THE EXISTENCE OF AN EXPONENTIAL ATTRACTOR IN MAGNETO-MICROPOLAR FLUID FLOW VIA THE ℓ-TRAJECTORIES METHOD

BY

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Abstract. We consider the magneto-micropolar fluid flow in a bounded domain $\Omega \subset \mathbb{R}^2$. The flow is modelled by a system of PDEs, a generalisation of the two–dimensional Navier–Stokes equations. Using the Galerkin method we prove the existence and uniqueness of weak solutions and then using the ℓ-trajectories method we prove the existence of the exponential attractor in the dynamical system associated with the model.

1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be a bounded subset with a smooth boundary $\partial\Omega$. For simplicity we assume that Ω is connected. We consider the two-dimensional magneto-micropolar fluid flow in Ω described by the following equations:

$$(1.1) \quad \begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - (\mu+\chi)\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{b}\cdot\nabla)\mathbf{b} + \nabla(p+\mathbf{b}\cdot\mathbf{b}) = \mathbf{f} + 2\chi\tilde{\nabla}\times\omega, \\[2mm] \dfrac{\partial \omega}{\partial t} - \alpha\Delta\omega + 4\chi\omega + (\mathbf{v}\cdot\nabla)\omega = g + 2\chi\nabla\times\mathbf{v}, \\[2mm] \dfrac{\partial \mathbf{b}}{\partial t} + \nu\tilde{\nabla}\times(\nabla\times\mathbf{b}) - \tilde{\nabla}\times(\mathbf{v}\,\tilde{\times}\,\mathbf{b}) = 0, \\[2mm] \nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{b} = 0, \\[2mm] \mathbf{v}(0) = \mathbf{v}_0, \quad \omega(0) = \omega_0, \quad \mathbf{b}(0) = \mathbf{b}_0, \end{cases}$$

with the boundary conditions

$$\mathbf{v}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \quad \nabla\times\mathbf{b}|_{\partial\Omega} = 0, \quad \mathbf{b}|_{\partial\Omega} = 0.$$

Here, the velocity field $\mathbf{v}$, the pressure p, the microrotation field ω and the magnetic field $\mathbf{b}$ are unknown. The external forces $\mathbf{f}$, g and the positive constants μ, χ, α and ν are given. In the whole paper vectors (with values in $\mathbb{R}^2$) will be set in boldface, the other letters will be used for scalars.

The operators $\nabla\times$, $\tilde{\nabla}$ and $\tilde{\times}$ are defined as follows:

$$\nabla \times \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \qquad \text{for } \mathbf{F}(x,y) = (F_1(x,y), F_2(x,y)),$$

(1.2)
$$\tilde{\nabla} \times f = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \qquad \text{for a scalar function } f,$$

$$\mathbf{a} \tilde{\times} \mathbf{b} = a_1 b_2 - a_2 b_1 \qquad \text{for } \mathbf{a} = (a_1, a_2) \text{ and } \mathbf{b} = (b_1, b_2).$$

The problems leading to consideration of micropolar fluids arise in natural sciences. Some physiological fluids behave like suspensions of particles in a Newtonian fluid. An example is blood, which consists of a mixture of blood cells in plasma. In this case some microrotational effects can be observed. In the case of a flow of a conducting fluid in the presence of a magnetic field, electric currents appear which in turn modify the electric field. This creates additional mechanical forces changing the fluid flow. This case belongs to the field of magnetohydrodynamics. The fluid combining the two properties above is a magneto-micropolar fluid, and again blood can be listed as a good example. The hemoglobin is an iron-containing oxygen-transport protein of red blood cells. The presence of iron particles renders them sensitive to the presence of a magnetic field.

We are interested in the long-time behaviour of solutions to (1.1); in particular we establish the existence of an exponential attractor. Problems of this type for 2D micropolar fluids were investigated for example in [8, 9]. In this work we will prove that for the system above there exist unique weak solutions, defined on arbitrarily long time intervals. Next, we will use the ℓ-trajectories method (see [10, 5]) to prove the existence of the global attractor $\mathcal{A}$ in a dynamical system, given by the equations above on a suitable space. This method will imply that we can extend this attractor to the exponential attractor $\mathcal{M}$.

It is easy to see that for $g = 0$ and $\chi = 0$, a particular solution to this system is $(\mathbf{v}, 0, 0)$, where $\mathbf{v}$ is a solution for the Navier–Stokes equations, well known from hydrodynamics:

(1.3)
$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v}|_{\partial\Omega} = 0. \end{cases}$$

For this equation it is already known that in the case of the 2D flow there exists an exponential attractor [1]. Nonlinearities in (1.1) are of the same type as those in the Navier–Stokes equations, so it is natural to ask whether in this case there also exists an exponential attractor. K. Matsuura [11] proved the existence of such an attractor by considering a discrete dynamical system in which the dynamics was given by a specific time value of the solution

semigroup for (1.1) with a different boundary condition for $\mathbf{b}$, namely $\mathbf{b} \cdot \mathbf{n}|_{\partial\Omega} = 0$. He used the criteria from [2].

Our goal is to obtain the result, using a new method, very general and abstract, which is the above-mentioned ℓ-trajectories method, introduced by Málek and Pražák [10]. The idea is briefly sketched in Section 2.2. It is a new, simple and elegant way to prove the existence of attractors. Although the method can be applied to the three-dimensional case of generalised Navier–Stokes equations with periodic boundary conditions and considered in spaces of type $\mathbf{L}_2$ and $\mathbf{W}^{1,p}$ for $p \geq 11/5$, it has no use in our case, where $p = 2$ and we do not assume periodic boundary conditions. Furthermore for the classical Navier–Stokes equations the question of existence of global solutions still remains open, so we cannot define the semigroup of solution operators.

The paper is organised as follows: in Section 2 we introduce function spaces and some useful tools. In Section 2.2 we introduce basic definitions and briefly sketch the idea of the ℓ-trajectories method. In Section 3 we simplify the original system (1.1) by rewriting it in an abstract evolutionary form, and prove some useful estimates. In Section 4 we sketch the proof of the existence and uniqueness of solutions to (3.3) and prove the regularity of strong solutions. We will need these results in Section 4.1, where we state the main results and prove that all assumptions for the ℓ-trajectories method are fulfilled.

2. Basic information

2.1. Function spaces and useful facts. Let $C_0^\infty(\Omega)$ be the space of smooth compactly supported functions in Ω. We denote by $L_p(\Omega)$ the usual Lebesgue space and by $W^{k,p}(\Omega)$ the Sobolev space of functions with integrable distributional derivatives up to order k. If $p = 2$ this Sobolev space is a Hilbert space denoted $H^k(\Omega)$. Additionally $\mathbf{C}_0^\infty = C_0^\infty \times C_0^\infty$ and analogously $\mathbf{L}_2$ and $\mathbf{H}^1$. Let $\mathbf{C}_\sigma^\infty = \{\mathbf{v} \in \mathbf{C}_0^\infty : \nabla \cdot \mathbf{v} = 0\}$ and let $\mathbf{L}_{2,\sigma}$ and $\mathbf{H}_\sigma^1$ be the closures of $\mathbf{C}_\sigma^\infty$ in $\mathbf{L}_2$ and $\mathbf{H}^1$ respectively. We set $\mathbf{H}_\sigma^2 = \mathbf{H}^2 \cap \mathbf{H}_\sigma^1$.

For simplicity we will use the same notation $(\cdot, \cdot)$ for the scalar product in $L_2(\Omega)$, $\mathbf{L}_2(\Omega)$ and $\mathbf{L}_2(\Omega; \mathbb{R}^2)$. We define the following Hilbert spaces:

$$H := \mathbf{L}_{2,\sigma}(\Omega) \times L_2(\Omega) \times \mathbf{L}_{2,\sigma}(\Omega),$$
$$V := \mathbf{H}_\sigma^1(\Omega) \times H_0^1(\Omega) \times \mathbf{H}_\sigma^1(\Omega),$$

with the corresponding scalar products and norms

$$(U_1, U_2)_H = (\mathbf{v}_1, \mathbf{v}_2) + (\omega_1, \omega_2) + (\mathbf{b}_1, \mathbf{b}_2), \quad \|U\|_H = \|U\| = (U,U)_H^{1/2},$$
$$(U_1, U_2)_V = (\nabla\mathbf{v}_1, \nabla\mathbf{v}_2) + (\nabla\omega_1, \nabla\omega_2) + (\nabla \times \mathbf{b}_1, \nabla \times \mathbf{b}_2),$$
$$\|U\|_V = (U,U)_V^{1/2},$$

where $U_i = (\mathbf{v}_i, \omega_i, \mathbf{b}_i)$, $i = 1, 2$. Additionally

$$V_2 := \mathbf{H}_\sigma^2 \times H^2 \times \mathbf{H}_\sigma^2.$$

We define on V a norm

$$\|U\|_\# = (\mu + \chi)\|\nabla\mathbf{v}\|_{\mathbf{L}_2}^2 + \alpha\|\nabla\omega\|_{L_2}^2 + \nu\|\nabla \times \mathbf{b}\|_{\mathbf{L}_2}^2,$$

which is equivalent to $\|\cdot\|_V$.

The compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L_2(\Omega)$ yields $V \hookrightarrow\hookrightarrow H$. Moreover, in our case $n = p = 2$, so from the theory of Sobolev spaces we know that $f \in W^{1,n}(\Omega)$ ($\Omega \subset \mathbb{R}^n$ - open and bounded) is p-integrable for all $p < \infty$. The embedding $H^1 \hookrightarrow L_q$ is compact for each $q \in [1, \infty)$ as a simple consequence of the Rellich–Kondrashov theorem.

It is easy to see that the operators $(\mathbf{v}, v) \mapsto (\mathbf{v} \cdot \nabla)v$, defined in (1.2), have the following property:

CLAIM 2.1.

$$\begin{aligned}
((\mathbf{v} \cdot \nabla)\mathbf{v}_1, \mathbf{v}_2) &= -((\mathbf{v} \cdot \nabla)\mathbf{v}_2, \mathbf{v}_1), \\
((\mathbf{v} \cdot \nabla)v_1, v_2) &= -((\mathbf{v} \cdot \nabla)v_2, v_1), \\
(\mathbf{v} \cdot \nabla)\mathbf{b} - (\mathbf{b} \cdot \nabla)\mathbf{v} &= -\tilde{\nabla} \times (\mathbf{v}\tilde{\times}\mathbf{b}),
\end{aligned}$$

(2.1)

for each $\mathbf{v}, \mathbf{b} \in \mathbf{H}_\sigma^1$ *and all* $\mathbf{v}_i$ *and* v_i ($i = 0, 1$) *from* $\mathbf{H}^1$ *and* H^1 *respectively.*

Now let us introduce some facts concerning functions $f : I \subset \mathbb{R} \to X$, where X is a Banach space and X^* its dual. More information about Bochner spaces can be found in [7, 12].

DEFINITION 2.1. Let $1 \le p \le \infty$ and $I \subset \mathbb{R}$. Then $L_p(I; X)$ is the set of all strongly measurable functions $f : I \to \mathbb{R}$ such that

$$\|f\|_{L_p(I;X)} = \left(\int_I \|f(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \le p < \infty,$$

$$\|f\|_{L_\infty(I;X)} = \operatorname{ess\,sup}_I \|f(t)\|_X < \infty \quad \text{for } p = \infty.$$

With the norms given by the expressions above, the spaces $L_p(I; X)$ are Banach spaces. Elements f and g are equal if $f(t) = g(t)$ a.e. in I.

We need to weaken the classical definition of the time derivative. We apply the language of Bochner spaces.

DEFINITION 2.2. Let $U, V \in L_{1,\text{loc}}(0, T; X)$. Then V is a *weak time derivative* of U, denoted $V = dU/dt$, if

$$\int_0^T U(t)\frac{d\phi}{dt}(t) \, dt = -\int_0^T V(t)\phi(t) \, dt \quad \text{for all } \phi \in C_0^\infty(0, T).$$

THEOREM 2.3. *Let $U \in L_2(0,T;V)$ and $dU/dt \in L_2(0,T;V^*)$. Then $U \in C([0,T];H)$ and*

$$\sup_{0 \le t \le T} \|U(t)\|_H \le C\left(\|U\|_{L_2(0,T;V)} + \left\|\frac{dU}{dt}\right\|_{L_2(0,T;V^*)}\right).$$

Moreover, for a.e. $0 \le t \le T$,

$$\frac{d}{dt}\|U(t)\|_H^2 = 2\left\langle \frac{dU}{dt}, U \right\rangle_{V^*,V}.$$

Now let us recall some useful facts and theorems.

LEMMA 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and $f \in L_4(\Omega) \cap H^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{L_4} \le C\|f\|_{L_2}^{1/2}\|f\|_{H^1}^{1/2}.$$

The space $\mathbf{L}_2(\Omega)$ can be decomposed into a sum of "gradients" and divergence-free vector fields (see [13]).

THEOREM 2.5. *Let Ω be a domain with Lipschitz boundary and outward normal unit vector $\mathbf{n}$. Then $\mathbf{L}_2(\Omega) = \overline{\mathbf{L}_{2,\sigma}}(\Omega) \oplus P$, where*

$$\overline{\mathbf{L}_{2,\sigma}}(\Omega) = \{\mathbf{v} \in \mathbf{L}_2(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in a weak sense,}$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of trace}\},$$

$$(\overline{\mathbf{L}_{2,\sigma}}(\Omega))^\perp =: P = \{\mathbf{v} \in \mathbf{L}_2(\Omega) : \mathbf{v} = \nabla p, \ p \in W^{1,2}(\Omega)\},$$

Now we define the orthogonal projection $\mathbf{P}$ known as the *Leray projection*,

(2.2) $$\mathbf{P} : \mathbf{L}_2(\Omega) \to \mathbf{L}_{2,\sigma}(\Omega).$$

We will also need the following Aubin–Lions Theorem (see [12]).

THEOREM 2.6. *Let $X \hookrightarrow\hookrightarrow H \hookrightarrow Y$ be Banach spaces, with X reflexive. Assume that $\{u_n\}_n$ is a bounded sequence in $L_2(0,T;X)$ and $\{du_n/dt\}_n$ is bounded in $L_p(0,T;Y)$ for some $p > 1$. Then $\{u_n\}_n$ has a subsequence strongly convergent in $L_2(0,T;H)$.*

2.2. Introduction to dynamical systems. Consider the following abstract, autonomous evolution equation:

(2.3) $$\begin{cases} u_t(t) = F(u(t)) & \text{in } X, \\ u(0) = u_0, \end{cases}$$

where X is a Banach space, $F : X \to X$ is nonlinear operator and $u_0 \in X$. If one assumes that for all $u_0 \in X$ there exists a unique and global in time solution $u : [0, \infty) \to X$ to (2.3), then one can associate with (2.3) a semigroup $\{S(t)\}_{t \ge 0}$ of nonlinear operators $S(t) : X \to X$ defined by $S(t)u_0 := u(t)$ for $t > 0$.

By investigating properties of $\{S(t)\}_{t\geq 0}$, one can obtain some basic properties of solutions to (2.3), such as asymptotic behaviour in time. The global attractor is one of the objects whose existence gives some information about the asymptotic behaviour of the whole system.

DEFINITION 2.7. A nonempty subset $\mathcal{A} \subset X$ is a *global attractor* for the semigroup $\{S(t)\}_{t\geq 0}$ if it is compact in X, is *invariant*, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and for all $\epsilon > 0$ and each bounded subset $B \subset X$ there exists a time $t_0 = t_0(\epsilon, B)$ such that for all $t \geq t_0$ the set $S(t)B$ is included in the ϵ-hull of $\mathcal{A}$.

If such an attractor $\mathcal{A}$ exists, then it follows directly from the definition that it is unique. It can have a complicated structure, but as a compact subset of infinite-dimensional Banach space it must have an empty interior. It appears that for many dynamical systems the attractor $\mathcal{A}$ has finite fractal dimension, which is reflected in the behaviour of the trajectories of the system.

DEFINITION 2.8. Let $\mathcal{C}$ be a compact subset of X. The *fractal dimension* of $\mathcal{C}$ is

$$d_f^X(\mathcal{C}) := \limsup_{\epsilon \to 0} \frac{\ln N_\epsilon^X(\mathcal{C})}{\ln(1/\epsilon)},$$

where $N_\epsilon^X(\mathcal{C})$ is the smallest number of balls in X with radius ϵ, needed to cover $\mathcal{C}$.

For many dynamical systems with a global attractor there also exists an exponential attractor.

DEFINITION 2.9. An *exponential attractor* for the semigroup $\{S(t)\})_{t\geq 0}$ in X is a set $\mathcal{M}$ that is: compact in X; *positively invariant*, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$; it has finite fractal dimension, i.e. $d_f^X(\mathcal{M}) < \infty$; and it *exponentially attracts* bounded subsets of X, i.e. for all bounded $B \subset X$ there exist positive constants c_1, c_2 and time $t_0 \geq 0$ such that for $t \geq t_0$ the image $S(t)B$ is included in the $\epsilon(t)$-hull of $\mathcal{M}$, with $\epsilon(t) = c_2 e^{-c_1 t}$.

We will use the ℓ-trajectories method, fully described in [10], to prove the existence of an exponential attractor.

Let X, Y and Z be three Banach spaces such that $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$, with X reflexive and separable. For $\tau > 0$ and p_1, p_2 such that $2 \leq p_1 < \infty$ and $1 \leq p_2 < \infty$ we denote $X_\tau := L_2(0, \tau; X)$, $Y_\tau := \{u \in L_{p_1}(0, \tau; Y) : du/dt \in L_{p_2}(0, \tau; Z)\}$. Let $C([0, \tau]; X_w)$ be the space of functions from the interval $[0, \tau]$ to X, continuous in the weak topology on X. Assume that all solutions to (2.3) belong at least to $C([0, T]; X_w)$ for each $T > 0$. Then by the ℓ-*trajectory* we mean part of the solution parametrised by $[0, l]$, i.e. if $u = u(t)$, $t > 0$ is a solution to (2.3), then $\chi = u|_{[0,l]}$ is an ℓ-trajectory. On

the set of all ℓ-trajectories we introduce a new semigroup $\{L_t\}_{t\geq 0}$ by the formula

$$L_t\chi(\tau) = S(t)u(\tau) = u(t+\tau)$$

for $\tau \in [0,l]$ and $t > 0$, where χ is an ℓ-trajectory and u the corresponding solution.

The method relies on checking ten assumptions labelled (**A1**)–(**A10**). They are formulated in [10] and recalled in Section 4.1 (Theorems 4.5 and 4.7).

The idea is to construct a global attractor $\mathcal{A}_\ell$ on the set of all ℓ-trajectories (using assumptions (**A1**)–(**A5**)), and then prove that this attractor is of finite fractal dimension (as ensured by assumption (**A6**) together with (**A1**)–(**A5**)). The next step is to map the global attractor $\mathcal{A}_\ell$ to the original space X so that it would become a global attractor $\mathcal{A}$ for the original semigroup. This is ensured by assumption (**A7**). The stronger version of (**A7**), assumption (**A8**), then transfers the finite fractal dimension of $\mathcal{A}_\ell$ in X_l to $\mathcal{A}$ in X. Further it is proven that if we assume that X is a Hilbert space then from additional assumptions (**A9**)–(**A10**) we obtain the existence of an exponential attractor for $\mathcal{A}_\ell$ in X_l. From the last step we see that assumption (**A8**) transfers also the exponential attraction property of $\mathcal{A}_\ell$ to the original space X. For more details we refer to [10].

3. Analysis of the equations. Now we reduce the system (1.1) to an abstract form. Let us define the operators

$$A : D(A) = \mathbf{H}_\sigma^2 \times (H^2 \cap H_0^1) \times \mathbf{H}_\sigma^2 \to H,$$

(3.1) $\quad AU = A(\mathbf{v}, \omega, \mathbf{b}) = \big(-(\mu+\chi)\mathbf{P}\Delta\mathbf{v}, -\alpha\Delta\omega, \nu\tilde{\nabla} \times (\nabla \times \mathbf{b})\big)$

$\quad$ for $U \in D(A)$,

(3.2) $\quad B(U) = B(\mathbf{v}, \omega, \mathbf{b})$

$$= \big(\mathbf{P}(-2\chi\tilde{\nabla} \times \omega + (\mathbf{v}{\cdot}\nabla)\mathbf{v} - (\mathbf{b}{\cdot}\nabla)\mathbf{b}),$$

$$4\chi\omega + (\mathbf{v}{\cdot}\nabla)\omega - 2\chi\nabla \times \mathbf{v}, (\mathbf{v}{\cdot}\nabla)\mathbf{b} - (\mathbf{b}{\cdot}\nabla)\mathbf{v}\big).$$

We may use the operator $\mathbf{P}$ on both sides of the equation for $\mathbf{v}$ in (1.1) to eliminate "gradient" terms. Set $F = (\mathbf{f}, g, 0)$. Then the system (1.1) takes the form

(3.3) $\quad \begin{cases} \dfrac{dU}{dt}(t) + AU(t) + B(U(t)) = F & \text{in } [0,T], \\ U(0) = U_0 \in H. \end{cases}$

REMARK 3.1. Due to the ellipticity of A, solutions U to the equation $AU = F$ belong to $V \cap V_2$.

We see that A is self-adjoint on V_2 and its eigenfunctions are smooth (namely $\mathbf{C}_\sigma^\infty \times C_0^\infty \times \mathbf{C}_\sigma^\infty$ functions). The operators A and B also define continuous linear operators on V given by the scalar products $(AU, \Phi)_V$ and $(B(U), \Phi)_V$ respectively. We also need the following estimate:

LEMMA 3.2. *For $U, W, \Phi \in V$,*

$$|\langle B_2(U) - B_2(W), \Phi\rangle_{V^*,V}|$$
$$\leq C(\|U\|_V^{1/2}\|U\|^{1/2} + \|W\|_V^{1/2}\|W\|^{1/2})\|U - W\|^{1/2}\|U - W\|_V^{1/2}\|\Phi\|_V.$$

Proof. Let $U = (\mathbf{v}, \omega, \mathbf{b})$ and $W = (\mathbf{w}, \eta, \mathbf{d})$. We will estimate the differences of the corresponding elements:

$$|((\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{w}\cdot\nabla)\mathbf{w}, \boldsymbol{\Phi}_1)| = \left| \int_\Omega \sum_{i,k=1}^2 \left(v_i \frac{\partial v_k}{\partial x_i} \Phi_{1,k} - w_i \frac{\partial w_k}{\partial x_i} \Phi_{1,k} \right) dx \right|$$

$$= \left| \int_\Omega \sum_{i,k=1}^2 v_i \left(\frac{\partial \Phi_{1,k}}{\partial x_i} v_k - \frac{\partial \Phi_{1,k}}{\partial x_i} w_k \right) dx \right|$$

$$= \left| \int_\Omega \sum_{i,k=1}^2 \left((v_i - w_i)\frac{\partial \Phi_{1,k}}{\partial x_i} v_k - w_i \frac{\partial \Phi_{1,k}}{\partial x_i}(w_k - v_k) \right) dx \right|$$

$$= |(((\mathbf{v} - \mathbf{w})\cdot\nabla)\boldsymbol{\Phi}_1, \mathbf{v}) - ((\mathbf{w}\cdot\nabla)\boldsymbol{\Phi}_1, \mathbf{w} - \mathbf{v})|$$

$$\leq \|\mathbf{v} - \mathbf{w}\|_4 \|\nabla\Phi_1\|_2 \|\mathbf{v}\|_4 + \|\mathbf{v} - \mathbf{w}\|_4 \|\nabla\Phi_1\|_2 \|\mathbf{w}\|_4.$$

This combined with (2.4) gives the desired property. ∎

4. Existence, uniqueness and properties of solutions

THEOREM 4.1. *For all times $T > 0$, each $U_0 \in H$ and all external forces $F \in L_2(0, T; V^*)$ there exists a weak solution U of (3.3) such that $U \in C([0,T]; H) \cap L_2(0, T; V)$ and $dU/dt \in L_2(0, T; V^*)$ with estimates of norms depending only on data.*

Proof. We will use the Galerkin method. The space H is a Hilbert space. The operator A^{-1} is self-adjoint and symmetric on H. It is also compact, for we have

$$A^{-1} : H \to V \hookrightarrow\hookrightarrow H,$$

which is a consequence of the Lax–Milgram Theorem. Then we know that there exists an orthonormal basis $\{\mathbf{w}_n\}_n$ of H such that $A^{-1}\mathbf{w}_n = \tilde{\lambda}_n\mathbf{w}_n$ for some $\tilde{\lambda}_n$ and $(\mathbf{w}_i, \mathbf{w}_j) = \delta_{ij}$. Moreover, each $\mathbf{w}_n$ is an eigenfunction of A with eigenvalue $\lambda_n = (\tilde{\lambda}_n)^{-1}$,

$$A^{-1}\mathbf{w}_n = \tilde{\lambda}_n\mathbf{w}_n, \quad \lambda_n\mathbf{w}_n = (\tilde{\lambda}_n)^{-1}\mathbf{w}_n = A\mathbf{w}_n.$$

The set $\{\mathbf{w}_n\}_n$ also forms an orthogonal basis of V. Moreover, $(\mathbf{w}_i, \mathbf{w}_j)_V = \lambda_i \delta_{ij}$.

We define $U_n(t) = (\mathbf{v}_n(t), \omega_n(t), \mathbf{b}_n(t)) = \sum_{k=1}^n \mathbf{w}_k c_k^n(t)$ to be the nth Galerkin approximation if

(4.1) $\qquad \left\langle \dfrac{dU_n}{dt}(t), \mathbf{w}_j \right\rangle + \langle AU_n(t), \mathbf{w}_j \rangle + \langle B(U_n(t)), \mathbf{w}_j \rangle = \langle F, \mathbf{w}_j \rangle,$

$$U_n(0) = \sum_{i=1}^n \langle U_0, \mathbf{w}_i \rangle \mathbf{w}_i.$$

From now on, we proceed as in the proof of existence of solutions to 2D Navier–Stokes equations (see [13]). ∎

Now we will investigate the uniqueness of solutions. Let U_1 and U_2 be any two solutions to (3.3) with initial data $U_{1,0}$ and $U_{2,0}$ respectively. From the weak formulation we get, for $0 \le s \le t \le T$ and $i = 1, 2$,

$$\int_s^t \left(\left\langle \dfrac{dU_i}{dt}(\tau), \Phi(\tau) \right\rangle + \langle AU_i(\tau), \Phi(\tau) \rangle + \langle B(U_i(\tau)), \Phi(\tau) \rangle \right) d\tau$$
$$= \int_s^t \langle F(\tau), \Phi(\tau) \rangle \, d\tau.$$

Now we subtract the two equalities and set $W := U_1 - U_2$. Then

(4.2) $\quad \displaystyle\int_s^t \left(\left\langle \dfrac{dW}{dt}, \Phi \right\rangle + \langle AW, \Phi \rangle \right) d\tau + \int_s^t \langle B_1(W), \Phi \rangle \, d\tau$
$$= \int_s^t \langle B_2(U_2) - B_2(U_1), \Phi \rangle \, d\tau.$$

Take $\Phi = W$. Together with the estimate in Lemma 3.2 we obtain

(4.3) $\quad \frac{1}{2} \|W(t)\|^2 + c \displaystyle\int_s^t \|W(\tau)\|_V^2 \, d\tau$
$$\le \frac{1}{2} \|W(s)\|^2 + C \int_s^t \Theta(U, \tilde{U}) \|W(\tau)\|^{1/2} \|W(\tau)\|_V^{1+1/2} \, d\tau$$

where $\Theta(U, \tilde{U}) = C(\|U\|^{1/2} \|U\|_V^{1/2} + \|\tilde{U}\|^{1/2} \|\tilde{U}\|_V^{1/2})$. Using the Young inequality for the integral on the right-hand side we get

(4.4) $\quad \|W(t)\|^2 + c \displaystyle\int_s^t \|W(\tau)\|_V^2 \, d\tau \le \|W(s)\|^2 + \int_s^t \Theta(U, \tilde{U})^4 \|W(\tau)\|^2 \, d\tau.$

Due to the bounds of solutions in $L_\infty(0, T; H)$ and $L_2(0, T; V)$ we conclude that $\Theta(U, \tilde{U}) \in L_4(0, T)$ with an estimate depending only on the data. Now define $f(t) := \|W(t)\|^2$ and $g(t) := \Theta(U(t), \tilde{U}(t))^4$. We may use the

Gronwall inequality for non-negative and integrable functions f and g to obtain

$$(4.5) \quad \|U(t) - \tilde{U}(t)\| \leq C(U_0, \tilde{U}_0, F)\|U(s) - \tilde{U}(s)\| \quad \text{for } 0 \leq s \leq t \leq T.$$

REMARK 4.2. In particular for $s = 0$ from (4.5) the uniqueness of solutions to (3.3) follows. Since the solutions are continuous functions from $[0, T]$ to H we can define the semigroup $\{S(t)\}_{t \geq 0}$ of solutions for the problem (3.3) with F independent of time. For a weak solution U with initial data $U_0 \in H$ we define $S(t)U_0 := U(t)$.

Theorem 4.1 guarantees that for $U_0 \in H$ the time derivative dU/dt has values in V^*. We will show that for $U_0 \in V$ the derivative has values in H. We will need the following lemma:

LEMMA 4.3. *Let Ω be an open and bounded subset with boundary of class C^2. Then there exist constants $c, C > 0$ such that for any $U \in V \cap V_2$,*

$$c\|U\|_{V_2} \leq \|AU\|_H \leq C\|U\|_{V_2}.$$

Proof. The proof is identical to that for the Stokes operator. ∎

THEOREM 4.4. *Let Ω be as in the lemma above. Moreover, assume $U_0 \in V$ and $F \in L_2(0, T; H)$. Then the solution to the problem (3.3) satisfies*

$$U \in L_2(0, T; V_2) \cap L_\infty(0, T; V), \qquad \frac{dU}{dt} \in L_2(0, T; H),$$

$$\|U\|_{L_2(0,T;V_2)} + \left\|\frac{dU}{dt}\right\|_{L_2(0,T;H)} + \|U\|_{L_\infty(0,T;V)}$$
$$\leq C(\|F\|_{L_2(0,T;H)}, \|U_0\|_{L_2(0,T;V)}).$$

In particular $U \in C([0, T]; V)$. If we only have $U_0 \in H$, then the estimates above are valid only on $(\delta, T]$, where $\delta > 0$ may be arbitrarily small.

Proof. Let U_n be the Galerkin approximations constructed in Theorem 4.1. Due to the elipticity of the operator A its eigenfunctions are of class $\mathbf{C}^\infty$. This and standard ODE theory ensure that U_n is at least C^1 in time and $\mathbf{C}^\infty$ in space. Since in the definition of weak solution we need a function from the space $L_2(0, T; V)$, we are allowed to use as a test function in (4.1) the function dU_n/dt and next AU_n to obtain

$$(4.6) \quad \left\|\frac{d}{dt}U_n(t)\right\|^2 + \left(AU_n(t), \frac{d}{dt}U_n(t)\right)_H$$
$$= \left(F, \frac{d}{dt}U_n(t)\right)_H - \left(B(U_n(t)), \frac{d}{dt}U_n(t)\right)_H,$$

$$(4.7) \quad \left(\frac{d}{dt}U_n(t), AU_n(t)\right)_H + \|AU_n(t)\|^2$$
$$= (F, AU_n(t))_H - (B(U_n(t)), AU_n(t))_H.$$

We see that

$$(4.8) \qquad \left(AU_n(t), \frac{d}{dt}U_n(t) \right)_H = \sum_{k,j} (Ac_j^n(t)\mathbf{w}_j, \dot{c}_k^n(t)\mathbf{w}_k)$$

$$= \frac{1}{2}\frac{d}{dt}\sum_j \lambda_j |c_j^n(t)|^2 = \frac{1}{2}\frac{d}{dt}\|U_n\|_\#^2.$$

The Cauchy inequality yields

$$(4.9) \qquad \left| \left(B(U_n(t)), \frac{d}{dt}U_n(t) \right)_H \right| \le C(\epsilon)\|BU_n(t)\|^2 + \epsilon \left\| \frac{d}{dt}U_n(t) \right\|^2.$$

We recall that $B = B_1 + B_2$ and $\|B_1 U_n(t)\|^2 \le C\|U_n\|_V^2$. For B_2 we use the interpolation inequality

$$(4.10)$$
$$\|B_2(U_n(t))\|^2 \le \|\mathbf{v}\|_{\mathbf{L}_4}^2 \|\nabla\mathbf{v}\|_{\mathbf{L}_4}^2 + \|\mathbf{b}\|_{\mathbf{L}_4}^2 \|\nabla\mathbf{b}\|_{\mathbf{L}_4}^2 + \|\mathbf{v}\|_{\mathbf{L}_4}^2 \|\nabla\omega\|_{\mathbf{L}_4}^2$$

$$+ \|\mathbf{v}\|_{\mathbf{L}_4}^2 \|\nabla\mathbf{b}\|_{\mathbf{L}_4}^2 + \|\mathbf{b}\|_{\mathbf{L}_4}^2 \|\nabla\mathbf{v}\|_{\mathbf{L}_4}^2$$

$$\le C(\|U_n\| \, \|U_n\|_V^2 \|U_n\|_{V_2}) \le \delta\|U_n\|_{V_2}^2 + C(\delta)\|U_n\|^2\|U_n\|_V^4.$$

Both estimates give

$$(4.11) \qquad \|B(U_n(t))\|^2 \le 2(\|B_1 U_n(t)\|^2 + \|B_2(U_n(t))\|^2)$$

$$\le \delta\|U_n\|_{V_2}^2 + C(\delta)\|U_n\|^2\|U_n\|_V^2(1 + \|U_n\|_V^2).$$

Now we add (4.6) and (4.7) and estimate the right-hand side using the inequalities above and Lemma 4.3 to obtain

$$(4.12) \qquad \frac{d}{dt}\|U_n(t)\|_\#^2 + \frac{1}{2}\left\|\frac{d}{dt}U_n(t)\right\|^2 + \frac{1}{2}\|AU_n(t)\|^2 \le 2\|F_n\|^2 + 2\|B(U_n(t))\|^2.$$

Now use (4.11), take $\delta = c/8$ and integrate over $[0, T]$ to get

$$(4.13) \qquad \|U_n(t)\|_\#^2 + \frac{1}{2}\int_0^t \left\|\frac{d}{dt}U_n(s)\right\|^2 ds + \frac{c}{4}\int_0^t \|U_n(s)\|_{V_2}^2 \, ds$$

$$\le \|U_n(0)\|_\#^2 + 2\int_0^t \|F_n(s)\|^2 \, ds$$

$$+ C\int_0^t \|U_n(s)\|^2 \|U_n(s)\|_V^2 (1 + \|U_n(s)\|_V^2) \, ds.$$

The Gronwall inequality implies that for $t \in [0, T]$,

$$(4.14) \qquad \|U_n(t)\|_\#^2$$
$$\le (\|U_0\|_\#^2 + 2\|F\|_{L_2(0,T;H)}^2)\exp(C(U_0, F)(T + \|U_n\|_{L_2(0,T;V)}^2))$$
$$\le C(U_0, F, T)(\|U_0\|_\#^2 + 2\|F\|_{L_2(0,T;H)}^2) \le C(U_0, F, T)(1 + \|U_0\|_V^2),$$

which leads to

$$(4.15) \qquad \sup_{t \in [0,T]} \|U_n(t)\|_\#^2 + \frac{1}{2} \int\limits_0^T \left\| \frac{d}{dt} U_n(s) \right\|^2 ds + \frac{c}{4} \int\limits_0^T \|U_n(s)\|_{V_2}^2 \, ds$$
$$\leq C(\|U_0\|_V, \|F\|_{L_2(0,T;H)}, U_0, T) = C(U_0, F, T).$$

Finally, for each n we have

$$(4.16) \qquad U_n \in L_\infty(0,T;V) \cap L_2(0,T;V_2), \quad \frac{d}{dt} U_n \in L_2(0,T;H)$$

with bounds uniform with respect to n, and passing to the limit as $n \to \infty$ completes the proof.

If the initial datum U_0 is not regular enough, take $\delta > 0$ and a function $\rho \in C^1([0,T])$ that $\rho|_{[0,\delta/2)} = 0$, $\rho|_{(\delta,T]} = 1$ and $\rho \geq 0$. Next, we multiply the (4.6) and (4.7) by ρ and apply the same estimates as above to obtain an analogue of (4.12). We continue the procedure and after integration by parts we see that the $\#$-norm of U_0 vanishes, hence the information about its regularity is no longer required. This yields a variant of (4.16) with an arbitrary $\delta > 0$ in place of zero. ∎

4.1. Existence of the attractor. Now we come to the proofs of the two main theorems of this paper. It is sufficient to check assumptions (**A1**)–(**A10**). Theorems 2.1–2.6 from [10] guarantee the existence of an exponential attractor. We emphasise that the force $F \in H$ does not depend on time. The method of checking assumption (**A6**), and the proof of (4.5), from which (**A4**) and (**A7**)–(**A9**) follow, comes from the book [5, §2.4]. The proof of (**A10**) comes directly from [10].

THEOREM 4.5. *For the semigroup $\{S(t)\}_{t \geq 0}$ associated with (3.3) there exists a global attractor $\mathcal{A}$ of finite fractal dimension.*

Proof. We will use Theorems 2.1–2.4 from [10], combined in

THEOREM 4.6 ([10, 2.1–2.4]). *Let (**A1**)–(**A8**) hold. Then there exists a global attractor $\mathcal{A}$ for semigroup $\{S(t)\}_{t \geq 0}$ in B^1, where $B^1 = e(\mathcal{B}_l^1)$, where $e : X_l \to X$ and $e(\chi) = \chi(l)$. Moreover, $\mathcal{A}$ has a finite fractal dimension.*

Hence it suffices to check assumptions (**A1**)–(**A8**), which we recall along the way.

ASSUMPTION (**A1**). *For any $U_0 \in X$ and $T > 0$ there exists (not necessarily unique) $U \in C([0,T]; X_w) \cap Y_T$, a solution to the evolution problem on $[0,T]$ with $U(0) = U_0$. Moreover, for any solution the estimates of $\|U\|_{Y_T}$ are uniform with respect to $\|U(0)\|_X$.*

In our case

$$X = H, \quad Y = V, \quad Z = V^*, \quad Y_T = L_2(0, T; V) \cap L_2(0, T; V^*).$$

From Theorem 4.1 we know that for all $U_0 \in H$ and $T > 0$ there exists a solution $U \in C([0, T]; H) \cap Y_T$ to (3.3) satisfying

$$\|U\|_{Y_T} = \|U\|_{L_2(0, T; V)} + \left\| \frac{dU}{dt} \right\|_{L_2(0, T; V^*)} \leq C(U_0, F, T).$$

ASSUMPTION (**A2**). *There exists a bounded set $B^0 \subset X$ with the following properties: if u is an arbitrary solution with the initial condition u_0 then (i) there exists $t_0 = t_0(\|u_0\|_X)$ such that $u(t) \in B^0$ for all $t \geq t_0$ and (ii) if $u_0 \in B^0$ then $u(t) \in B^0$ for all $t \geq 0$.*

The weak formulation and Galerkin approximation imply

$$\frac{d}{dt} \|U(t)\|^2 + c\|U(t)\|_\# \leq C\|F\|^2.$$

The embedding $V \hookrightarrow\hookrightarrow H$ yields, for some $\kappa > 0$,

$$\frac{d}{dt} \|U\|^2 + \kappa \|U\|^2 \leq C(F).$$

Multiplying by $e^{\kappa t}$ and integrating on $[0, t]$ we get

$$(4.17) \quad \|U(t)\|^2 \leq e^{-\kappa t} \|U_0\|^2 + e^{-\kappa t} C(F) \frac{e^{\kappa t} - 1}{\kappa} \leq e^{-\kappa t} \|U_0\|^2 + \frac{C(F)}{\kappa}.$$

Now if we set $r_0 = 2(C(F)/\kappa)^{1/2}$ then the ball $B_H(0, r_0)$ satisfies item (i) of assumption (**A2**). Let $t_* = \inf\{t \geq 0 : S(t)B_H(0, r_0) \subset B_H(0, r_0)\}$, the first time when the image of this ball is absorbed by itself. Then we define

$$B^0 := \overline{\bigcup_{0 \leq t \leq t_*} S(t)B_H(0, r_0)}^H$$

and this set satisfies both (i) and (ii), hence (**A2**).

ASSUMPTION (**A3**). *Each ℓ-trajectory has a unique continuation to a solution.*

From Corollary 4.2 we know that solutions to problem (3.3) are unique. Since ℓ-trajectories are fragments of solutions parametrised on $[0, l]$, it is obvious that each of them has a unique continuation.

ASSUMPTION (**A4**). *For all $t > 0$ the operators $L_t : X_l \to X_l$ are continuous on $\mathcal{B}_l^0$, the set of all ℓ-trajectories starting from any point in B^0 from assumption (**A2**).*

We use the inequality (4.5), which gives a stronger property. Let $\chi, \tilde{\chi} \in X_l$, U and $\tilde{U}$ be two solutions to problem (3.3) with initial conditions U_0 and $\tilde{U}_0$ respectively such that $\chi = U|_{[0,l]}$ and $\tilde{\chi} = \tilde{U}|_{[0,l]}$. Then

$$
\begin{aligned}
\|L_t\chi - L_t\tilde{\chi}\|_{X_l}^2 &= \int_0^l \|L_t\chi(s) - L_t\tilde{\chi}(s)\|_X^2 \, ds = \int_0^l \|L_tU(s) - L_t\tilde{U}(s)\|_X^2 \, ds \\
&= \int_0^l \|U(s+t) - \tilde{U}(s+t)\|_X^2 \, ds \\
&\leq \int_0^l C(\mathcal{B}_l^0, F)^2 \|U(s) - \tilde{U}(s)\|_X^2 \, ds = C(\mathcal{B}_l^0, F)^2 \|\chi - \tilde{\chi}\|_{X_l}^2.
\end{aligned}
$$

Hence

(4.18) $$\|L_t\chi - L_t\tilde{\chi}\|_{X_l} \leq C(\mathcal{B}_l^0, F)\|\chi - \tilde{\chi}\|_{X_l},$$

which means that operators $L_t : X_l \to X_l$ are Lipschitz continuous on the set of all ℓ-trajectories for $t > 0$, in particular, on $\mathcal{B}_l^0$.

ASSUMPTION (**A5**). *For some $\tau_0 > 0$ the closure of the set $L_{\tau_0}(\mathcal{B}_l^0)$ in X_l is contained in $\mathcal{B}_l^0$.*

The set B^0 is closed, so the trajectories starting from any point of this set remain in it. Thus $L_\tau \mathcal{B}_l^0 \subset \mathcal{B}_l^0$ for all $\tau \geq 0$. Now we check that $\mathcal{B}_l^0$ is closed.

Let $\chi_n \in \mathcal{B}_l^0$ be a sequence of ℓ-trajectories convergent in H_l with corresponding solutions U_n. Let χ and U denote the relevant limits. Then χ is also an ℓ-trajectory. We have

$$\forall_n \forall_{t \in [0,l]} \quad \chi_n(t) = U_n(t) \in B^0.$$

Since $U_n \to U$ in $L_2(0, l; H)$, we also have convergence a.e. along a subsequence, i.e. $U_k(t) \to U(t)$ for a.e. $t \in [0, l]$. Then $U(t) \in B^0$ for a.e. $t \in [0, l]$. The solutions are continuous functions from $[0, l]$ into H, so closedness of B^0 implies $U(t) \in B^0$ for all $t \in (0, l]$.

ASSUMPTION (**A6**). *There exists a space W_l with $W_l \hookrightarrow\hookrightarrow X_l$ and $\tau > 0$ such that $L_\tau : X_l \to W_l$ is Lipschitz continuous on $\mathcal{B}_l^1 = \overline{L_{\tau_0}(\mathcal{B}_l^0)}$ for τ_0 from (**A5**).*

Let $W_l := \{U \in L_2(0, l; V) : dU/dt \in L_2(0, l; V^*)\}$ endowed with the norm $\|U\|_{W_l} := \|U\|_{L_2(0,l;V)} + \|dU/dt\|_{L_2(0,l;V^*)}$. As a consequence of the Aubin–Lions Theorem 2.6 we obtain compactness of the embedding $W_l \hookrightarrow H_l$. Now we have to prove that for some $\tau > 0$ the operator $L_\tau : H_l \to W_l$ is Lipschitz continuous on $\mathcal{B}_l^1$.

Fix $\tau > 0$. We will use (4.4), for fixed $s \in (0, l)$ and $t = \tau + l$, i.e.

$$\|W(\tau + l)\|^2 + c \int_s^{\tau+l} \|W(t)\|_V^2 \, dt \leq \|W(s)\|^2 + \int_s^{\tau+l} \Theta(U, \tilde{U})^4 \|W(t)\|^2 \, dt,$$

where $W = U - \tilde{U}$ and $U, \tilde{U}, \chi, \tilde{\chi}$ are the solutions and corresponding ℓ-trajectories. For $\|W(t)\|^2$ on the right-hand side we use (4.5) for $s \leq t$ and integrate both sides with respect to $s \in [0, l]$ to get

$$c \int_0^l \int_s^{\tau+l} \|W(t)\|_V^2 \, dt \, ds \leq C \int_0^l \|W(s)\|^2 \, ds.$$

The left-hand side can be estimated by

$$\int_0^l \int_s^{\tau+l} \|W(t)\|_V^2 \, dt \, ds \geq c \int_0^l \|U(s + \tau) - \tilde{U}(s + \tau)\|_V^2 \, ds,$$

which implies

$$(4.19) \qquad c\|L_\tau \chi - L_\tau \tilde{\chi}\|_{L_2(0,l;V)}^2 \leq C \int_0^l \|U(s) - \tilde{U}(s)\|^2 \, ds = C\|\chi - \tilde{\chi}\|_{H_l}.$$

Next we employ the dual definition of the norm in $L_2(0, l; V^*)$ and (3.3) to get

$$(4.20) \qquad \left\| \frac{d}{dt} W \right\|_{L_2(0,l;V^*)} = \sup_{\substack{\phi \in L_2(0,l;V) \\ \|\phi\|_{L_2(0,l;V)} \leq 1}} \int_0^l \left\langle \frac{dW}{dt}, \phi \right\rangle_{V^*,V} dt$$

$$= \sup_{\substack{\phi \in L_2(0,l;V) \\ \|\phi\|_{L_2(0,l;V)} \leq 1}} \int_0^l (\langle -AW, \phi \rangle_{V^*,V} + \langle B(\tilde{U}) - B(U), \phi \rangle_{V^*,V}) \, dt.$$

First, we estimate the linear term:

$$\left| \int_0^l \langle -AW, \phi \rangle_{V^*,V} \, ds \right| \leq C \int_0^l \|W\|_V \|\phi\|_V \, ds \leq C\|W\|_{L_2(0,l;V)}.$$

From Theorem 4.1 it follows that the weak solution to (3.3) satisfies $U \in L_2(0, l; V)$, which means $U(t) \in V$ for a.e. $t \in [0, l]$. Theorem 4.4 gives in turn $U \in L_\infty(\delta, l; V)$ for each $\delta > 0$, or $U \in L_\infty(0, l; V)$ whenever $U_0 \in V$. This implies that each ℓ-trajectory mapped by L_τ will be in $L_\infty(0, l; V)$. So by the use of Lemma 3.2 to estimate the nonlinearity we get

$$\left| \int_0^l \langle B(\tilde{U}) - B(U), \phi \rangle_{V^*,V} \, ds \right| \leq C \int_0^l \Theta(U, \tilde{U}) \|W\|^{1/2} \|W\|_V^{1/2} \|\phi\|_V \, ds$$

$$\leq C(\|U_0, \tilde{U}_0\|_V, \|U_0, \tilde{U}_0\|_H, F, l) \int_0^l \|W\|_V \|\phi\|_V \, ds \leq C(\mathcal{B}_l^1) \|W\|_{L_2(0,l;V)}.$$

Both inequalities yield

$$\left\|\frac{d}{dt}\chi - \frac{d}{dt}\tilde{\chi}\right\|_{L_2(0,l;V^*)} \le C\|\chi - \tilde{\chi}\|_{L_2(0,l;V)}.$$

Substituting $\chi, \tilde{\chi} \mapsto L_\tau\chi, L_\tau\tilde{\chi}$ into (4.19) we get the desired estimate

$$\left\|\frac{d}{dt}L_\tau\chi - \frac{d}{dt}L_\tau\tilde{\chi}\right\|_{L_2(0,l;V^*)} \le C(\mathcal{B}_l^1, \tau)\|\chi - \tilde{\chi}\|_{H_l}.$$

ASSUMPTION (**A7**). *The mapping $e : X_l \to X$ is continuous on $\mathcal{B}_l^1$, the closure of $L_\tau(\mathcal{B}_l^0)$ in X_l.*

Let $\chi, \tilde{\chi}$ and $U, \tilde{U}$ be as in (**A4**). Again we use inequality (4.5) to obtain

$$\|e(\chi) - e(\tilde{\chi})\|_H^2 = \|\chi(l) - \tilde{\chi}(l)\|_H^2$$
$$= \|U(l) - \tilde{U}(l)\|_H^2 \le C^2\|U(s) - \tilde{U}(s)\|_H^2$$

for any $0 \le s \le l$. After integration in $s \in [0, l]$, we arrive at

(4.21) $$\|e(\chi) - e(\tilde{\chi})\|_H \le Cl^{-1/2}\|\chi - \tilde{\chi}\|_{H_l}.$$

ASSUMPTION (**A8**). *The mapping $e : X_l \to X$ is Hölder continuous on $\mathcal{B}_l^1$.*

This follows from (4.21) in (**A7**).

Thus Theorem 4.5 is proved. ∎

Now we use Theorems 2.5–2.6 to show that the attractor $\mathcal{A}$ has the property of exponential attraction.

THEOREM 4.7. *The dynamical system associated with $\{S(t)\}_{t\ge 0}$ given by the equation (3.3) has an exponential attractor $\mathcal{M}$.*

Proof. It is sufficient to use Theorems 2.5–2.6 from [10], which are combined in

THEOREM 4.8 ([10, 2.5–2.6]). *Let X be a Hilbert space and let assumptions (**A1**)–(**A6**) and (**A8**)–(**A10**) hold. Then for the set $B^1 = e(\mathcal{B}_l^1)$ the dynamical system (S_t, B^1) possesses an exponential attractor $\mathcal{M}$.*

It is clear that the space $H = X$ defined in 2.1 is a Hilbert space. As before, we need only check assumptions (**A9**)–(**A10**).

ASSUMPTION (**A9**). *For all $\tau > 0$ the operators $L_t : X_l \to X_l$ are uniformly Lipschitz continuous on $\mathcal{B}_l^1$ (with respect to $t \in [0, \tau]$).*

This follows from (4.18) in (**A4**).

ASSUMPTION (**A10**). *For all $\tau > 0$ there exist $c > 0$ and $\beta \in (0,1]$ such that for all $\chi \in \mathcal{B}_l^1$ and $t_1, t_2 \in [0, \tau]$,*

$$\|L_{t_1}\chi - L_{t_2}\chi\|_{X_l} \leq c|t_1 - t_2|^\beta.$$

Fix $\tau > 0$. On $\mathcal{B}_l^1$ we have $dU/dt \in L_2((0, T]; H)$ for any $T > 0$. Let $U, \tilde{U}, \chi, \tilde{\chi}$ be as usual and $0 < t_1, t_2 < \tau < T - l$ for sufficiently large $T > 0$. We have

$$\|L_{t_1}\chi - L_{t_2}\chi\|_{H_l}^2 = \int\limits_0^l \|U(s + t_1) - \tilde{U}(s + t_2)\|^2 \, ds$$

$$= \int\limits_0^l \left\|\int\limits_{s+t_1}^{s+t_2} \frac{d}{dt}U(t')\, dt'\right\|^2 ds \leq \int\limits_0^l \left(\int\limits_{s+t_1}^{s+t_2} \left\|\frac{d}{dt}U(t')\right\| dt'\right)^2 ds$$

$$\leq \int\limits_0^l |t_2 - t_1| \int\limits_{s+t_1}^{s+t_2} \left\|\frac{d}{dt}U(t')\right\|^2 dt' \, ds \leq l|t_2 - t_1| \left\|\frac{d}{dt}U\right\|_{L_2(0,\tau+l;H)}^2.$$

This implies that (**A10**) holds with $c = C(\mathcal{B}_l^1, \tau, l)$ and $\beta = 1/2$.

Hence the proof of Theorem 4.7 is complete. ∎

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