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ON SUMS OF POWERS OF THE POSITIVE INTEGERS

BY

A. SCHINZEL (Warszawa)

Abstract. The pairs (k, m) are studied such that for every positive integer n we have $1^k + 2^k + \cdots + n^k \mid 1^{km} + 2^{km} + \cdots + n^{km}$.

W. Bednarek asked in a letter for a characterization of pairs $\langle k, m\rangle$ of positive integers such that for every positive integer n,

$$1^k + 2^k + \ldots + n^k \mid 1^{km} + 2^{km} + \ldots + n^{km}. \tag{1}$$

The following theorem contains a partial answer with the help of Bernoulli numbers B_n. Recall that

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_{2l+1} = 0,$$

and the Bernoulli polynomial $\sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}$ is denoted by $B_n(x)$. We shall prove

THEOREM 1. *If the divisibility* (1) *holds for every positive integer n, then m is odd and*

$$B_{km}/B_k \in \mathbb{Z} \textit{ for } k \textit{ even}, \quad mB_{km-1}/B_{k-1} \in \mathbb{Z} \textit{ for } k \textit{ odd } \geq 3. \tag{2}$$

The condition is sufficient for $k \leq 3$, but insufficient for $k = 4$ and infinitely many m.

In fact we propose

CONJECTURE. *For $k > 3$ the divisibility* (1) *holds for every positive integer n only for $m = 1$.*

To support this conjecture we shall prove

THEOREM 2. *For $k = 4$, $n \equiv 58966743 \pmod{5^6 \cdot 11251^2}$ the divisibility* (1) *holds only for $m = 1$.*

THEOREM 3. *For $m = n = 3$ the divisibility* (1) *holds only for $k \leq 3$.*

LEMMA 1. *For all positive integers k and n,*

$$1^k + \cdots + (n-1)^k =: S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}).$$

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Proof. See [1, Chapter V, §6, Theorem 3]. ∎

LEMMA 2. *If $P, Q \in \mathbb{Q}[x]$ and $P(n)/Q(n) \in \mathbb{Z}$ for all sufficiently large integers n then $r(x) = P(x)/Q(x)$ is an integer-valued polynomial.*

Proof. We may assume that $P, Q \in \mathbb{Z}[x]$ and $D(x) = (P(x), Q(x))_{\mathbb{Z}}$, $P = DP_1$, $Q = DQ_1$, $P_1, Q_1 \in \mathbb{Z}[x]$, $(P_1, Q_1) = 1$ and $P_1(n)/Q_1(n) \in \mathbb{Z}$ for $n > n_1$. Let R be the resultant of P_1, Q_1. We have $R = AP_1 + BQ_1$, where $A, B \in \mathbb{Z}[x]$. If $\deg Q_1 > 0$, then $|Q_1(n)| > |R|$ for $n > n_2$. Choosing $n > \max\{n_1, n_2\}$ we infer that $Q(n) \mid R$, which is impossible. Thus $\deg Q_1 = 0$, $r \in \mathbb{Q}[x]$. Suppose that $r(n_0) \in \mathbb{Z}$ and let $Q(n_0) = q$, $P(n_0) = p \not\equiv 0 \pmod{q}$. For $n > n_1$, $n \equiv n_0 \pmod{q}$, we have $P(n) \equiv p \pmod{q}$, $Q(n) \equiv 0 \pmod{q}$, so that $P_1(n)/Q_1(n) \notin \mathbb{Z}$, a contradiction. ∎

LEMMA 3. *If $3^\nu \parallel 2N$, where $N = n, n+1$ or $n + 1/2$ and $\nu \geq 1$, then for every positive integer m,*

$$3^{\nu-1} \mid S_{2m}(n+1). \tag{3}$$

Proof. Let $3^\mu \parallel m$. We distinguish two cases: $\nu \leq \mu + 2$ and $\nu > \mu + 2$. In the former case, for every integer i,

$$i^{2m} \equiv \begin{cases} 1 \pmod{3^{\mu+1}} & \text{if } i \not\equiv 0 \pmod{3}, \\ 0 \pmod{3^{\mu+1}} & \text{if } i \equiv 0 \pmod{3}. \end{cases}$$

Hence

$$S_{2m}(n+1) \equiv \left\lceil \frac{2n}{3} \right\rceil = \frac{2N}{3} \pmod{3^{\mu+1}}$$

and (3) holds.

In the latter case, for every integer $i \not\equiv 0 \pmod{3}$ there exists just one integer $j \equiv 1 \pmod{3^{\mu+1}}$, $0 < j < 3^\nu$, such that

$$i^{2m} \equiv j \pmod{3^\nu}. \tag{4}$$

To every $j \equiv 1 \pmod{3^{\mu+1}}$, $0 < j < 3^\nu$, there correspond $2\cdot 3^{\mu-\nu}N$ values of $i \not\equiv 0 \pmod{3}$, $1 \leq i \leq n$, satisfying (4). Hence

$$\sum_{\substack{i=1 \\ i\not\equiv 0 \,(\mathrm{mod}\, 3)}}^{n} i^{2m} \equiv 2 \cdot 3^{\mu-\nu} N \sum_{\substack{j\equiv 1 \,(\mathrm{mod}\, 3^{\mu+1}) \\ 0<j<3^\nu}} j \pmod{3^\nu}.$$

However,

$$\sum_{\substack{j\equiv 1 \,(\mathrm{mod}\, 3^{\mu+1}) \\ 0<j<3^\nu}} j = 3^{\nu-\mu-1} + \frac{3^\nu(3^{\nu-\mu-1} - 1)}{2} \equiv 3^{\nu-\mu-1} \pmod{3^\nu},$$

thus

$$\sum_{\substack{i=1\\ i\not\equiv 0 \pmod 3}}^{n} i^{2m} \equiv 2\,\frac{N}{3} \pmod{3^{\nu}}, \qquad \sum_{\substack{i=1\\ i\not\equiv 0 \pmod 3}}^{n} i^{2m} \equiv 0 \pmod{3^{\nu-1}}.$$

Similarly for $k < \nu - \mu - 2$,

$$\sum_{\substack{i=1\\ i\not\equiv 0 \pmod 3}}^{\lfloor n/3^k \rfloor} i^{2m} = \sum_{\substack{i=1\\ i\not\equiv 0 \pmod 3}}^{\lfloor N/3^k \rfloor} i^{2m} \equiv 2\,\frac{N}{3^{k+1}} \pmod{3^{\nu-k}},$$

thus

$$3^{2km} \sum_{\substack{i=1\\ i\not\equiv 0 \pmod 3}}^{\lfloor n/3^k \rfloor} i^{2m} \equiv 0 \pmod{3^{\nu-1}}$$

and

$$\sum_{\substack{i=1\\ i\not\equiv 0 \pmod{3^{\nu-\mu-2}}}}^{n} i^{2m} \equiv 0 \pmod{3^{\nu-1}}. \tag{5}$$

However, if $i \equiv 0 \pmod{3^{\nu-\mu-2}}$ and $\nu - \mu - 2 > 0$, then since

$$2m(\nu-\mu-2) \geq 2\cdot 3^{\mu}(\nu-\mu-2) \geq \nu - 1,$$

we have $i^{2m} \equiv 0 \pmod{3^{\nu-1}}$ and (5) implies (3). ■

LEMMA 4. *If $2^{\nu} \parallel N$, where $N = n$ or $n+1$ and $\nu \geq 1$, then for every positive integer $r > 2$,*

$$2^{\nu-1} \mid S_{2r}(n+1). \tag{6}$$

REMARK. The lemma is also true for $r \leq 2$, but this will not be needed in what follows.

Proof. Let $2^{\rho} \parallel r$. We distinguish two cases: $\nu \leq \rho + 4$ and $\nu > \rho + 4$.

In the former case, for every integer i,

$$i^{2r} \equiv \begin{cases} 1 \pmod{2^{\rho+3}} & \text{if } i \equiv 1 \pmod 2, \\ 0 \pmod{2^{\rho+3}} & \text{if } i \equiv 0 \pmod 2 \end{cases}$$

(here we use $r > 1$). Hence

$$S_{2r}(n+1) \equiv \left\lceil \frac{n}{2} \right\rceil = \frac{N}{2} \pmod{2^{\rho+3}}$$

and (6) follows.

In the latter case, for every integer $i \equiv 1 \pmod 2$ there exists just one integer $j \equiv 1 \pmod{2^{\rho+3}}$, $0 < j < 2^{\nu}$, such that

$$i^{2r} \equiv j \pmod{2^{\nu}}.$$

To every $j \equiv 1 \pmod{2^{\rho+3}}$, $0 < j < 2^\nu$, there correspond $2^{\rho+2-\nu}N$ values of $i \equiv 1 \pmod 2$, $1 \le i \le n$. Hence

$$\sum_{\substack{i=1\\ i\equiv 1 \,(\mathrm{mod}\, 2)}}^{n} i^{2r} \equiv 2^{\rho+2-\nu} N \sum_{\substack{j\equiv 1 \,(\mathrm{mod}\, 2^{\rho+3})\\ 0<j<2^\nu}} j \pmod{2^\nu}.$$

However,

$$\sum_{\substack{j\equiv 1 \,(\mathrm{mod}\, 2^{\rho+3})\\ 0<j<2^\nu}} j = 2^{\nu-\rho-3} + \frac{2^\nu(2^{\nu-\rho-3}-1)}{2} \equiv 2^{\nu-\rho-3} \pmod{2^{\nu-1}},$$

thus

$$\sum_{\substack{i=1\\ i\equiv 1 \,(\mathrm{mod}\, 2)}}^{n} i^{2\nu} \equiv \frac{N}{2} \equiv 0 \pmod{2^{\nu-1}}.$$

Similarly, for $k < \nu - \rho - 4$,

$$\sum_{\substack{i=1\\ i\equiv 1 \,(\mathrm{mod}\, 2)}}^{\lfloor n/2^k \rfloor} i^{2r} = \sum_{\substack{i=1\\ i\equiv 1 \,(\mathrm{mod}\, 2)}}^{N/2^k} i^{2r} \equiv 0 \pmod{2^{\nu-1-k}},$$

thus

$$2^{2kr} \sum_{i=1}^{\lfloor n/2^k \rfloor} i^{2r} \equiv 0 \pmod{2^{\nu-1}}$$

and

$$\sum_{\substack{i=1\\ i\not\equiv 0 \,(\mathrm{mod}\, 2^{\nu-\rho-4})}}^{n} i^{2r} \equiv 0 \pmod{2^{\nu-1}}. \tag{7}$$

However, if $i \equiv 0 \pmod{2^{\nu-\rho-4}}$ and $\nu - \rho - 4 > 0$, then since $r > 2$,

$$2r(\nu-\rho-4) \ge \max\{2^{\rho+1}, 6\}(\nu-\rho-4) \ge \nu - 1,$$

we have $i^{2r} \equiv 0 \pmod{2^{\nu-1}}$ and (7) implies (6). ∎

LEMMA 5. *If a prime p satisfies $p-1 \nmid k$, then p does not divide the denominator of B_k. If $p-1 \mid k$, then p occurs in the denominator of B_k in the first power only.*

Proof. This is the von Staudt theorem, see [1, Chapter V, §6, Theorem 4]. ∎

Proof of Theorem 1. Necessity. Since (1) holds for $n = 2$ we obtain $m \equiv 1 \pmod 2$. Consider now k even. By Lemma 1 we have

$$S_k(n) = \frac{1}{k+1} B_{k+1}(n), \qquad S_{k+1}(n) = \frac{1}{km+1} B_{km+1}(n),$$

hence, for all integers $n > 1$, $B_{k+1}(n) > 0$ and

$$\frac{k+1}{km+1}\,\frac{B_{km+1}(n)}{B_{k+1}(n)} \in \mathbb{Z}.$$

By Lemma 2,

$$r(x) = \frac{k+1}{km+1}\,\frac{B_{km+1}(x)}{B_{k+1}(x)}$$

is an integer-valued polynomial and, since $r(0) = B_{km}/B_k$, (2) follows.

Consider next $k \geq 3$ odd. By Lemma 1 we have

$$S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}), \qquad S_{km}(n) = \frac{1}{km+1}(B_{km+1}(n) - B_{km+1}),$$

hence, for all integers $n > 1$, $B_{k+1}(n) > B_{k+1}$ and

$$\frac{k+1}{km+1}\,\frac{B_{km+1}(n) - B_{km+1}}{B_{k+1}(n) - B_{k+1}} \in \mathbb{Z}.$$

By Lemma 2,

$$r(x) = \frac{k+1}{km+1}\,\frac{B_{km+1}(x) - B_{km+1}}{B_{k+1}(x) - B_{k+1}}$$

is an integer-valued polynomial and, since $r(0) = mB_{km-1}/B_{k-1}$, (2) follows.

Sufficiency. We consider separately $k = 1, 2, 3$.

$\underline{k = 1}$. If $m \equiv 1 \pmod 2$, then for $n > 0$,

$$i^m + (n-i)^m \equiv 0 \pmod n,$$

hence

$$2S_m(n) \equiv 0 \pmod n \quad \text{and also} \quad 2S_m(n+1) \equiv 0 \pmod n.$$

Thus

$$\frac{n}{(n,2)} \mid S_m(n+1). \tag{8}$$

Similarly

$$i^m + (n+1-i)^m \equiv 0 \pmod{n+1} \qquad (1 \leq i \leq n),$$

thus

$$2S_m(n+1) \equiv 0 \pmod{n+1}$$

and

$$\frac{n+1}{(n+1,2)} \mid S_m(n+1). \tag{9}$$

It follows from (8) and (9) that

$$S_1(n+1) = \frac{n(n+1)}{2} = \frac{n}{(n,2)} \cdot \frac{n+1}{(n+1,2)} \mid S_m(n+1).$$

k = 2. Let $\varepsilon = 0$ or 1 or $\frac{1}{2}$. Then

$$2(n+\varepsilon) = 2^{\alpha_\varepsilon} \prod_{p>2} p^{e_{p\varepsilon}},$$

where $p > 2$ is a prime. Put

$$g_\varepsilon \equiv \begin{cases} 5 \pmod 8 \\ g_p \pmod p \quad \text{if } p > 2 \text{ and } e_{p\varepsilon} > 0, \end{cases} \tag{10}$$

where g_p is a primitive root mod p.

For every positive $i < n + \varepsilon$ we have

$$g_e i \equiv \pm j \pmod{2(n+\varepsilon)},$$

where $0 < j < n+\varepsilon$; here j and the sign are uniquely determined. It follows that

$$g_\varepsilon^{2m} i^{2m} \equiv j^{2m} \pmod{4(n+\varepsilon)}$$

and, since to different i correspond different j,

$$g_\varepsilon^{2m} S_{2m}(\lceil n+\varepsilon \rceil) \equiv S_{2m}(\lceil n+\varepsilon \rceil) \pmod{4(n+\varepsilon)},$$
$$4(n+\varepsilon) \mid (g_\varepsilon^{2m} - 1) S_{2m}(\lceil n+\varepsilon \rceil).$$

However, by (2) and Lemma 5, and since $B_2 = \frac{1}{6}$, for every prime p we have either $p - 1 \nmid 2m$ or $p \le 3$. Therefore, by (10),

$$(g_\varepsilon^{2m} - 1, 4(n+\varepsilon)) = 2(2n+2\varepsilon, 4)3^{\beta_\varepsilon}, \qquad \beta_\varepsilon \le e_{3\varepsilon},$$

thus for $\varepsilon = 0, 1$,

$$\frac{n+\varepsilon}{(n+\varepsilon, 2)3^{\beta_\varepsilon}} \mid S_{2m(n+\varepsilon)},$$

while for $\varepsilon = \frac{1}{2}$,

$$\frac{2n+1}{3^{\beta_\varepsilon}} \mid S_{2m}(n+1).$$

If $e_{3\varepsilon} > 0$, by Lemma 3 we have

$$3^{e_{3\varepsilon}-1} \mid S_{2m}(n+1),$$

thus for $\varepsilon = 0, 1$,

$$\frac{n+\varepsilon}{(n+\varepsilon, 6)} \mid S_{2m}(n+1),$$

while for $\varepsilon = \frac{1}{2}$,

$$\frac{2n+1}{(2n+1, 3)} \mid S_{2m}(n+1).$$

It follows that

$$\tag{11} \begin{aligned} &\frac{n}{(n,6)} \mid S_{2m}(n+1), \qquad \frac{n+1}{(n+1,6)} \mid S_{2m}(n+1),\\ &\frac{2n+1}{(2n+1,3)} \mid S_{2m}(n+1), \end{aligned}$$

hence

$$S_2(n+1) = \frac{n(n+1)(2n+1)}{6} = \frac{n}{(n,6)}\,\frac{n+1}{(n+1,6)}\,\frac{2n+1}{(2n+1,3)} \mid S_{2m}(n+1).$$

$\underline{k=3}$. For $m=1$ the condition (2) is clearly sufficient. Thus we assume $m \ge 3$, $3m-1 = 2r$, $r \ge 4$. Let $\varepsilon = 0$ or 1, and

$$n+\varepsilon = 2^{\alpha_{2+\varepsilon}} \prod_{p>2} p^{e_{p,2+\varepsilon}},$$

where $p > 2$ is a prime. Put

$$\tag{12} h_\varepsilon \equiv \begin{cases} 5 \pmod 8 \\ g_{p^r} \pmod{p^r} & \text{if } p>2 \text{ and } e_{p,2+\varepsilon} > 0, \end{cases}$$

where g_{p^r} is a primitive root mod p^r.

For every positive $i < n+\varepsilon$ we have

$$h_\varepsilon i \equiv j \pmod{n+\varepsilon},$$

where $0 < j < n+\varepsilon$. It follows that

$$h_\varepsilon^{2r} i^{2r} \equiv j^{2r} \pmod{n+\varepsilon},$$

and, since to different i correspond different j,

$$\begin{gathered} h_\varepsilon^{2r} S_{2r}(n+\varepsilon) \equiv S_{2r}(n+\varepsilon) \pmod{n+\varepsilon},\\ n+\varepsilon \mid (h_\varepsilon^{2r}-1) S_{2r}(n+\varepsilon). \end{gathered}$$

However, by (2) and Lemma 5, and since $B_2 = \frac{1}{6}$, for every prime p we have either $p-1 \nmid 2r$ or $p \mid 6m$. By (12),

$$(n+\varepsilon, h_\varepsilon^{2r}-1) = 2^{\beta_{2+\varepsilon}} \prod_{\substack{p \mid 3m \\ p-1 \mid 2r}} p^{\min\{e_{p,2+\varepsilon},1\}} =: 2^{\beta_{2+\varepsilon}} \Pi_\varepsilon,$$

thus

$$\frac{n+\varepsilon}{2^{\beta_{2+\varepsilon}}(n+\varepsilon, \Pi_\varepsilon)} \mid S_{2r}(n+\varepsilon), \quad \beta_{2+\varepsilon} \le \alpha_{2+\varepsilon}.$$

If $\alpha_{2+\varepsilon} > 0$, by Lemma 4 we have

$$2^{\alpha_{2+\varepsilon}-1} \mid S_{2r}(n+1),$$

thus in any case

$$\tag{13} \frac{n+\varepsilon}{(n+\varepsilon, 2\Pi_\varepsilon)} \mid S_{2r}(n+1).$$

Now, for every integer i,

$$i^{3m} + (n+\varepsilon-i)^{3m} \equiv 3m(n+\varepsilon)i^{2r} \pmod{(n+\varepsilon)^2},$$

hence for every positive integer n,

$$2S_{3m}(n+\varepsilon) \equiv 3m(n+\varepsilon)S_{2r}(n+\varepsilon) \pmod{(n+\varepsilon)^2},$$

and by (13),

$$\frac{(n+\varepsilon)^2}{((n+\varepsilon)^2,4)} \mid S_{3m}(n+1). \tag{14}$$

It follows that

$$\frac{n^2}{(n^2,4)} \mid S_{3m}(n+1) \quad \text{and} \quad \frac{(n+1)^2}{((n+1)^2,4)} \mid S_{3m}(n+1),$$

hence

$$S_3(n+1) = \frac{n^2(n+1)^2}{4} \mid S_{3m}(n+1).$$

Insufficiency. Take m to be a prime $\equiv 17 \pmod{30}$. The condition (2) is fulfilled, since $B_{4m}/B_4 = -30B_{4m} \in \mathbb{Z}$. Indeed, by Lemma 5, B_{4m} has in the denominator only the first powers of primes p such that $p-1 \mid 4m$. The divisibility gives $p = 2, 3, 5, 2m+1$ or $4m+1$. Now, $2\cdot 3\cdot 5 = 30$, $2m+1$ is divisible by 5 and $4m+1$ by 3. It follows from Theorem 2 that $S_4(n+1) \nmid S_{4m}(n+1)$ for a positive integer n. ∎

LEMMA 6. *If p is a prime, $k' \equiv k \not\equiv 0 \pmod{p-1}$ and $n' \equiv n \pmod{p}$, then*

$$S_{k'}(n') \equiv S_k(n) \pmod{p}.$$

Proof. This follows from the well-known congruence

$$1^k + \cdots + (p-1)^k \equiv 0 \pmod{p}$$

provided $k \not\equiv 0 \pmod{p-1}$ (see [2, p. 95]), and from the Fermat theorem. ∎

LEMMA 7. *If $p > 2$ is a prime, $k \geq \alpha > 1$, $k' \geq \alpha$, $k \not\equiv 0 \pmod{p(p-1)}$, $k' \equiv k \pmod{p^{\alpha-1}(p-1)}$ and $n' \equiv n \pmod{p^{\alpha+1}}$, then*

$$S_{k'}(n') \equiv S_k(n) \pmod{p^\alpha}. \tag{15}$$

Proof. Let g be a primitive root modulo $p^{\alpha+1}$. The transformation $i \mapsto gi \pmod{p^{\alpha+1}}$ maps the set of residues modulo $p^{\alpha+1}$ onto itself. Hence

$$g^k(S_k(n') - S_k(n)) \equiv S_k(n') - S_k(n) \pmod{p^{\alpha+1}},$$

thus

$$(g^k - 1)(S_k(n') - S_k(n)) \equiv 0 \pmod{p^{\alpha+1}}$$

and, since by the assumption on k, $(g^k-1, p^2) = p$, we obtain

$$S_k(n') \equiv S_k(n) \pmod{p^\alpha}.$$

The congruence (15) now follows by Euler's theorem. ∎

LEMMA 8. *If* $n \equiv 58966743 \pmod{11251^2}$, *then*

$$S_{4m}(n+1) \equiv 0 \pmod{11251}$$

only if $m \equiv 1 \pmod{5625}$.

Proof. The number $p = 11251$ is a prime and

$$n \equiv 252 \pmod{p}, \qquad \left\lfloor \frac{n}{p} \right\rfloor \equiv 5241 \pmod{p}.$$

If $4m \equiv 0 \pmod{p-1}$, then

$$S_{4m}(n+1) \equiv n - \left\lfloor \frac{n}{p} \right\rfloor \equiv -4989 \not\equiv 0 \pmod{p}.$$

If $4m \not\equiv 0 \pmod{p-1}$, it suffices by Lemma 6 to verify the congruence $S_{4m}(252) \equiv 0 \pmod{p}$ for m in the interval $[1, 11249]$. The verification has been performed by J. Browkin. ■

LEMMA 9. *If* $n \equiv 58966743 \pmod{5^6}$, *then*

$$S_{4m}(n+1) \equiv 0 \pmod{5^5}$$

only if $m = 1$ *or* $m \equiv 501 \pmod{625}$.

Proof. We have $58966743 \equiv 13618 \pmod{5^6}$. If $m \equiv 0 \pmod{5}$, then

$$S_{4m}(n+1) \equiv n - \left\lfloor \frac{n}{5} \right\rfloor \equiv 13618 - 2723 = 10895 \not\equiv 0 \pmod{25}.$$

If $m \not\equiv 0 \pmod{5}$, it suffices by Lemma 7 to verify the congruence $S_{4m}(13619) \equiv 0 \pmod{5^5}$ for m in the interval $[1, 626]$. The verification has been performed by J. Browkin. ■

Proof of Theorem 2. Since for $n \equiv 58966743 \pmod{5^6 \cdot 11251^2}$ we have

$$S_4(n+1) \equiv 0 \pmod{5^5 \cdot 11251}$$

the theorem follows from Lemmas 8 and 9. ■

LEMMA 10. *For every positive integer* k,

$$(2^k, 1 + 2^k + 3^k) \le 4, \tag{16}$$

$$(3^{k+1}, 1 + 2^k + 3^k) \le 3k, \tag{17}$$

Proof. We have $3^k \equiv 1 \pmod{4}$ for k even and $3^k \equiv 3 \pmod{8}$ for k odd, which implies (16). Further

$$\mathrm{ord}_3(1 + 2^k) = \begin{cases} 0 & \text{for } k \text{ even}, \\ \mathrm{ord}_3 k + 1 & \text{for } k \text{ odd}, \end{cases}$$

which implies (17). ■

Proof of Theorem 3. We have

$$1 + 2^{3k} + 3^{3k} - 2^k \cdot 3^{k+1} = (1 + 2^k + 3^k)(1 + 2^{2k} + 3^{2k} - 2^k - 3^k - 6^k),$$

thus if (1) holds, then

$$1 + 2^k + 3^k \mid 2^k \cdot 3^{k+1}. \tag{18}$$

By (16) and (17), $(2^k \cdot 3^{k+1}, 1 + 2^k + 3^k) \le 12k$, thus by (18),

$$1 + 2^k + 3^k \le 12k,$$

which implies $k \le 3$. ■

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A. Schinzel
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: schinzel@impan.pl

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