the finite cyclic group of order n. For a fixed prime p denote by $\mathbb{Z}(p^\infty)$ the set of rational numbers of the form $\{k/p^n : k = 0, 1, \ldots, p^n - 1, n = 0, 1, \ldots\}$ equipped with addition modulo 1. Then $\mathbb{Z}(p^\infty)$ is an Abelian group, which we endow with the discrete topology. Denote by $\mathrm{Aut}(X)$ the group of topological automorphisms of the group X.

Put $\boldsymbol{a} = (a_0, a_1, \ldots)$, where all $a_j \in \mathbb{Z}$, $a_j > 1$. First we recall the definition of the group $\Delta_{\boldsymbol{a}}$ of $\boldsymbol{a}$-adic integers [10, (10.2)]. As a set $\Delta_{\boldsymbol{a}}$ coincides with the Cartesian product $\mathbf{P}_{n=0}^{\infty}\{0,1,\dots,a_n-1\}$. For $\mathbf{x}=(x_0,x_1,\dots)$, $\mathbf{y}=(y_0,y_1,\dots)\in\varDelta_{\boldsymbol{a}}$ let $\mathbf{z}=\mathbf{x}+\mathbf{y}$ be defined as follows. Let $x_0+y_0=t_0a_0+z_0$, where $z_0\in\{0,1,\dots,a_0-1\}$, $t_0\in\{0,1\}$. Assume that the numbers $z_0,z_1,\dots,z_k$ and $t_0,t_1,\dots,t_k$ have already been determined. Then set $x_{k+1}+y_{k+1}+t_k=t_{k+1}a_{k+1}+z_{k+1}$, where $z_{k+1}\in\{0,1,\dots,a_{k+1}-1\}$ and $t_{k+1}\in\{0,1\}$. This defines by induction a sequence $\mathbf{z}=(z_0,z_1,z_2,\dots)$. The set $\varDelta_{\boldsymbol{a}}$ with addition defined as above is an Abelian group, whose neutral element is the identically zero sequence. Endow $\varDelta_{\boldsymbol{a}}$ with the product topology. The resulting group is called the *$\boldsymbol{a}$-adic integers*. If all of the integers a_j are equal to some fixed prime number p, we write $\varDelta_p$ instead of $\varDelta_{\boldsymbol{a}}$, and call this object the *group of p-adic integers*. Note that $\varDelta_p^*\approx\mathbb{Z}(p^\infty)$ (see [10, (25.2)]).

Consider the group $\mathbb{R}\times\varDelta_{\boldsymbol{a}}$. Let B be the subgroup of $\mathbb{R}\times\varDelta_{\boldsymbol{a}}$ of the form $B=\{(n,n\mathbf{u})\}_{n=-\infty}^{\infty}$, where $\mathbf{u}=(1,0,0,\dots)$. The factor group $\varSigma_{\boldsymbol{a}}=(\mathbb{R}\times\varDelta_{\boldsymbol{a}})/B$ is called the *$\boldsymbol{a}$-adic solenoid*. It is compact connected and has dimension one [10, (10.12), (10.13), (24.28)]. The character group of $\varSigma_{\boldsymbol{a}}$ is topologically isomorphic to the subgroup $H_{\boldsymbol{a}}\subset\mathbb{Q}$ of the form

$$H_{\boldsymbol{a}}=\left\{\frac{m}{a_0a_1\dots a_n}:n=0,1,\dots;\ m\in\mathbb{Z}\right\}.$$

We will assume without loss of generality that if $X=\varSigma_{\boldsymbol{a}}$ then $Y=X^*=H_{\boldsymbol{a}}$.

Let Y be an Abelian group, f be a function on Y, and $h\in Y$. Denote by $\varDelta_h$ the finite difference operator

$$\varDelta_h f(y)=f(y+h)-f(y).$$

A function $f(\cdot)$ on Y is called a *polynomial* if

$$\varDelta_h^{n+1}f(y)=0$$

for some n and for all $y,h\in Y$. If Y is a subgroup of $\mathbb{Q}$ then this definition of a polynomial coincides with the classical one.

Let $M^1(X)$ be the convolution semigroup of probability distributions on X, let

$$\hat\mu(y)=\int\limits_X (x,y)\,d\mu(x)$$

be the characteristic function of a distribution $\mu\in M^1(X)$, and let $\sigma(\mu)$ be the support of μ. If K is a closed subgroup of X and $\sigma(\mu)\subset K$, then $\hat\mu(y+h)=\hat\mu(y)$ for all $y\in Y$ and $h\in A(Y,K)$. If E is a closed subgroup of Y and $\hat\mu(y)=1$ for $y\in E$, then $\hat\mu(y+h)=\hat\mu(y)$ for all $y\in Y$ and $h\in E$, and we have $\sigma(\mu)\subset A(X,E)$. For $\mu\in M^1(X)$ we define the distribution $\bar\mu\in M^1(X)$ by the rule $\bar\mu(B)=\mu(-B)$ for all Borel sets $B\subset X$. Observe that $\hat{\bar\mu}(y)=\overline{\hat\mu(y)}$.

A distribution $\gamma \in M^1(X)$ is called *Gaussian* ([16, §4.6]) if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y)\exp\{-\varphi(y)\}, \quad y \in Y,$$

for some $x \in X$ and some continuous nonnegative function φ satisfying

$$\varphi(u+v) + \varphi(u-v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \tag{2.1}$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on X. It is easy to see that any nonnegative function φ on $H_{\boldsymbol{a}}$ satisfying equation (2.1) is of the form $\varphi(y) = \lambda y^2$, where $\lambda \geq 0$, $y \in H_{\boldsymbol{a}}$. It is well-known that the support of a Gaussian distribution on a locally compact Abelian group X is a coset of a connected subgroup of X. Since all factor groups of $H_{\boldsymbol{a}}$ are periodic, the set of proper nonzero closed connected subgroups of $\Sigma_{\boldsymbol{a}}$ is empty. Thus, if γ is a nondegenerate Gaussian distribution on $X = \Sigma_{\boldsymbol{a}}$ then $\sigma(\gamma) = X$.

Denote by $I(X)$ the set of idempotent distributions on X, i.e. the set of shifts of Haar distributions m_K of compact subgroups K of X. Observe that the characteristic function of m_K is

$$\hat{m}_K(y) = \begin{cases} 1, & y \in A(Y, K), \\ 0, & y \notin A(Y, K). \end{cases} \tag{2.2}$$

We note that if $\mu \in \Gamma(X) * I(X)$, i.e. $\mu = \gamma * m_K$, where $\gamma \in \Gamma(X)$, then μ is invariant with respect to a compact subgroup $K \subset X$ and under the natural homomorphism $X \to X/K$, μ induces a Gaussian distribution on the factor group X/K.

3. The Heyde theorem (the general case). Let ξ_1, ξ_2 be independent random variables with values in $X = \Sigma_{\boldsymbol{a}}$ and with distributions μ_1, μ_2. Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ and $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Assume that the conditional distribution of L_2 given L_1 is symmetric. Taking into consideration new independent random variables $\xi_j' = \alpha_j\xi_j$ we reduce the study of the possible distributions μ_j on X to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_j \in \mathrm{Aut}(X)$ and $\delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. Note that any topological automorphism δ of X is of the form

$$\delta = f_p f_q^{-1}$$

for some relatively prime p and q, where $f_p, f_q \in \mathrm{Aut}(X)$. Note that for any $\delta \in \mathrm{Aut}(X)$ the conditional distribution of L_2 given L_1 is symmetric if and only if the conditional distribution of δL_2 given L_1 is symmetric. Hence without loss of generality, we can assume from the beginning that $L_1 = \xi_1 + \xi_2$ and $L_2 = p\xi_1 + q\xi_2$, where $p, q \in \mathbb{Z}$, $pq \neq 0$, p and q are relatively prime, $f_p, f_q, f_{p\pm q} \in \mathrm{Aut}(X)$.

Now we formulate the main result of this article.

THEOREM 3.1. *Let $X = \Sigma_{\boldsymbol{a}}$. Assume that $f_p, f_q, f_{p\pm q} \in \mathrm{Aut}(X)$, and p and q are relatively prime. The following statements hold:*

(i) *Assume that $pq = -3$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. If the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then at least one of the distributions μ_j is in $\Gamma(X) * I(X)$.*

(ii) *Assume that $pq \neq -3$. Then there exist independent random variables ξ_1, ξ_2 with values in X and distributions μ_1, μ_2 such that the conditional distribution of $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric and $\mu_j \notin \Gamma(X) * I(X)$, $j = 1, 2$.*

Theorem 3.1 can be regarded as a group analogue of the Heyde theorem for $\boldsymbol{a}$-adic solenoids.

REMARK 3.2. Let $X = \Sigma_{\boldsymbol{a}}$. Note that in Theorem 3.1 we suppose that there exist relatively prime p and q such that f_p and f_q are automorphisms and $f_{p\pm q} \in \mathrm{Aut}(X)$. It is easy to prove that X has this property if and only if $f_2, f_3 \in \mathrm{Aut}(X)$.

To prove Theorem 3.1 we need some lemmas.

LEMMA 3.3. *Let X be a locally compact second countable Abelian group. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where α_j, β_j are continuous endomorphisms of X. The conditional distribution of L_2 given L_1 is symmetric if and only if the characteristic functions of the distributions μ_j satisfy the equation*

$$\hat{\mu}_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v)\hat{\mu}_2(\tilde{\alpha}_2 u + \tilde{\beta}_2 v) = \hat{\mu}_1(\tilde{\alpha}_1 u - \tilde{\beta}_1 v)\hat{\mu}_2(\tilde{\alpha}_2 u - \tilde{\beta}_2 v), \quad u, v \in Y. \tag{3.1}$$

Lemma 3.3 was proved in [5, §16.1] in the case where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. The same proof is valid for arbitrary continuous endomorphisms α_j, β_j of X.

LEMMA 3.4. *Let either $|q| = 2$ or $q = 4m + 3$, where m is an integer. Let $X = \Delta_2$. Then there exist independent identically distributed random variables ξ_1, ξ_2 with values in X and distribution $\mu \notin I(X)$ such that the conditional distribution of $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.*

Proof. Since $X = \Delta_2$, we have $Y \approx \mathbb{Z}(2^\infty)$.

Let g_0 be an arbitrary positive definite function on $Y_{(2)}$. Set

$$g(y) = \begin{cases} g_0(y), & y \in Y_{(2)}, \\ 0, & y \notin Y_{(2)}. \end{cases}$$

Then g is positive definite on Y ([11, §32]). By the Bochner theorem there exists a distribution $\mu \in M^1(X)$ such that $\hat{\mu} = g$. It is clear that g_0 can be chosen in such a way that $\mu \notin I(X)$.

Let ξ_1, ξ_2 be independent identically distributed random variables with values in X and distribution μ. We check that the conditional distribution of $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. By Lemma 3.3 it suffices to show that the characteristic function $\hat{\mu}$ satisfies (3.1) which takes the form

$$\hat{\mu}(u+v)\hat{\mu}(u+qv) = \hat{\mu}(u-v)\hat{\mu}(u-qv), \qquad u, v \in Y. \tag{3.2}$$

It is clear that if $u, v \in Y_{(2)}$ then (3.2) holds.

If either $u \in Y_{(2)}, v \notin Y_{(2)}$ or $u \notin Y_{(2)}, v \in Y_{(2)}$ then $u \pm v \notin Y_{(2)}$. Hence $\hat{\mu}(u+v) = \hat{\mu}(u-v) = 0$ and (3.2) is satisfied.

Let $u, v \notin Y_{(2)}$. Suppose that the left-hand side of (3.2) does not vanish. Then

$$u + v \in Y_{(2)}, \qquad u + qv \in Y_{(2)}. \tag{3.3}$$

Let $q = 2$. It follows from (3.3) that $v \in Y_{(2)}$, contrary to the choice of v. Hence the left-hand side of (3.2) is zero. Similarly, we prove that the right-hand side is zero.

Let $q = -2$. It follows from (3.3) that $3v \in Y_{(2)}$. Since $f_3 \in \mathrm{Aut}(Y)$ and $Y_{(2)}$ is a characteristic subgroup, we have $v \in Y_{(2)}$, contrary to the choice of v. Hence the left-hand side of (3.2) is zero. Similarly, we prove that the right-hand side is zero.

Let $q = 4m+3$. It follows from (3.3) that $(q-1)v \in Y_{(2)}$. Since $q - 1 = 2(2m+1)$ and $f_{2m+1} \in \mathrm{Aut}(Y)$, we have $2v \in Y_{(2)}$. Hence v is an element of order 4. So, $qv = -v$. It follows that (3.2) holds.

Assume now that the right-hand side of (3.2) does not vanish. Similarly, we prove that in this case (3.2) is an equality.

LEMMA 3.5. *Let $q = 4m+1$ where $m \notin \{0, -1\}$. Let $|2m+1| = p_1^{l_1} \cdots p_k^{l_k}$ be a prime decomposition of $|2m+1|$. Let $X = \varDelta_{p_1} \times \cdots \times \varDelta_{p_k}$. Then there exist independent identically distributed random variables ξ_1, ξ_2 with values in X and distribution $\mu \notin I(X)$ such that the conditional distribution of $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.*

Proof. Since $X = \varDelta_{p_1} \times \cdots \times \varDelta_{p_k}$, we have $Y \approx \mathbb{Z}(p_1^\infty) \times \cdots \times \mathbb{Z}(p_k^\infty)$.

Let g_0 be an arbitrary positive definite function on $Y_{(2m+1)}$. Set

$$g(y) = \begin{cases} g_0(y), & y \in Y_{(2m+1)}, \\ 0, & y \notin Y_{(2m+1)}. \end{cases}$$

Then g is positive definite on Y ([11, §32]). By the Bochner theorem there exists a distribution $\mu \in M^1(X)$ such that $\hat{\mu} = g$. It is clear that g_0 can be chosen so that $\mu \notin I(X)$.

Let ξ_1, ξ_2 be independent identically distributed random variables with values in X and distribution μ. We check that the conditional distribution of $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. By Lemma 3.3 it suffices to show that $\hat\mu$ satisfies (3.1) which takes the form (3.2).

Let $u, v \in Y_{(2m+1)}$. Then $qv = (q+1)v - v = -v$ and (3.2) holds.

If either $u \in Y_{(2m+1)}, v \notin Y_{(2m+1)}$ or $u \notin Y_{(2m+1)}, v \in Y_{(2m+1)}$ then $u \pm v \notin Y_{(2m+1)}$. Hence $\hat\mu(u+v) = \hat\mu(u-v) = 0$ and (3.2) is satisfied.

Let $u, v \notin Y_{(2m+1)}$. Suppose that the left-hand side of (3.2) does not vanish. Then $u + v \in Y_{(2m+1)}$, $u + qv \in Y_{(2m+1)}$. Hence $(q-1)v \in Y_{(2m+1)}$. Since $q - 1 = 4m$ and $f_{4m} \in \mathrm{Aut}(Y)$, we have $v \in Y_{(2m+1)}$, contrary to the choice of v. Hence the left-hand side of (3.2) is zero. Similarly, we prove that the right-hand side is zero.

LEMMA 3.6. *Let $X = \Sigma_{\boldsymbol{a}}$. If $f_n \in \mathrm{Aut}(X)$, where $n = p_1^{l_1} \cdots p_k^{l_k}$ is a prime decomposition, then X contains a subgroup topologically isomorphic to $\Delta_{p_1} \times \cdots \times \Delta_{p_k}$.*

Proof. Since $X = \Sigma_{\boldsymbol{a}}$, the character group $Y = H_{\boldsymbol{a}}$ is a subgroup of $\mathbb{Q}$. It is well known that the factor-group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the weak direct product

$$\mathop{\mathbf{P}^*}_{p\in\mathcal{P}} \mathbb{Z}(p^\infty),$$

where $\mathcal{P}$ is the set of prime numbers ([9, §8]). Since $Y \subset \mathbb{Q}$, we have $Y/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. The condition $f_n \in \mathrm{Aut}(X)$ implies that all f_{p_j} are in $\mathrm{Aut}(X)$, hence in $\mathrm{Aut}(Y)$. It is obvious that if p is a prime and $f_p \in \mathrm{Aut}(Y)$ then $F_p \subset Y/\mathbb{Z}$, where $F_p \approx \mathbb{Z}(p^\infty)$. Hence $L \subset Y/\mathbb{Z}$, where $L \approx \mathbb{Z}(p_1^\infty) \times \cdots \times \mathbb{Z}(p_k^\infty)$. It is clear that $Y/\mathbb{Z} = L \times M$, where M is a subgroup of $Y/\mathbb{Z}$. Since $(Y/\mathbb{Z})^* \approx A(X, \mathbb{Z}) \subset X$ and $(Y/\mathbb{Z})^* \approx L^* \times M^*$, the group X contains a subgroup topologically isomorphic to L^*. The conclusion now follows from the form of L.

LEMMA 3.7. *Let X be a locally compact second countable Abelian group. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where α_j, β_j are continuous endomorphisms of X. The linear forms L_1 and L_2 are independent if and only if the characteristic functions of the distributions μ_j satisfy the equation*

$$\hat\mu_1(\tilde\alpha_1 u + \tilde\beta_1 v)\hat\mu_2(\tilde\alpha_2 u + \tilde\beta_2 v) = \hat\mu_1(\tilde\alpha_1 u)\hat\mu_1(\tilde\beta_1 v)\hat\mu_2(\tilde\alpha_2 u)\hat\mu_2(\tilde\beta_2 v), \quad u, v \in Y. \tag{3.4}$$

Lemma 3.7 was proved in [5, §10.1] in the case where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. The same proof is valid for arbitrary continuous endomorphisms α_j, β_j of X.

LEMMA 3.8. *Let X be a locally compact second countable Abelian group, and δ_1, δ_2 be continuous endomorphisms of X. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. If the conditional distribution of the linear form $L_2 = \delta_1\xi_1 + \delta_2\xi_2$ given $L_1 = \xi_1 + \xi_2$ is* *symmetric then the linear forms* $L_1' = (\delta_1+\delta_2)\xi_1 + 2\delta_2\xi_2$ *and* $L_2' = 2\delta_1\xi_1 + (\delta_1+\delta_2)\xi_2$ *are independent.*

Proof. By Lemma 3.3 the symmetry of the conditional distribution of L_2 given L_1 implies that the $\hat\mu_j(\cdot)$ satisfy

$$\hat\mu_1(u+\varepsilon_1 v)\hat\mu_2(u+\varepsilon_2 v) = \hat\mu_1(u-\varepsilon_1 v)\hat\mu_2(u-\varepsilon_2 v), \quad u,v\in Y, \tag{3.5}$$

where $\varepsilon_j = \tilde{\delta}_j$.

Putting $u=\varepsilon_2 y, v=-y$ and then $u=-\varepsilon_1 y, v=y$ into (3.5) we obtain

$$\hat\mu_1((\varepsilon_2-\varepsilon_1)y) = \hat\mu_1((\varepsilon_1+\varepsilon_2)y)\hat\mu_2(2\varepsilon_2 y), \qquad y\in Y, \tag{3.6}$$

$$\hat\mu_2((\varepsilon_2-\varepsilon_1)y) = \hat\mu_1(-2\varepsilon_1 y)\hat\mu_2(-(\varepsilon_1+\varepsilon_2)y), \quad y\in Y. \tag{3.7}$$

Let $t,s\in Y$. Putting $u=\varepsilon_1 s+\varepsilon_2 t$, $v=s+t$ into (3.5) we obtain

$$\hat\mu_1((\varepsilon_1+\varepsilon_2)t+2\varepsilon_1 s)\hat\mu_2(2\varepsilon_2 t+(\varepsilon_1+\varepsilon_2)s) = \hat\mu_1((\varepsilon_2-\varepsilon_1)t)\hat\mu_2(-(\varepsilon_2-\varepsilon_1)s) \tag{3.8}$$

for all $s,t\in Y$. In view of (3.6) and (3.7), equation (3.8) can be written in the form

$$\begin{aligned}\hat\mu_1((\varepsilon_1+\varepsilon_2)t+2\varepsilon_1 s)&\hat\mu_2(2\varepsilon_2 t+(\varepsilon_1+\varepsilon_2)s)\\ &= \hat\mu_1((\varepsilon_1+\varepsilon_2)t)\hat\mu_2(2\varepsilon_2 t)\hat\mu_1(2\varepsilon_1 s)\hat\mu_2((\varepsilon_1+\varepsilon_2)s), \quad s,t\in Y.\end{aligned} \tag{3.9}$$

Lemma 3.7 and (3.9) imply that the linear forms $L_1' = (\delta_1+\delta_2)\xi_1+2\delta_2\xi_2$ and $L_2' = 2\delta_1\xi_1+(\delta_1+\delta_2)\xi_2$ are independent. ∎

REMARK 3.9. Lemma 3.8 implies that the Heyde theorem on the group $\mathbb{R}$ for $n=2$ can be obtained from the Skitovich–Darmois theorem.

Proof of Theorem 3.1. By Lemma 3.3 the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions of the distributions μ_j satisfy (3.1) which takes the form

$$\hat\mu_1(u+pv)\hat\mu_2(u+qv) = \hat\mu_1(u-pv)\hat\mu_2(u-qv), \quad u,v\in Y. \tag{3.10}$$

We will study the solutions of this equation.

Consider first the case where $pq=-3$. Obviously, without loss of generality we can assume that $p=1$ and $q=-3$, that is, $L_1=\xi_1+\xi_2$ and $L_2=\xi_1-3\xi_2$. Lemma 3.8 implies that the linear forms $L_1'=-2\xi_1-6\xi_2$ and $L_2'=2\xi_1-2\xi_2$ are independent. Making the substitution $\zeta_1=2\xi_1$ and $\zeta_2=-2\xi_2$, we find that the linear forms $L_1''=-\zeta_1+3\zeta_2$ and $L_2''=\zeta_1+\zeta_2$ are also independent. As has been proved in [6], the independence of L_1'' and L_2'' implies that the distribution of at least one of the random variables ζ_j belongs to $\Gamma(X)*I(X)$. Returning to the random variables ξ_j, we obtain statement (i) of Theorem 3.1.

Now assume that $pq\neq-3$. Two cases are possible: pq is either composite or prime.

We prove that in both cases there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the conditional distribution of L_2 given L_1 is symmetric.

CASE 1: *pq is composite.* In this case we follow the scheme of the proof of the analogous case in [6, Theorem 3.1].

Put $s = p - q$, and decompose $|s|$ into prime factors, $|s| = s_1^{k_1} \cdots s_l^{k_l}$. Since $f_s \in \mathrm{Aut}(X)$, the group Y contains the subgroup

$$H = \left\{ \frac{m}{s_{j_1}^{n_1} \cdots s_{j_r}^{n_r}} : m, n_j \in \mathbb{Z} \right\}.$$

If $|s| = 1$ we suppose that $H = \mathbb{Z}$. Set $G = H^*$.

CASE 1a: $|p| > 1$, $|q| > 1$. Since p and s are relatively prime, and so are q and s, we have $H^{(p)} \neq H$ and $H^{(q)} \neq H$. Assume that $\lambda_j \in M^1(G)$ and $\sigma(\lambda_1) \subset A(G, H^{(p)})$, $\sigma(\lambda_2) \subset A(G, H^{(q)})$. It follows that $\hat{\lambda}_1(y) = 1$ for $y \in H^{(p)}$, and $\hat{\lambda}_2(y) = 1$ for $y \in H^{(q)}$. Therefore

$$\hat{\lambda}_1(u + pv) = \hat{\lambda}_1(u),\ \hat{\lambda}_2(u + qv) = \hat{\lambda}_2(u), \quad u, v \in H. \tag{3.11}$$

Consider the functions g_j on Y of the form

$$g_j(y) = \begin{cases} \hat{\lambda}_j(y), & y \in H, \\ 0, & y \notin H. \end{cases} \tag{3.12}$$

These functions are positive definite ([11, §32]). By the Bochner theorem there exist distributions $\mu_j \in M^1(X)$ such that $\hat{\mu}_j = g_j$, $j = 1, 2$. We will show that the characteristic functions $\hat{\mu}_j$ satisfy (3.10).

We deduce from (3.11) and (3.12) that if $u, v \in H$, then (3.10) holds.

Let $u \notin H, v \in H$. Then $u \pm pv \notin H$ and hence $\hat{\mu}_1(u + pv) = \hat{\mu}_1(u - pv) = 0$, and (3.10) is satisfied.

Let $v \notin H$. Suppose that the left-hand side of (3.10) does not vanish. Then $u + pv \in H$ and $u + qv \in H$. Hence $sv \in H$. Therefore $v \in H$, contrary to the choice of v. Hence the left-hand side of (3.10) is zero. Similarly, we prove that the right-hand side is zero.

So, the characteristic functions $\hat{\mu}_j$ satisfy (3.10). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then by Lemma 3.3 the conditional distribution of L_2 given L_1 is symmetric. It is clear that λ_j can be chosen in such a way that $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$. The desired statement in Case 1a is proven.

CASE 1b: *Either* $|p| = 1, |q| > 1$ *or* $|p| > 1, |q| = 1$. Assume for definiteness that $|p| = 1$. Without loss of generality, we suppose $p = 1$. Let $q = q_1 q_2$ be a decomposition of q, where $|q_j| > 1$, $j = 1, 2$. It is obvious that if $f_q \in \mathrm{Aut}(X)$, then $f_{q_1}, f_{q_2} \in \mathrm{Aut}(X)$. Note that the conditional distribution of $L_2 = \xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if the conditional distribution of $L'_2 = \frac{1}{q_1}\xi_1 + q_2\xi_2$ given $L'_1 = \xi_1 + \xi_2$ is symmetric. Making the substitution $\zeta_1 = \frac{1}{q_1}\xi_1$, we reduce the problem to the case when $L''_1 = q_1\xi_1 + \xi_2$, $L''_2 = \xi_1 + q_2\xi_2$. Equation (3.10) in this case takes the form

$$\hat{\mu}_1(q_1u + v)\hat{\mu}_2(u + q_2v) = \hat{\mu}_1(q_1u - v)\hat{\mu}_2(u - q_2v), \quad u, v \in Y. \tag{3.13}$$

Assume that $\lambda_j \in M^1(G)$ and $\sigma(\lambda_j) \subset A(G, H^{(q_j)})$, $j = 1, 2$. It is obvious that $\hat{\lambda}_j(y) = 1$ for $y \in H^{(q_j)}$. Hence

$$\hat{\lambda}_1(q_1u + v) = \hat{\lambda}_1(v), \quad \hat{\lambda}_2(u + q_2v) = \hat{\lambda}_2(u), \quad u, v \in H. \tag{3.14}$$

In the same manner as in Case 1a we define the functions g_j by (3.12) and the distributions $\mu_j \in M^1(X)$. We will show that the characteristic functions $\hat{\mu}_j$ satisfy (3.13).

We conclude from (3.14) and (3.12) that if $u, v \in H$, then (3.13) holds.

Let $u \in H, v \notin H$. Then $q_1u \pm v \notin H$ and hence $\hat{\mu}_1(q_1u+v) = \hat{\mu}_1(q_1u-v) = 0$, and equation (3.13) is satisfied.

Let $u \notin H$. Suppose that the left-hand side of (3.13) does not vanish. Then $q_1u+v \in H$ and $u+q_2v \in H$. Hence $su \in H$. Therefore $u \in H$, contrary to the choice of u. Hence the left-hand side of (3.13) is zero. Reasoning similarly we show that the right-hand side is also zero.

So, the characteristic functions $\hat{\mu}_j$ satisfy (3.13). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then by Lemma 3.3 the conditional distribution of the linear form L_2 given L_1 is symmetric. It is clear that λ_j can be chosen in such a way that $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$. The desired statement in Case 1b is proven.

Case 2: *pq is a prime, i.e. either $|p|=1$, $|q|>1$ or $|p|>1$, $|q|=1$.* Assume for definiteness that $p = 1$ and q is a prime, i.e. $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 + q\xi_2$. Equation (3.10) takes the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + qv) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - qv), \quad u, v \in Y. \tag{3.15}$$

Case 2a: $|q| = 2$. Since $f_2 \in \mathrm{Aut}(X)$, Lemma 3.6 implies that X contains a subgroup topologically isomorphic to Δ_2. Then Theorem 3.1(ii) follows from Lemma 3.4.

Let q be an odd number. There exist two possibilities: 1) $q = 4m + 3$; 2) $q = 4m + 1$.

Case 2b: $q = 4m + 3$. Note that since q is odd and $f_{q+1} \in \mathrm{Aut}(X)$, the endomorphism f_2 is in $\mathrm{Aut}(X)$. Hence Lemma 3.6 implies that X contains a subgroup topologically isomorphic to Δ_2. Then Theorem 3.1(ii) follows from Lemma 3.4.

CASE 2c: $q = 4m+1$ $(m \neq -1)$. Since $f_{q+1} \in \mathrm{Aut}(X)$ and $q+1=2(2m+1)$, f_{2m+1} is a topological automorphism of X. Let $|2m+1| = p_1^{l_1} \cdots p_k^{l_k}$ be a prime decomposition. Lemma 3.6 implies that X contains a subgroup topologically isomorphic to $\Delta_{p_1} \times \cdots \times \Delta_{p_k}$. Then Theorem 3.1(ii) follows from Lemma 3.5.

REMARK 3.10. Theorem 3.1(i) cannot be strengthened. Namely, if $pq = -3$ then there exist independent random variables ξ_1, ξ_2 with values in X and distributions μ_1, μ_2 such that the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric and one of the distributions μ_j is not in $\Gamma(X) * I(X)$.

It suffices to consider the case when $p = 1$, $q = -3$. We shall construct solutions of equation (3.1) which takes the form

$$\hat{\mu}_1(u+v)\hat{\mu}_2(u-3v) = \hat{\mu}_1(u-v)\hat{\mu}_2(u+3v), \qquad u, v \in Y. \tag{3.16}$$

Let γ_1 and γ_2 be Gaussian distributions on X with characteristic functions $\hat{\gamma}_1(y) = e^{-3y^2}$ and $\hat{\gamma}_2(y) = e^{-y^2}$. It is easy to verify that these functions satisfy (3.16).

Since $f_{p+q} \in \mathrm{Aut}(X)$, we have $f_2 \in \mathrm{Aut}(X)$. Hence $f_2 \in \mathrm{Aut}(Y)$. Therefore Y contains a subgroup of dyadic rational numbers, say H. Since $f_q \in \mathrm{Aut}(X)$, we have $f_3 \in \mathrm{Aut}(X)$. Hence $f_3 \in \mathrm{Aut}(Y)$. Therefore $1/3 \in Y$. Denote by L the subgroup in Y generated by H and the element $1/3$. Observe that $L = \{H, 1/3 + H, 2/3 + H\}$. Let $G = A(X, H)$, $K = A(X, L)$. Let $\omega_1 = \frac{1}{2}[m_G + m_K]$ and $\omega_2 = m_G$. It follows from (2.2) that

$$\hat{\omega}_1(y) = \begin{cases} 1, & y \in H, \\ 1/2, & y \in L \setminus H, \\ 0, & y \notin L, \end{cases} \qquad \hat{\omega}_2(y) = \begin{cases} 1, & y \in H, \\ 0, & y \notin H. \end{cases} \tag{3.17}$$

We verify that these functions satisfy (3.16).

It is clear that if $u \in H$, $v \in L$ then equation (3.16) is an equality.

If $u \in L \setminus H$, $v \in L$ then $u \pm 3v \notin H$. Hence $\hat{\omega}_2(u-3v) = \hat{\omega}_2(u+3v) = 0$ and (3.16) holds.

If either $u \in L$, $v \notin L$ or $u \notin L$, $v \in L$ then $u \pm v \notin L$. Hence $\hat{\omega}_1(u+v) = \hat{\omega}_2(u-v) = 0$ and (3.17) is satisfied.

Let $u, v \notin L$. Suppose that the left-hand side of (3.16) does not vanish. Then $u+v \in L$ and $u-3v \in H$. Hence $4v \in L$. We deduce that $v \in L$, contrary to the choice of v. Hence the left-hand side of (3.16) is zero. Reasoning similarly we show that the right-hand side is also zero.

Put $\mu_j = \gamma_j * \omega_j$, $j = 1, 2$. It is obvious that the functions $\hat{\mu}_j(y)$ satisfy (3.16). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then Lemma 3.3 implies that the conditional distribution of the linear form $L_2 = \xi_1 - 3\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. By construction $\mu_1 \notin \Gamma(X) * I(X)$ and $\mu_2 \in \Gamma(X) * I(X)$.

REMARK 3.11. We note that if in Theorem 3.1 the distributions μ_1, μ_2 have nonvanishing characteristic functions, then $\mu_1, \mu_2 \in \Gamma(X)$. Indeed, it follows from the conditions on the coefficients of the linear forms that one of the numbers $p, q, p \pm q$ is even. So, $f_2 \in \mathrm{Aut}(X)$. Hence the group $X = \Sigma_{\boldsymbol{a}}$ does not contain elements of order two. The following theorem (see [3]) implies the desired statement: Let X be a locally compact second countable Abelian group containing no elements of order two. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 with nonvanishing characteristic functions. Consider the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_j, \delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. If the conditional distribution of L_2 given L_1 is symmetric then $\mu_1, \mu_2 \in \Gamma(X)$.

4. The Heyde theorem (the special case). We prove in this section that Theorem 3.1 can be essentially strengthened if we assume in addition that the support of at least one of the distributions μ_j is not contained in a coset of a proper closed subgroup of X.

Let $\mu \in M^1(X)$. It is easy to see that $\sigma(\mu)$ is not contained in a coset of a proper closed subgroup of X if and only if

$$\{y \in Y : |\hat{\mu}(y)| = 1\} = \{0\}. \tag{4.1}$$

THEOREM 4.1. *Let $X = \Sigma_{\boldsymbol{a}}$. Assume that $f_p, f_q, f_{p\pm q} \in \mathrm{Aut}(X)$ for some p and q relatively prime. The following statements hold:*

- (i) *Assume $pq > 0$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 such that at least one support $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of X. If the conditional distribution of the linear form $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then $\mu_1 = \mu_2 = m_X$.*
- (ii) *Assume $pq = -3$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2 such that at least one support $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of X. If the conditional distribution of $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric then at least one of the distributions μ_j is in $\Gamma(X) * I(X)$.*
- (iii) *Assume $pq < 0$ and $pq \neq -3$. Then there exist independent random variables ξ_1, ξ_2 with values in X and distributions μ_1, μ_2 such that the conditional distribution of $L_2 = p\xi_1 + q\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, the distributions μ_j are not in $\Gamma(X) * I(X)$, and none of the supports $\sigma(\mu_j)$ is contained in a coset of a proper closed subgroup of X.*

To prove Theorem 4.1 we need some lemmas. Lemmas 4.2 and 4.3 below were proved in [6] in the case $a = 1$. For $a \neq 1$ the proofs follow the same scheme.

LEMMA 4.2. *Let Y be an arbitrary Abelian group, let $a, b \in \mathbb{Z}$, $ab \neq 0$, and let g_1 and g_2 be functions on Y satisfying the equation*

$$g_1(u + av)g_2(u + bv) = g_1(u)g_1(av)g_2(u)g_2(bv), \quad u, v \in Y, \tag{4.2}$$

and the conditions

$$g_1(-y) = \overline{g_1(y)}, \quad g_2(-y) = \overline{g_2(y)}, \quad y \in Y, \quad g_1(0) = g_2(0) = 1. \tag{4.3}$$

Set $c = a - b$. If $g_1(cz_0)g_2(cz_0) \neq 0$ for a certain $z_0 \in Y$ then $g_1(y)g_2(y) \neq 0$ for all y in the subgroup $M = \{kabz_0\}_{k \in \mathbb{Z}}$.

Proof. Putting $u = -by$, $v = y$ and then $u = ay$, $v = -y$ in (4.2) we get

$$g_1(cy) = g_1(-by)g_1(ay)g_2(-by)g_2(by), \quad y \in Y, \tag{4.4}$$

$$g_2(cy) = g_1(ay)g_1(-ay)g_2(ay)g_2(-by), \quad y \in Y. \tag{4.5}$$

Substituting $y = z_0$ into (4.4) and (4.5) we conclude that

$$g_1(az_0) \neq 0, \quad g_1(bz_0) \neq 0, \quad g_2(az_0) \neq 0, \quad g_2(bz_0) \neq 0. \tag{4.6}$$

Putting $u = az_0$, $v = kz_0$ and then $u = bz_0$, $v = kz_0$, $k \in \mathbb{Z}$, in (4.2) we obtain

$$g_1((k+1)az_0)g_2((bk+a)z_0) = g_1(az_0)g_1(akz_0)g_2(az_0)g_2(bkz_0), \tag{4.7}$$

$$g_1((ak+b)z_0)g_2((k+1)bz_0) = g_1(bz_0)g_1(akz_0)g_2(bz_0)g_2(bkz_0). \tag{4.8}$$

Taking into account (4.6), it follows by induction from (4.7) and (4.8) that $g_1(kaz_0) \neq 0$, $g_2(kbz_0) \neq 0$, for all $k \in \mathbb{Z}$, as desired.

LEMMA 4.3. *Let M be an arbitrary subgroup in $\mathbb{Q}$, and g_1 and g_2 be functions on M satisfying* (4.2) *and* (4.3) *with $Y = M$, and the conditions*

$$0 < g_1(y) \leq 1, \quad 0 < g_2(y) \leq 1, \quad y \in M. \tag{4.9}$$

Put $c = b - a$. Then on the subgroup $M^{(cab)}$ the following representation holds:

$$g_1(y) = \exp\{-\lambda_1 y^2\}, \quad g_2(y) = \exp\{-\lambda_2 y^2\}, \tag{4.10}$$

for some $\lambda_j \geq 0$.

Proof. Set $\varphi_1(y) = -\ln g_1(y)$ and $\varphi_2(y) = -\ln g_2(y)$. It follows from (4.2) that

$$\varphi_1(u + av) + \varphi_2(u + bv) = A(u) + B(v), \quad u, v \in M, \tag{4.11}$$

where $A(u) = \varphi_1(u) + \varphi_2(u)$, $B(v) = \varphi_1(av) + \varphi_2(bv)$.

We use the finite difference method to solve (4.11).

Let $h_1 \in M$. Substitute $u + bh_1$ for u and $v - h_1$ for v in (4.11) and subtract (4.11) from the resulting equation. We get

$$\varDelta_{ch_1}\varphi_1(u + av) = \varDelta_{bh_1}A(u) + \varDelta_{-h_1}B(v). \tag{4.12}$$

Putting $v = 0$ in (4.12) and subtracting the resulting equation from (4.12) we obtain

$$\varDelta_{av}\varDelta_{ch_1}\varphi_1(u) = \varDelta_{-h_1}B(v) - \varDelta_{-h_1}B(0). \tag{4.13}$$

Let $h_2 \in M$. Substitute $u + h_2$ for u in (4.13) and subtract (4.13) from the resulting equation. We get

$$\varDelta_{h_2}\varDelta_{av}\varDelta_{ch_1}\varphi_1(u) = 0. \tag{4.14}$$

We conclude from (4.14) that the function φ_1 satisfies

$$\varDelta_h^3\varphi_1(y) = 0, \quad y \in M,\ h \in M^{(ca)}. \tag{4.15}$$

Reasoning similarly we get

$$\varDelta_h^3\varphi_2(y) = 0, \quad y \in M,\ h \in M^{(cb)}. \tag{4.16}$$

It follows from (4.15) and (4.16) that the φ_j are polynomials of degree 2 on the subgroup $M^{(cab)}$. Taking into account (4.3) and (4.9) we get $\varphi_j(y) = \lambda_j y^2$ where $y \in M^{(cab)}$ and $\lambda_j \geq 0$.

Proof of Theorem 4.1. Let $pq > 0$. Lemma 3.8 implies that the linear forms $L_1' = (p+q)\xi_1 + 2q\xi_2$ and $L_2' = 2p\xi_1 + (p+q)\xi_2$ are independent. Making the substitution $\xi_1' = (p+q)\xi_1$ and $\xi_2' = 2q\xi_2$, we find that the linear forms $L_1' = \xi_1' + \xi_2'$ and $L_2' = \frac{2p}{p+q}\xi_1' + \frac{p+q}{2q}\xi_2'$ are also independent. We also note that if $\delta \in \mathrm{Aut}(X)$ then linear forms L_1 and L_2 are independent if and only if so are L_1 and δL_2. Thus we may assume without loss of generality that $L_1' = \xi_1' + \xi_2'$ and $L_2' = 4pq\xi_1' + (p+q)^2\xi_2'$. Denote by μ_j' the distributions of the random variables ξ_j'. Since $f_2, f_p, f_q, f_{p+q} \in \mathrm{Aut}(X)$, it suffices to prove that $\mu_j' = m_X$.

By Lemma 3.7 the independence of L_1' and L_2' implies that the characteristic functions of the distributions μ_j' satisfy (3.4) which takes the form

$$\hat\mu_1'(u+4pqv)\hat\mu_2'(u+(p+q)^2v) = \hat\mu_1'(u)\hat\mu_1'(4pqv)\hat\mu_2'(u)\hat\mu_2'((p+q)^2v), \quad u, v \in Y. \tag{4.17}$$

It is clear that the characteristic functions of the distributions $\bar\mu_j'$ also satisfy (4.17). Therefore the characteristic functions of the distributions $\nu_j = \mu_j' * \bar\mu_j'$ satisfy (4.17). Note that $\hat\nu_j(y) = |\hat\mu_j'(y)|^2 \geq 0$, $j = 1, 2$. Moreover, since at least one support $\sigma(\mu_j)$ is not contained in a coset of a proper closed subgroup of X, at least one support $\sigma(\nu_j)$ is not contained in any such coset. It follows from (4.1) that for at least one j,

$$\{y \in Y : \hat\nu_j(y) = 1\} = \{0\}. \tag{4.18}$$

Putting $u = -(p+q)^2 y, v = y$ and then $u = -4pqy, v = y$ into (4.17) we obtain

$$\hat{\nu}_1((p-q)^2 y) = \hat{\nu}_1((p+q)^2 y)\hat{\nu}_1(4pqy)\hat{\nu}_2^2((p+q)^2 y), \quad y \in Y, \tag{4.19}$$

$$\hat{\nu}_2((p-q)^2 y) = \hat{\nu}_1(4pqy)\hat{\nu}_2(4pqy)\hat{\nu}_2((p+q)^2 y), \quad y \in Y. \tag{4.20}$$

Assume first that $\hat{\nu}_1(y)\hat{\nu}_2(y) = 0$ for all $y \in Y \setminus \{0\}$. It follows from (4.19) that $\hat{\nu}_1((p-q)^2 y) = 0$ for all $y \in Y \setminus \{0\}$. Since $f_{p-q} \in \mathrm{Aut}(X)$, we conclude that $\hat{\nu}_1(y) = 0$ for all $y \in Y \setminus \{0\}$. Hence $\nu_1 = m_X$, so that $\mu_1' = m_X$. Similarly, (4.20) implies that $\mu_2' = m_X$.

Assume now that $\hat{\nu}_1(y_0)\hat{\nu}_2(y_0) \neq 0$ for some nonzero $y_0 \in Y$. Since $f_{p-q} \in \mathrm{Aut}(X)$, we have $Y^{((p-q)^2)} = Y$. We apply Lemma 4.2 to obtain a subgroup $M \subset Y$ such that $\hat{\nu}_1(y)\hat{\nu}_2(y) \neq 0$ for all $y \in M$. By Lemma 4.3 the restrictions of the characteristic functions $\hat{\nu}_1$ and $\hat{\nu}_2$ to $M^{(4pq(p+q)^2)(p-q)^2)}$ have form (4.10). Substituting these representations into (4.17) we get

$$4pq\lambda_1 + (p+q)^2\lambda_2 = 0.$$

Since $pq > 0$, this equality implies that $\lambda_1 = \lambda_2 = 0$. Hence $\hat{\nu}_1(y) = \hat{\nu}_2(y) = 1$ for $y \in M^{(4pq(p+q)^2)(p-q)^2)}$, which contradicts (4.18).

Let $pq = -3$. The desired statement follows from Theorem 3.1(i).

Let $pq < 0$, $pq \neq -3$. Denote by ω_j the distributions constructed in the proof of Theorem 3.1 in the corresponding cases. We note that $\hat{\omega}_1$ and $\hat{\omega}_2$ satisfy (3.10). Denote by the γ_j Gaussian distributions on X with $\hat{\gamma}_1(y) = e^{-\lambda y^2}$ and $\hat{\gamma}_2(y) = e^{\frac{p}{q}\lambda y^2}$, where $\lambda > 0$. It is easy to verify that $\hat{\gamma}_1$ and $\hat{\gamma}_2$ satisfy (3.10). Put $\mu_j = \omega_j * \gamma_j$. It is obvious that $\hat{\mu}_1$ and $\hat{\mu}_2$ also satisfy (3.10). Since the support of a symmetric nondegenerate Gausssian distribution is a connected closed subgroup, we have $\sigma(\gamma_j) = X$. Hence $\sigma(\mu_j) = X$. By construction, $\mu_j \notin \Gamma(X) * I(X)$. Thus the μ_j are as desired.

Remark 4.4. The distributions constructed in Remark 3.9 show that Theorem 4.1(ii) cannot be strengthened to the statement that both μ_j is in $\Gamma(X) * I(X)$.

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