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CYCLE-FINITE ALGEBRAS WITH ALMOST ALL INDECOMPOSABLE MODULES OF PROJECTIVE OR INJECTIVE DIMENSION AT MOST ONE

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Abstract. We describe the structure of artin algebras for which all cycles of indecomposable modules are finite and almost all indecomposable modules have projective or injective dimension at most one.

1. Introduction and the main result. Throughout the paper by an *algebra* we mean an artin algebra over a fixed commutative artin ring K, which we will assume to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by all indecomposable modules.

The Jacobson radical rad_A of $\operatorname{mod} A$ is the ideal generated by all non-invertible homomorphisms between indecomposable modules, and the infinite radical $\operatorname{rad}_A^{\infty}$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. By a result of Auslander [5], $\operatorname{rad}_A^{\infty} = 0$ if and only if A is of finite representation type, that is, $\operatorname{ind} A$ admits only a finite number of pairwise non-isomorphic modules (see [17] for an alternative proof). On the other hand, if A is of infinite representation type then $(\operatorname{rad}_A^{\infty})^2 \neq 0$, by a result proved in [9] (see [10] for the structure of module categories $\operatorname{mod} A$ of algebras A with $(\operatorname{rad}_A^{\infty})^3 = 0$ and [17], [34] for other results and open problems concerning the Jacobson radical power series of module categories).

We denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We identify a module in $\operatorname{ind} A$ with the corresponding vertex of Γ_A.

A prominent rôle in the representation theory of algebras is played by cycles of modules, or more generally cycles of complexes of modules. Recall that a *cycle* in the module category $\operatorname{mod} A$ of an algebra A is a sequence

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = X_0$$

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of non-zero non-isomorphisms in $\operatorname{ind} A$, and such a cycle is said to be *finite* if the homomorphisms $f_1, \ldots, f_r$ do not belong to $\operatorname{rad}_A^\infty$. Following Ringel [32], a module in $\operatorname{ind} A$ which does not lie on the cycle in $\operatorname{mod} A$ is called *directing*. It has been proved independently by Peng and Xiao [29] and Skowroński [39] that every Auslander–Reiten quiver Γ_A has at most finitely many τ_A-orbits containing directing modules. Moreover, by a result of Ringel [32], the support algebra of a directing module is a tilted algebra. On the other hand, the support algebras of non-directing indecomposable modules depend on properties of cycles containing these modules.

Following Assem and Skowroński [3], an algebra is said to be *cycle-finite* if all cycles in $\operatorname{mod} A$ are finite. The class of cycle-finite algebras is wide and contains the following classes of algebras: algebras of finite representation type, tame tilted algebras [16], [32], tame double tilted algebras [30], tame generalized double tilted algebras [31], tubular algebras [32], [33], tame quasitilted algebras [19], [45], tame coil and multicoil algebras [3], [4], tame generalized multicoil algebras [26], and strongly simply connected algebras of polynomial growth [43]. It has also been proved in [2] that the class of algebras A for which the derived category $D^b(\operatorname{mod} A)$ of bounded complexes over $\operatorname{mod} A$ is cycle-finite coincides with the class of piecewise hereditary algebras of Dynkin, Euclidean, and tubular type, and consequently these algebras are also cycle-finite.

Moreover, frequently an algebra A admits a Galois covering $R \to R/G = A$, where R is a cycle-finite locally bounded category and G is an admissible group of automorphisms of R, which allows the representation theory of A to be reduced to the representation theory of cycle-finite algebras which are convex subcategories of R (see [28] and [44] for some general results). For example, every finite-dimensional selfinjective algebra A of polynomial growth over an algebraically closed field K admits a canonical standard form $\bar{A}$ (geometric socle deformation of A) such that $\bar{A}$ has a Galois covering $R \to R/G = \bar{A}$, where R is a cycle-finite selfinjective locally bounded category and G is an admissible infinite cyclic group of automorphisms of R, and the Auslander–Reiten quiver $\Gamma_{\bar{A}}$ is the orbit quiver Γ_R/G with respect to the induced action of G (see [48]).

We are concerned with the problem of describing the structure of algebras A for which all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$ have projective dimension $\operatorname{pd}_A X \leq 1$ or injective dimension $\operatorname{id}_A X \leq 1$. This class contains all algebras A of small homological dimension (briefly, *shod algebras*) for which every module X in $\operatorname{ind} A$ satisfies $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$ (see [8]). It is known ([14]) that any shod algebra A has $\operatorname{gl.dim} A \leq 3$. Moreover, it has been shown by Happel, Reiten and Smalø[14] that A is a shod algebra with $\operatorname{gl.dim} A \leq 2$ if and only if A is a *quasitilted algebra*, that is, $A = \operatorname{End}_{\mathcal{H}}(T)$ for a tilting object T in an abelian hereditary K-category $\mathcal{H}$.

Following [14], denote by $\mathcal{L}_A$ the full subcategory of $\operatorname{ind} A$ formed by all modules X such that $\operatorname{pd}_A Y \leq 1$ for every predecessor Y of X in $\operatorname{ind} A$, and by $\mathcal{R}_A$ the full subcategory of $\operatorname{ind} A$ formed by all modules X such that $\operatorname{id}_A Z \leq 1$ for every successor Z of X in $\operatorname{ind} A$. Coelho and Lanzilotta [8] proved that A is a shod algebra if and only if $\operatorname{ind} A = \mathcal{L}_A \cup \mathcal{R}_A$.

An important class of quasitilted algebras is formed by *tilted algebras*, that is, algebras of the form $\operatorname{End}_H(T)$, where T is a tilting object in the module category $\operatorname{mod} H$ of a hereditary algebra H [15]. It has been proved by Happel and Reiten [13] (in the tame case by Skowroński [45]) that the remaining class of quasitilted algebras is formed by *quasitilted algebras of canonical type* (see also [11], [19], [45] for the representation theory of this class of algebras). Further, Reiten and Skowroński proved in [30] that A is a shod algebra with $\operatorname{gl.dim} A = 3$ if and only if A is a *strictly double tilted algebra*. Characterizations of tilted and double tilted algebras using homological properties of directing modules have been established in [46]. This completes the classification of algebras with small homological dimension (equivalently, with $\operatorname{ind} A = \mathcal{L}_A \cup \mathcal{R}_A$).

In [31] a wide class of algebras A with the property that $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$, called *generalized double tilted algebras*, has been introduced and investigated. In particular, Skowroński [47] proved that, for an algebra A, $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$ if and only if A is a generalized double tilted algebra or a quasitilted algebra. Moreover, the following problem was raised by Skowroński in [47]:

PROBLEM 1.1. Let A be an algebra such that for all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$. Is then $\mathcal{L}_A \cup \mathcal{R}_A$ cofinite in $\operatorname{ind} A$?

We note that this was proved in [40, Theorems 3.1 and 3.2] to be the case if for all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leq 1$ (respectively, $\operatorname{id}_A X \leq 1$).

The aim of this paper is to provide a positive solution of Problem 1.1 for cycle-finite algebras. The main result of this paper is the following theorem.

THEOREM 1.2. *For a cycle-finite algebra A, the following statements are equivalent:*

(i) *For all but finitely many isomorphism classes of modules X in* $\operatorname{ind} A$, *we have* $\operatorname{pd}_A X \leq 1$ *or* $\operatorname{id}_A X \leq 1$.
(ii) *A is a generalized double tilted algebra or a quasitilted algebra of canonical type.*

For basic background on the representation theory of algebras we refer the reader to the books [1], [6], [32], [35], [36], [50].

2. Preliminaries. We recall some notation and concepts on algebras and modules needed in our further considerations.

Let A be an algebra and $e_1, \dots, e_n$ a set of pairwise orthogonal primitive idempotents of A with $1_A = e_1 + \dots + e_n$. Then:

- $P_i = e_i A$, $i \in \{1, \dots, n\}$, is a complete set of pairwise non-isomorphic indecomposable projective modules in $\operatorname{mod} A$.
- $I_i = D(Ae_i)$, $i \in \{1, \dots, n\}$, is a complete set of pairwise non-isomorphic indecomposable injective modules in $\operatorname{mod} A$.
- $S_i = \operatorname{top}(P_i) = e_i A / e_i \operatorname{rad} A$, $i \in \{1, \dots, n\}$, is a complete set of pairwise non-isomorphic simple modules in $\operatorname{mod} A$.
- $S_i = \operatorname{soc}(I_i)$, for any $i \in \{1, \dots, n\}$.

Moreover, $F_i := \operatorname{End}_A(S_i) \cong e_i A e_i / e_i(\operatorname{rad} A)e_i$, for $i \in \{1, \dots, n\}$, are division algebras. The *quiver* Q_A of A is the valued quiver defined as follows:

- The vertices of Q_A are the indices $1, \dots, n$ of the chosen set $e_1, \dots, e_n$ of primitive idempotents of A.
- Given a pair of vertices i and j in Q_A, there is an arrow $i \to j$ from i to j in Q_A if and only if $e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j \neq 0$. Moreover, one equips the arrow $i \to j$ in Q_A with the valuation (d_{ij}, d'_{ij}), so we have in Q_A the valuated arrow
$$i \xrightarrow{(d_{ij}, d'_{ij})} j$$
where
$$d_{ij} = \dim_{F_j} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j,$$
$$d'_{ij} = \dim_{F_i} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j.$$

An algebra A is called *triangular* if its quiver Q_A is acyclic (i.e. there is no oriented cycle in Q_A). We identify an algebra A with the associated category A^* whose objects are the vertices of the quiver Q_A, $\operatorname{Hom}_{A^*}(i, j) = e_j A e_i$ for any objects i and j of A^*, and the composition of morphisms in A^* is given by multiplication in A. For a module M in $\operatorname{mod} A$, we denote by $\operatorname{supp}(M)$ the full subcategory of $A = A^*$ given by all objects i such that $Me_i \neq 0$, and call it the *support* of M. More generally, for a family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ of components of Γ_A, we denote by $\operatorname{supp}(\mathcal{C})$ the full subcategory of A given by all objects i such that $Xe_i \neq 0$ for some indecomposable module X in $\mathcal{C}$, and call it the *support* of $\mathcal{C}$. Then a module M in $\operatorname{mod} A$ (respectively, a family of components $\mathcal{C}$ in Γ_A) is said to be *sincere* if $\operatorname{supp}(M) = A$ (respectively, if $\operatorname{supp}(\mathcal{C}) = A$). Finally, a full subcategory B of A is said to be *convex* if every path in Q_A with source and target in Q_B lies entirely in Q_B. Observe that, for every convex subcategory B of A, there is a fully faithful embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$ such that $\operatorname{mod} B$ is the full subcategory of $\operatorname{mod} A$

consisting of all modules M with $Me_i = 0$ for all vertices i of Q_A which are not vertices of Q_B.

We will use the following lemma.

LEMMA 2.1. *Let R and S be algebras, M an S-R-bimodule, $A = \left[\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right]$ the matrix algebra defined by the bimodule ${}_SM_R$, and Y a module in* $\operatorname{mod} A$ *represented by a triple (Y_0, Y_1, φ) with $\varphi \neq 0$. Then, for every indecomposable direct summand Z of an R-module Y_1, we have* $\operatorname{Hom}_R(M, Z) \neq 0$.

Proof. This follows immediately from the arguments in the proof of [36, Lemma XV.1.8]. ∎

3. Auslander–Reiten components. We introduce various types of components of Auslander–Reiten quivers and prove a result on the shape of Auslander–Reiten components with infinite cyclic part, needed in the proof of the main theorem.

Let A be an algebra. We recall that a component $\mathcal{C}$ of Γ_A is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module, and *semiregular* if $\mathcal{C}$ does not contain both a projective and an injective module. It has been proved in [20] and [51] that a regular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a *stable tube*, that is, $\mathcal{C}$ is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ for some $r \geq 1$. Moreover, Liu proved in [21] that a semiregular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a *semiregular tube*, that is, a *ray tube* (obtained from a stable tube by a finite number (possibly zero) of ray insertions) or a *coray tube* (obtained from a stable tube by a finite number (possibly zero) of coray insertions). A component $\mathcal{P}$ of Γ_A is called *postprojective* if $\mathcal{P}$ is acyclic (without oriented cycles) and every module in $\mathcal{P}$ lies in the τ_A-orbit of a projective module. Dually, a component $\mathcal{Q}$ of Γ_A is called *preinjective* if $\mathcal{Q}$ is acyclic and every module in $\mathcal{Q}$ lies in the τ_A-orbit of an injective module. Following [25], a full translation subquiver Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ, there is an infinite sectional path $P = X_1 \to X_2 \to \cdots$.
(C2) For each injective module I in Γ, there is an infinite sectional path $\cdots \to Y_2 \to Y_1 = I$.

Further, a component $\mathcal{C}$ of Γ_A is called *almost cyclic* if all but finitely many modules in $\mathcal{C}$ lie on oriented cycles in Γ_A. We note that the stable tubes, ray tubes and coray tubes of Γ_A are semiregular, almost cyclic, and coherent. Following Skowroński [38], a component $\mathcal{C}$ of Γ_A is said to be *generalized standard* if $\operatorname{rad}_A^\infty(X, Y) = 0$ for all modules X and Y from $\mathcal{C}$. It has been proved in [38, Theorem 2.3] that every generalized standard component $\mathcal{C}$

of Γ_A is almost periodic, that is, all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic.

A family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be a *separating family* in mod A if all components in Γ_A split into three disjoint families $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that the following conditions are satisfied:

(S1) $\mathcal{C}^A$ is a sincere family of pairwise orthogonal generalized standard components.
(S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$.
(S3) Any homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in mod A factorizes through $\mathrm{add}(\mathcal{C}^A)$.

Moreover, if (S1), (S2) and the condition

(S3*) any homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in mod A factorizes through $\mathrm{add}(\mathcal{C}_i)$ for any $i \in I$,

are satisfied, then $\mathcal{C}$ is said to be a *strongly separating family* in mod A (see [26], [27], [32]).

For a component $\mathcal{C}$ of Γ_A, we denote by ${}_l\mathcal{C}$ the *left stable* part of $\mathcal{C}$, obtained by deleting from $\mathcal{C}$ all τ_A-orbits containing projective modules, and by ${}_r\mathcal{C}$ the *right stable* part of $\mathcal{C}$, obtained by deleting from $\mathcal{C}$ all τ_A-orbits containing injective modules. Finally, we denote by ${}_c\Gamma_A$ the *cyclic part* of Γ_A, obtained by removing from Γ_A all acyclic modules and the arrows attached to them. The connected components of ${}_c\Gamma_A$ are called the *cyclic components* of Γ_A (see [25]).

A prominent role in the proof of the main theorem is played by the following proposition.

PROPOSITION 3.1. *Let A be a cycle-finite algebra such that, for all but finitely many isomorphism classes of modules X in* ind A, *we have* $\mathrm{pd}_A X \le 1$ *or* $\mathrm{id}_A X \le 1$. *Then every infinite cyclic component $\mathcal{D}$ of Γ_A is the cyclic part ${}_c\mathcal{C}$ of a semiregular tube $\mathcal{C}$ of Γ_A.*

Proof. Let $\mathcal{D}$ be an infinite cyclic component of Γ_A, and $\mathcal{C}$ be the component of Γ_A containing the translation quiver $\mathcal{D}$. Since $\mathcal{D}$ is infinite and cyclic, it follows from [24, Corollary 2.8] that ${}_l\mathcal{C}$ or ${}_r\mathcal{C}$ contains a connected component Γ containing an oriented cycle and infinitely many modules of $\mathcal{D}$. We will prove that $\mathcal{C}$ is a semiregular tube of Γ_A, by considering three cases.

(1) Assume first that Γ is contained in the stable part ${}_s\mathcal{C} = {}_l\mathcal{C} \cap {}_r\mathcal{C}$ of $\mathcal{C}$. Then Γ is an infinite stable translation quiver containing an oriented cycle, and hence Γ is a stable tube, by the main result of [51]. We claim that $\mathcal{C} = \Gamma$. Suppose that $\mathcal{C} \ne \Gamma$. Then $\mathcal{C}$ contains a finite τ_A-orbit

$$P, \tau_A^{-1}P, \dots, \tau_A^{-r+1}P, \tau_A^{-r}P = I,$$

with $r \geq 0$, P a projective module, I an injective module, and such that an immediate predecessor X of P and an immediate successor Y of I lie in Γ. Hence, there are infinite sectional paths in Γ of the forms

$$\Sigma:\ \cdots \to X_1 \to X_0 = X, \qquad \Omega:\ Y = Y_0 \to Y_1 \to \cdots .$$

Since Γ is a stable tube, these two paths intersect in infinitely many modules of Γ. So there are pairwise distinct modules Z_k, $k \in \mathbb{N}$, in Γ such that $\tau_A Z_k$ lies on Ω and $\tau_A^{-1} Z_k$ lies on Σ, for any $k \in \mathbb{N}$. Therefore, for any $k \in \mathbb{N}$, we have $\operatorname{Hom}_A(D(A), \tau_A Z_k) \neq 0$ and $\operatorname{Hom}_A(\tau_A^{-1} Z_k, A) \neq 0$, because there are sectional paths in $\mathcal{C}$ of the forms $I \to Y \to \cdots \to \tau_A Z_k$ and $\tau_A^{-1} Z_k \to \cdots \to X \to P$. Applying [1, Lemma IV.2.7], we conclude that $\operatorname{pd}_A Z_k \geq 2$ and $\operatorname{id}_A Z_k \geq 2$, for all $k \in \mathbb{N}$, which contradicts the assumption on A. Hence $\mathcal{C} = \Gamma$. Obviously $\mathcal{C}$ is then a stable tube and $\mathcal{C} = {}_c\mathcal{C} = \mathcal{D}$.

(2) Assume that Γ is a component of ${}_l\mathcal{C}$ containing at least one injective module. Then it follows from [21, (2.2) and (2.3)] that Γ contains an infinite sectional path

$$\cdots \to \tau_A^{2r} X_1 \to \tau_A^{r} X_s \to \cdots \to \tau_A^{r} X_1 \to X_s \to \cdots \to X_1,$$

where $r > s \geq 1$, X_i is injective for some $i \in \{1, \dots, s\}$, and each module in Γ belongs to the τ_A-orbit of one of the modules $X_1, \dots, X_s$. Hence, there exists a non-negative integer t such that $\tau_A^{-t} X_s$ is an injective module I in Γ, and Γ admits an infinite sectional path of the form

$$\Omega:\ I = V_0 \to V_1 \to \cdots$$

We denote by Γ^* the full translation subquiver of Γ given by all modules which are the targets of infinite sectional paths of Γ. We claim that no module in Γ^* is an immediate predecessor of a projective module of $\mathcal{C}$. Indeed, otherwise Γ admits an infinite sectional path

$$\Sigma:\ \cdots \to U_1 \to U_0 = R,$$

with R a direct predecessor of a projective module P in $\mathcal{C}$. Then the infinite sectional paths Ω and Σ intersect in infinitely many modules of Γ, and we conclude as in (1) that there are pairwise distinct modules Z_k in Γ with $\operatorname{pd}_A Z_k \geq 2$ and $\operatorname{id}_A Z_k \geq 2$, for all $k \in \mathbb{N}$, a contradiction. Therefore, we conclude that Γ^* is a left stable full translation subquiver of $\mathcal{C}$ which is closed under predecessors. Moreover, $\mathcal{D}$ is the cyclic part ${}_c\Gamma^*$ of Γ^*, because ${}_c\Gamma^*$ contains all modules of $\mathcal{D}$, and $\mathcal{D}$ is a component of ${}_c\Gamma_A$.

Observe also that Γ^* is a maximal almost cyclic and coherent full translation subquiver of $\mathcal{C}$. Since Γ^* contains no projective module, applying [25, Theorem A], we conclude that Γ^*, viewed as a translation quiver, can be obtained from a stable tube by an iterated application of admissible operations of types (ad 1*). Finally, using the fact that Γ^* does not contain immediate predecessors of projective modules in $\mathcal{C}$, we conclude that $\mathcal{C} = \Gamma^*$ is a coray tube (with at least one injective module) and $\mathcal{D}$ is its cyclic part ${}_c\mathcal{C}$.

(3) Assume finally that Γ is a component of ${}_r\mathcal{C}$ containing at least one projective module. Applying arguments dual to those in (2), we find that $\mathcal{C}$ is a ray tube (with at least one projective module) and $\mathcal{D}$ is its cyclic part. ∎

4. Cycle-finite quasitilted algebras of canonical type. In this section we recall the structure of the Auslander–Reiten quivers of representation-infinite tilted algebras of Euclidean type and tubular algebras, and then describe the structure of the Auslander–Reiten quivers of cycle-finite quasitilted algebras of canonical type. At the end of the section we recall the notion of a coherent sequence of cycle-finite quasitilted algebras of canonical type and present some theorems on the structure of the Auslander–Reiten quivers of algebras associated to such sequences (see [7] for more details).

By a *tame concealed algebra* we mean a tilted algebra $C = \mathrm{End}_H(T)$, where H is a hereditary algebra of Euclidean type $\tilde{\mathbb{A}}_{11}$, $\tilde{\mathbb{A}}_{12}$, $\tilde{\mathbb{A}}_m$, $\tilde{\mathbb{B}}_m$, $\tilde{\mathbb{C}}_m$, $\widetilde{\mathbb{BC}}_m$, $\widetilde{\mathbb{BD}}_m$, $\widetilde{\mathbb{CD}}_m$, $\tilde{\mathbb{D}}_m$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, $\tilde{\mathbb{E}}_8$, $\tilde{\mathbb{F}}_{41}$, $\tilde{\mathbb{F}}_{42}$, $\tilde{\mathbb{G}}_{21}$, or $\tilde{\mathbb{G}}_{22}$ (see [12]) and T is a (multiplicity-free) tilting H-module from the additive category of the postprojective component of Γ_H. The Auslander–Reiten quiver Γ_C of a tame concealed algebra C is of the form

$$\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C,$$

where $\mathcal{P}^C$ is a postprojective component containing all indecomposable projective C-modules, $\mathcal{Q}^C$ is a preinjective component containing all indecomposable injective C-modules, and $\mathcal{T}^C$ is an infinite family of pairwise orthogonal generalized standard stable tubes strongly separating $\mathcal{P}^C$ from $\mathcal{Q}^C$ (see [36, Theorem XVII.3.5]).

More generally, by a *tilted algebra of Euclidean type* we mean a tilted algebra $B = \mathrm{End}_H(T)$, where H is a hereditary algebra of Euclidean type and T is a (multiplicity-free) tilting module in $\mathrm{mod}\, H$. Assume that B is a representation-infinite tilted algebra of Euclidean type. Then one of the following holds:

(1) B is a domestic tubular (branch) extension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B = \mathcal{P}^C$ is the postprojective component of Γ_C, $\mathcal{T}^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes, obtained from $\mathcal{T}^C$ by ray insertions, $\mathcal{Q}^B$ is a preinjective component containing all indecomposable injective B-modules, and $\mathcal{T}^B$ strongly separates $\mathcal{P}^B$ from $\mathcal{Q}^B$;

(2) B is a domestic tubular (branch) coextension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B$ is a postprojective component containing all indecomposable projective B-modules, $\mathcal{T}^B$ is an infinite family of pairwise orthogonal generalized standard coray tubes, obtained from $\mathcal{T}^C$ by coray insertions, $\mathcal{Q}^B = \mathcal{Q}^C$ is the preinjective component of Γ_C, and $\mathcal{T}^B$ strongly separates $\mathcal{P}^B$ from $\mathcal{Q}^B$.

By a *tubular algebra* we mean a tubular (branch) extension (equivalently, tubular (branch) coextension) of a tame concealed algebra with Euler quadratic form positive semidefinite of corank 2 (see [18], [32], [33]). By general theory, a tubular algebra B admits two different tame concealed convex subcategories C_0 and C_∞ such that B is a tubular (branch) extension of C_0 and a tubular (branch) coextension of C_∞, and

$$\Gamma_B = \mathcal{P}_0^B \cup \mathcal{T}_0^B \cup \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^B \Big) \cup \mathcal{T}_\infty^B \cup \mathcal{Q}_\infty^B,$$

where $\mathcal{P}_0^B = \mathcal{P}^{C_0}$ is the postprojective component of Γ_{C_0}, $\mathcal{T}_0^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes with at least one projective module, obtained from the family $\mathcal{T}^{C_0}$ of stable tubes of Γ_{C_0} by ray insertions, $\mathcal{Q}_\infty^B = \mathcal{Q}^{C_\infty}$ is the preinjective component of Γ_{C_∞}, $\mathcal{T}_\infty^B$ is an infinite family of pairwise orthogonal generalized standard coray tubes with at least one injective module, obtained from the family $\mathcal{T}^{C_\infty}$ of stable tubes of Γ_{C_∞} by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers) $\mathcal{T}_q^B$ is an infinite family of pairwise orthogonal generalized standard stable tubes. Moreover, for any $q \in \mathbb{Q}^+ \cup \{0, \infty\}$, the family $\mathcal{T}_q^B$ strongly separates $\mathcal{P}^B \cup \bigcup_{p<q} \mathcal{T}_p^B$ from $\bigcup_{p>q} \mathcal{T}_p^B \cup \mathcal{Q}^B$. We also mention that, for a tubular algebra B, the convex subcategories C_0 and C_∞ have a common vertex in Q_A.

The following characterization of tame concealed and tubular algebras has been established in [42, Theorem 4.1].

THEOREM 4.1. *Let A be an algebra. The following statements are equivalent:*

(i) *A is a cycle-finite algebra and Γ_A admits a sincere stable tube.*
(ii) *A is either a tame concealed or a tubular algebra.*

An algebra is said to be *minimal representation-infinite* if A is representation-infinite but every proper convex subcategory of A is of finite representation type. We have the following characterization of minimal representation-infinite cycle-finite algebras given in [42, Corollary 4.4].

THEOREM 4.2. *Let A be an algebra. The following statements are equivalent:*

(i) *A is a minimal representation-infinite and cycle-finite algebra.*
(ii) *A is a tame concealed algebra.*

In particular, every representation-infinite cycle-finite algebra A admits a tame concealed convex subcategory C.

Let C be a tame concealed algebra and $\mathcal{T}^C$ the family of all stable tubes in Γ_C. By a *semiregular branch enlargement of* C we mean an algebra of the form

$$B = \begin{bmatrix} F & M & 0 \\ 0 & C & D(N) \\ 0 & 0 & G \end{bmatrix},$$

where

$$B^{(r)} = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix} \quad \text{and} \quad B^{(l)} = \begin{bmatrix} C & D(N) \\ 0 & G \end{bmatrix}$$

are respectively a tubular extension of C and a tubular coextension of C in the sense of [32, (4.7)], and no tube in $\mathcal{T}^C$ admits both a direct summand of M and a direct summand of N (see [19], [45]). Then B is a quasitilted algebra, and $B^{(r)}$ and $B^{(l)}$ are called the *right part* and the *left part* of B, respectively. Moreover, following [45], B is said to be a *tame semiregular branch enlargement of* C (or a *tame quasitilted algebra of canonical type*) if $B^{(r)}$ and $B^{(l)}$ are tilted algebras of Euclidean type or tubular algebras.

The following characterization of cycle-finite quasitilted algebras of canonical type follows from [19, Theorem 2.3] and [45, Theorem A].

THEOREM 4.3. *Let A be an algebra. The following statements are equivalent:*

(i) *A is cycle-finite and quasitilted of canonical type.*
(ii) *A is a tame semiregular branch enlargement of a tame concealed algebra C.*
(iii) *A is cycle-finite and Γ_A admits a separating family of semiregular tubes.*
(iv) *A is cycle-finite and Γ_A admits a strongly separating family of semiregular tubes.*

As a consequence, we obtain the following theorem on the structure of the Auslander–Reiten quiver of a tame quasitilted algebra of canonical type.

THEOREM 4.4. *Let B be a tame quasitilted algebra of canonical type. Then the Auslander–Reiten quiver Γ_B is a disjoint union*

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where:

(1) *$\mathcal{T}^B$ is a sincere family of pairwise orthogonal generalized standard semiregular tubes strongly separating $\mathcal{P}^B$ from $\mathcal{Q}^B$.*

(2) *If $B^{(l)}$ is a tilted algebra of Euclidean type, then $\mathcal{P}^B$ is the unique postprojective component $\mathcal{P}^{B^{(l)}}$ of $\Gamma_{B^{(l)}}$ containing all indecomposable projective $B^{(l)}$-modules.*

(3) *If $B^{(l)}$ is a tubular algebra, then*

$$\mathcal{P}^B = \mathcal{P}_0^{B^{(l)}} \cup \mathcal{T}_0^{B^{(l)}} \cup \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^{B^{(l)}} \Big),$$

and contains all indecomposable projective $B^{(l)}$-modules.

(4) *If $B^{(r)}$ is a tilted algebra of Euclidean type, then $\mathcal{Q}^B$ is the unique preinjective component $\mathcal{Q}^{B^{(r)}}$ of $\Gamma_{B^{(r)}}$ containing all indecomposable injective $B^{(r)}$-modules.*

(5) *If $B^{(r)}$ is a tubular algebra, then*

$$\mathcal{Q}^B = \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^{B^{(r)}} \Big) \cup \mathcal{T}_\infty^{B^{(r)}} \cup \mathcal{Q}_\infty^{B^{(r)}},$$

and contains all indecomposable injective $B^{(r)}$-modules.

(6) *Every indecomposable projective B-module belongs to $\mathcal{P}^B \cup \mathcal{T}^B$.*

(7) *Every indecomposable injective B-module belongs to $\mathcal{T}^B \cup \mathcal{Q}^B$.*

Let $\mathbb{B} = (B_1, \dots, B_n)$ be a sequence of algebras, $n \geq 1$. Following [7, Section 3], $\mathbb{B}$ is said to be a *coherent sequence of tame quasitilted algebras of canonical type* provided that:

(1) $B_1, \dots, B_n$ are tame quasitilted algebras of canonical type,

(2) if $n \geq 2$, then $B_i^{(r)} = B_{i+1}^{(l)}$ is a tubular algebra for all $i \in \{1, \dots, n-1\}$.

For a coherent sequence $\mathbb{B} = (B_1, \dots, B_n)$ of tame quasitilted algebras of canonical type, we define the algebra $A(\mathbb{B})$ in the following way: $A(\mathbb{B}) = B_1$ for $n = 1$, and $A(\mathbb{B})$ is the pushout sum

$$B_1 \underset{B_1^{(r)}}{\sqcup} \cdots \underset{B_{n-1}^{(r)}}{\sqcup} B_n = B_1 \underset{B_2^{(l)}}{\sqcup} \cdots \underset{B_n^{(l)}}{\sqcup} B_n$$

for $n \geq 2$.

The following recent result [7, Theorem 1.1] gives a characterization of cycle-finite algebras with all Auslander–Reiten components semiregular.

Theorem 4.5. *Let A be an algebra. Then the following statements are equivalent:*

(i) *A is a cycle-finite algebra and every component of Γ_A is semiregular.*

(ii) *A is isomorphic to the algebra $A(\mathbb{B})$ associated to a coherent sequence $\mathbb{B} = (B_1, \dots, B_n)$ of tame quasitilted algebras of canonical type.*

As a direct consequence of [7, Theorem 3.5], we obtain the following theorem, which describes the structure of the Auslander–Reiten quiver Γ_A of the algebra $A = A(\mathbb{B})$ associated to a coherent sequence $\mathbb{B}$ of tame quasitilted algebras of canonical type.

THEOREM 4.6. *Let $\mathbb{B} = (B_1, \ldots, B_n)$ be a coherent sequence of tame quasitilted algebras of canonical type and $A = A(\mathbb{B})$ the associated algebra. Then:*

(i) *A is a cycle-finite algebra and every component of Γ_A is semiregular.*
(ii) *Γ_A has the disjoint union form*

$$\Gamma_A = \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}_n^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}},$$

where $\bar{\mathbb{Q}}_n^1 = [1, n] \cap \mathbb{Q}$ and:

(a) *If $B_1^{(l)}$ is a tilted algebra of Euclidean type, then $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$ is a unique postprojective component of Γ_A.*
(b) *If $B_1^{(l)}$ is a tubular algebra, then*

$$\mathcal{P}^{\mathbb{B}} = \mathcal{P}_0^{B_1^{(l)}} \cup \mathcal{T}_0^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B_1^{(l)}} \Big)$$

and $\mathcal{P}_0^{B_1^{(l)}}$ is a unique postprojective component of Γ_A.
(c) *If $B_n^{(r)}$ is a tilted algebra of Euclidean type, then $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}}$ is a unique preinjective component of Γ_A.*
(d) *If $B_n^{(r)}$ is a tubular algebra, then*

$$\mathcal{Q}^{\mathbb{B}} = \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B_n^{(r)}} \Big) \cup \mathcal{T}_\infty^{B_n^{(r)}} \cup \mathcal{Q}_\infty^{B_n^{(r)}}$$

and $\mathcal{Q}_\infty^{B_n^{(r)}}$ is a unique preinjective component of Γ_A.
(e) *For each $r \in \{1, \ldots, n\}$, $\mathcal{T}_r^{\mathbb{B}} = \mathcal{T}^{B_r}$ is a family $(\mathcal{T}_{r,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_r}$ of pairwise orthogonal generalized standard semiregular tubes.*
(f) *For each $q \in \bar{\mathbb{Q}}_n^1 \setminus \{1, \ldots, n\}$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_q}$ of pairwise orthogonal generalized standard stable tubes.*
(g) *For each $q \in \bar{\mathbb{Q}}_n^1$, we have*

$$\operatorname{Hom}_A \Big(\Big(\bigcup_{p>q} \mathcal{T}_p^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}}, \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{p<q} \mathcal{T}_p^{\mathbb{B}} \Big) \Big) = 0.$$

(h) *For each $q \in \bar{\mathbb{Q}}_n^1$, every homomorphism from $\mathcal{P}^{\mathbb{B}} \cup (\bigcup_{p<q} \mathcal{T}_p^{\mathbb{B}})$ to $(\bigcup_{p>q} \mathcal{T}_p^{\mathbb{B}}) \cup \mathcal{Q}^{\mathbb{B}}$ factorizes through $\operatorname{add}(\mathcal{T}_{q,\lambda}^{\mathbb{B}})$ for any $\lambda \in \Lambda_q$.*

5. Proof of the main theorem: semiregular case. The following theorem implies Theorem 1.2 in the semiregular case.

THEOREM 5.1. *Let A be a cycle-finite algebra such that every component of Γ_A is semiregular. Then the following statements are equivalent:*

(i) *For all but finitely many isomorphism classes of modules X in* $\operatorname{ind} A$, *we have* $\operatorname{pd}_A X \le 1$ *or* $\operatorname{id}_A X \le 1$.
(ii) *A is a tame quasitilted algebra of canonical type.*

Proof. The implication (ii)⇒(i) follows from the homological characterization of quasitilted algebras given in [14].

To prove (i)⇒(ii), applying Theorem 4.5, we may assume that $A = A(\mathbb{B})$ for a coherent sequence $\mathbb{B} = (B_1, \dots, B_n)$ of tame quasitilted algebras of canonical type. By Theorem 4.3, B_i is a semiregular branch enlargement of a tame concealed algebra C_i, for any $i \in \{1, \dots, n\}$.

Suppose that A is not a quasitilted tilted algebra of canonical type. Then Theorems 4.3, 4.4 and 4.6 imply that $n \ge 2$ and there exists $i \in \{1, \dots, n-1\}$ such that, in the notation of Theorem 4.6,

$$\Gamma_A = \mathcal{P}_i^{\mathbb{B}} \cup \mathcal{T}_i^{\mathbb{B}} \cup \bigcup_{q \in \mathbb{Q} \cap (i, i+1)} \mathcal{T}_q^{\mathbb{B}} \cup \mathcal{T}_{i+1}^{\mathbb{B}} \cup \mathcal{Q}_{i+1}^{\mathbb{B}},$$

where:

- $\mathcal{P}_i^{\mathbb{B}} = \mathcal{P}^{\mathbb{B}} \cup \bigcup_{q \in \mathbb{Q} \cap [1, i)} \mathcal{T}_q^{\mathbb{B}}$.
- $\mathcal{Q}_{i+1}^{\mathbb{B}} = \bigcup_{q \in \mathbb{Q} \cap (i+1, n]} \mathcal{T}_q^{\mathbb{B}} \cup \mathcal{Q}^{\mathbb{B}}$.
- $\mathcal{T}_i^{\mathbb{B}}$ is a family $(\mathcal{T}_{i,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_i}$ of semiregular tubes, containing an indecomposable projective module and an indecomposable injective module.
- $\mathcal{T}_{i+1}^{\mathbb{B}}$ is a family $(\mathcal{T}_{i+1,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_{i+1}}$ of semiregular tubes, containing an indecomposable projective module and an indecomposable injective module.
- For each $q \in \mathbb{Q} \cap (i, i+1)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_q}$ of stable tubes from the Auslander–Reiten quiver of the tubular algebra $B_i^{(r)} = B_{i+1}^{(l)}$.

Take now a coray tube $\mathcal{T}_{i,\xi}^{\mathbb{B}}$ with $\xi \in \Lambda_i$, containing an indecomposable injective module, a ray tube $\mathcal{T}_{i+1,\mu}^{\mathbb{B}}$ with $\mu \in \Lambda_{i+1}$, containing an indecomposable projective module, and a stable tube $\mathcal{T}_{q,\eta}^{\mathbb{B}}$ with $q \in \mathbb{Q} \cap (i, i+1)$ and $\eta \in \Lambda_q$. We note that $\mathcal{T}_{i,\xi}^{\mathbb{B}}$ is obtained from the stable tube $\mathcal{T}_{\xi}^{C_i}$ of the unique separating family $\mathcal{T}^{C_i} = (\mathcal{T}_{\lambda}^{C_i})_{\lambda \in \Lambda_i}$ of stable tubes of Γ_{C_i} by a finite number of coray insertions. Similarly, $\mathcal{T}_{i+1,\mu}^{\mathbb{B}}$ is obtained from the stable tube $\mathcal{T}_{\mu}^{C_{i+1}}$ of the unique separating family $\mathcal{T}^{C_{i+1}} = (\mathcal{T}_{\lambda}^{C_{i+1}})_{\lambda \in \Lambda_{i+1}}$ of stable tubes of $\Gamma_{C_{i+1}}$ by a finite number of ray insertions. Then the coray tube $\mathcal{T}_{i,\xi}^{\mathbb{B}}$ contains an indecomposable injective module I and an indecomposable module M from $\mathcal{T}_{\xi}^{C_i}$ such that M is a direct summand of $I/\operatorname{soc} I$, and hence there is an epimorphism $I \to M$. Further, the ray tube $\mathcal{T}_{i+1,\mu}^{\mathbb{B}}$ contains an indecomposable projective module P and an indecomposable module N from $\mathcal{T}_\mu^{C_{i+1}}$ such that N is a direct summand of $\operatorname{rad} P$, and hence there is a monomorphism $N \to P$.

Consider now an injective envelope $f : M \to I(M)$ of M in $\operatorname{mod} A$. Since M is an indecomposable C_i-module, $I(M)$ has no direct summand lying in $\mathcal{T}_i^{\mathbb{B}}$, and hence all indecomposable direct summands of $I(M)$ are in $\mathcal{T}_{i+1}^{\mathbb{B}} \cup \mathcal{Q}_{i+1}^{\mathbb{B}}$. Now, Theorem 4.6 implies that $f : M \to I(M)$ factorizes through a module in $\operatorname{add}(\mathcal{T}_{q,\eta}^{\mathbb{B}})$. Hence $\operatorname{Hom}_A(M, U) \neq 0$ for an indecomposable module U in $\mathcal{T}_{q,\eta}^{\mathbb{B}}$. Clearly then $\operatorname{Hom}_A(I, U) \neq 0$, because we have an epimorphism $I \to M$. Applying now [41, Lemma 3.9], we conclude that $\operatorname{Hom}_A(I, X) \neq 0$ for all indecomposable modules X in $\mathcal{T}_{q,\eta}^{\mathbb{B}}$ of quasi-length $\geq r_{q,\eta}$, where $r_{q,\eta}$ is the rank of $\mathcal{T}_{q,\eta}^{\mathbb{B}}$.

Dually, consider a projective cover $g : P(N) \to N$ of N in $\operatorname{mod} A$. Since N is an indecomposable C_{i+1}-module, $P(N)$ has no direct summand lying in $\mathcal{T}_{i+1}^{\mathbb{B}}$, and hence all indecomposable direct summands of $P(N)$ are in $\mathcal{P}_i^{\mathbb{B}} \cup \mathcal{T}_i^{\mathbb{B}}$. Applying Theorem 4.6 again, we conclude that $g : P(N) \to N$ factorizes through a module in $\operatorname{add}(\mathcal{T}_{q,\eta}^{\mathbb{B}})$. Then $\operatorname{Hom}_A(V, N) \neq 0$ for an indecomposable module V in $\mathcal{T}_{q,\eta}^{\mathbb{B}}$. As before, we also have $\operatorname{Hom}_A(V, P) \neq 0$, because there is a monomorphism $N \to P$. Therefore, by [41, Lemma 3.9] again, $\operatorname{Hom}_A(X, P) \neq 0$ for all indecomposable modules X in $\mathcal{T}_{q,\eta}^{\mathbb{B}}$ of quasi-length $\geq r_{q,\eta}$.

Summing up, for all indecomposable modules X in $\mathcal{T}_{q,\eta}^{\mathbb{B}}$ of quasi-length $\geq r_{q,\eta}$, we have $\operatorname{Hom}_A(I, \tau_A X) \neq 0$ and $\operatorname{Hom}_A(\tau_A^{-1} X, P) \neq 0$, and consequently $\operatorname{pd}_A X \geq 2$ and $\operatorname{id}_A X \geq 2$ (see [1, Lemma IV.2.7]). This shows that for infinitely many indecomposable modules X in $\mathcal{T}_{q,\eta}^{\mathbb{B}}$, we have $\operatorname{pd}_A X \geq 2$ and $\operatorname{id}_A X \geq 2$. Hence (i) implies (ii). ∎

6. Proof of the main theorem: non-semiregular case. This section is devoted to proving the main theorem in the remaining case, where Γ_A admits a non-semiregular component. First, we prove some preparatory lemmas on hereditary and tilted algebras.

LEMMA 6.1. *Let H be a hereditary algebra of Euclidean type and E a module on the mouth of a stable tube of Γ_H. Moreover, assume that the valued quiver Q_H of H is a tree* (*oriented canonically, as in* [12]). *Then, for every infinite path*

$$(*) \qquad \cdots \to Y_1 \to Y_0$$

in the preinjective component $\mathcal{Q}^H$ of Γ_H, there is an infinite sqeuence $n_0 < n_1 < n_2 < \cdots$ of non-negative integers such that $\operatorname{Hom}_H(E, Y_{n_k}) \neq 0$ *for all $k \geq 0$.*

Proof. Denote by $r \geq 1$ the rank of the stable tube $\mathcal{T}$ of Γ_H containing E. We denote by Σ the section in $\mathcal{Q}^H$ formed by all indecomposable injective modules in $\operatorname{mod} H$. Recall that $\Sigma \cong Q_H^{\mathrm{op}}$ as valued quivers. Moreover, we denote by Q^0 the set of all vertices of Q_H, and by I_a the indecomposable injective module in $\operatorname{mod} H$ corresponding to a vertex a in Q^0.

Consider the set $\mathcal{S}(E)$ formed by all modules X of $\mathcal{Q}^H$ with $\operatorname{Hom}_H(E, X) \neq 0$. Observe that, if $r = 1$ ($\mathcal{T}$ is a homogeneous tube), then E is a sincere H-module, thus $\operatorname{Hom}_H(\tau_H^s E, I_a) = \operatorname{Hom}_H(E, I_a) \neq 0$ for all $s \geq 0$ and $a \in Q^0$. Therefore $\mathcal{S}(E)$ contains all modules of $\mathcal{Q}^H$, and the claim follows. Hence, we assume that the stable tube $\mathcal{T}$ has rank $r \geq 2$.

First we investigate the case where one of the mouth modules of $\mathcal{T}$, say $E' = \tau_H^{-s} E$ with $s \geq 0$, is a sincere H-module. Then, as H is a hereditary algebra, $\operatorname{Hom}_H(E, \tau_H^s I_a) \cong \operatorname{Hom}_H(E', I_a) \neq 0$ for all $a \in Q^0$, and consequently $\operatorname{Hom}_H(E, \tau_H^{\alpha r+s} I_a) \cong \operatorname{Hom}_H(E, \tau_H^s I_a) \neq 0$ for all $a \in Q^0$ and every integer $\alpha \geq 0$. Hence, if there is a path of the form $(*)$, then there is a sequence $n_0 < n_1 < \cdots$ of non-negative integers such that, for each $k \geq 0$, there are a vertex $a_k \in Q^0$ and an integer $\alpha_k \geq 0$ with $Y_{n_k} \cong \tau_H^{\alpha_k r+s} I_{a_k}$. Thus Y_{n_k} is in $\mathcal{S}(E)$ for every $k \geq 0$, and we are done.

Therefore the statement holds if H is of one of the Euclidean types $\tilde{\mathbb{A}}_{11}$, $\tilde{\mathbb{A}}_{12}$, $\tilde{\mathbb{B}}_m$, $\tilde{\mathbb{C}}_m$, $\widetilde{\mathbb{BC}}_m$, $\widetilde{\mathbb{CB}}_m$, $\tilde{\mathbb{G}}_{21}$, or $\tilde{\mathbb{G}}_{22}$. Indeed, if H is of type $\tilde{\mathbb{A}}_{11}$ or $\tilde{\mathbb{A}}_{12}$, then all stable tubes of Γ_H are homogeneous. For the remaining types, there is a unique stable tube of Γ_H of rank $r \geq 2$ which contains a sincere module on the mouth (see [12, 6. Tables]).

Now, we consider the remaining Euclidean types: $\tilde{\mathbb{D}}_m$ with $m \geq 4$, $\widetilde{\mathbb{BD}}_m$ or $\widetilde{\mathbb{CD}}_m$ with $m \geq 3$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, $\tilde{\mathbb{E}}_8$, $\tilde{\mathbb{F}}_{41}$, or $\tilde{\mathbb{F}}_{42}$. In each case, we proceed only for a module E lying in the stable tube of rank $r \geq 2$, not containing a sincere module. Observe also that, if we prove our claim for an arbitrarily chosen module E on the mouth of $\mathcal{T}$, then it holds for any other mouth module of $\mathcal{T}$. Note that Q_H is assumed to have a cannonical orientation (as in [12, 6. Tables]); for vertices of Q_H we use the same notations as in [12, 6. Tables].

(1) Let H be of type $\tilde{\mathbb{D}}_m$ with $m \geq 4$. Then, by [12, 6. Tables], the stable tube of Γ_H of rank $m-2$ contains a sincere module. Therefore, we may assume that E lies in one of two stable tubes of Γ_H of rank 2.

Assume first that $E = E'_0$ (in the notation of [12, 6. Tables]). Then $\operatorname{Hom}_H(E, I_a) \neq 0$ for all vertices a in $\{z_1, \dots, z_{m-3}, a_2, b_2\}$. Similarly, we have $\operatorname{Hom}_H(\tau_H^{-1} E, I_a) \neq 0$ for all $a \in \{z_1, \dots, z_{m-3}, a_1, b_1\}$. Therefore $\mathcal{S}(E)$ contains the set

$$\mathcal{S}_0^*(E) = \{I_{a_2}, I_{b_2}, I_{z_1}, \dots, I_{z_{m-3}}, \tau_H I_{b_1}, \tau_H I_{a_1}, \tau_H I_{z_1}, \dots, \tau_H I_{z_{m-3}}\}.$$

Since $\mathcal{T}$ is a stable tube of rank 2, $\mathcal{S}_\alpha^*(E) := \tau_H^{2\alpha} \mathcal{S}_0^*(E)$ is a subset of $\mathcal{S}(E)$ for any $\alpha \geq 0$. Thus, if we have a path of the form $(*)$ in $\mathcal{Q}^H$, and there is a module Y_{t_0} not lying in $\mathcal{S}(E)$, then either $Y_{t_0} \cong \tau_H^{2\alpha} I_a$ with $a \in \{a_1, b_1\}$ and $\alpha \geq 0$, or $Y_{t_0} \cong \tau_H^{2\alpha+1} I_a$ with $a \in \{a_2, b_2\}$ and $\alpha \geq 0$. In both cases, we have $Y_{t_0+1} \in \mathcal{S}^*_\alpha(E)$, hence the path $(*)$ admits infinitely many modules from $\mathcal{S}(E)$.

In a similar way, if $E = E''_0$, then for any $\alpha \geq 0$, the set $\mathcal{S}(E)$ contains the subset $\mathcal{S}^*_\alpha(E) = \tau_H^{2\alpha}\mathcal{S}^*_0(E)$, where

$$\mathcal{S}^*_0(E) = \{I_{a_2}, I_{b_1}, I_{z_1}, \ldots, I_{z_{m-3}}, \tau_H I_{b_2}, \tau_H I_{a_1}, \tau_H I_{z_1}, \ldots, \tau_H I_{z_{m-3}}\},$$

and again, if there is a path of the form $(*)$ then, for every module Y_{t_0} in $\tau_H^{2\alpha+1}\Sigma \cup \tau_H^{2\alpha}\Sigma$ not lying in $\mathcal{S}^*_\alpha(E)$, the module Y_{t_0+1} belongs to $\mathcal{S}^*_\alpha(E)$.

(2) Assume now that H is of type $\widetilde{\mathbb{BD}}_m$. Then $E = E'_0$ lies in the stable tube of rank 2 (other tubes admit a sincere mouth module), and a straightforward calculation shows that, for every $\alpha \geq 0$, the set $\mathcal{S}(E)$ contains $\mathcal{S}^*_\alpha(E) = \tau_H^{2\alpha}\mathcal{S}^*_0(E)$, where

$$\mathcal{S}^*_0(E) = \{I_a;\ a \in Q^0 \setminus a_1\} \cup \{\tau_H I_a;\ a \in Q^0 \setminus a_2\}.$$

Observe that the unique direct predecessor in $\mathcal{Q}^H$ of a module $Y = \tau_H^{2\alpha} I_{a_1}$, $\alpha \geq 0$, not lying in $\mathcal{S}^*_\alpha(E)$, is of the form $Y' = \tau_H^{2\alpha} I_{z_1}$, because I_{a_1} is a sink in Σ. Hence Y' is in $\mathcal{S}^*_\alpha(E)$. In a similar way, the unique direct predecessor of $Y = \tau_H^{2\alpha+1} I_{a_2}$, $\alpha \geq 0$, is of the form $Y' = \tau_H^{2\alpha+1} I_{z_1}$, and so Y' lies in $\mathcal{S}^*_\alpha(E)$. Therefore, for each infinite path of the form $(*)$, there is a sequence $n_0 < n_1 < \cdots$ of non-negative integers such that, for every $k \geq 0$, there is an integer $\alpha_k \geq 0$ with Y_{n_k} in $\mathcal{S}^*_{\alpha_k}(E) \subset \mathcal{S}(E)$. The proof in this case is thus complete. In a similar way, we prove that the claim holds for the Euclidean type $\widetilde{\mathbb{CD}}_m$.

(3) Now, we consider the types $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ and $\tilde{\mathbb{E}}_8$. Assume first that H is of type $\tilde{\mathbb{E}}_6$. Because the unique stable tube of Γ_H of rank 2 contains a sincere module, we may assume that E lies in one of the remaining tubes of rank 3. Let $E = E'_0$. Then using [12, 6. Tables], we easily find that the set

$$\mathcal{S}^*_0(E) = \bigcup_{i=0}^{2} \{\tau_H^i I_a;\ a \in Q^0 \setminus Q^{0,i}\},$$

where $Q^{0,0} = \{a_1, c_1, c_2\}$, $Q^{0,1} = \{c_1, b_1, b_2\}$, and $Q^{0,2} = \{b_1, a_1, a_2\}$, is a subset of $\mathcal{S}(E)$. Observe that, for every $\alpha \geq 0$, we also have the inclusion $\mathcal{S}^*_\alpha(E) := \tau_H^{3\alpha}\mathcal{S}^*_0(E) \subset \mathcal{S}(E)$, because $\mathcal{T}$ is of rank 3. Now, consider an arbitrary path of the form $(*)$ in $\mathcal{Q}^H$, and let $Y_{t_0} = \tau_H^{3\alpha+i} I_a$ with $\alpha \geq 0$ and $i \in \{0,1,2\}$ be a module not lying in $\mathcal{S}^*_\alpha(E)$. If $i = 0$, then $a \in \{a_1, c_1, c_2\}$. Assume that $a = a_1$. Then $Y_{t_0+1} \in \mathcal{S}^*_\alpha(E)$. If $a = c_q$, $q \in \{1,2\}$, then $Y_{t_0+3-q} \in \mathcal{S}^*_\alpha(E)$ or $Y_{t_0+4-q} \in \mathcal{S}^*_\alpha(E)$. Further, if $i = 1$, then for $a = c_2$, we have $Y_{t_0+1} \in \mathcal{S}^*_\alpha(E)$, and if $a = b_q$, $q \in \{1,2\}$, then $Y_{t_0+3-q} \in \mathcal{S}^*_\alpha(E)$ or $Y_{t_0+4-q} \in \mathcal{S}^*_\alpha(E)$. Finally, assume that $i = 2$. If $a = b_1$, then $Y_{t_0+1} \in \mathcal{S}^*_\alpha(E)$, and, if $a = a_q$, $q \in \{1,2\}$, then $Y_{t_0+3-q} \in \mathcal{S}^*_\alpha(E)$. This shows that there are infinitely many integers $n_k \geq 0$, $k \geq 0$, such that $Y_{n_k} \in \mathcal{S}(E)$ for any $k \geq 0$. Similar arguments prove the claim when $E = E''_0$ is contained in the second tube of rank 3 of Γ_H.

Now, let H be of type $\tilde{\mathbb{E}}_7$. We may assume that E is contained either in the stable tube of rank 2, or in the stable tube of rank 4, because the stable tube of rank 3 admits a sincere mouth module E_1. First, let $E = E_0''$ be a module in the tube of rank 2. Then we easily find that

$$\mathcal{S}_0^*(E) = \{I_a;\ a \in Q^0 \setminus \{a_1\}\} \cup \{\tau_H I_a;\ a \in Q^0 \setminus \{b_1\}\}$$

is a subset of $\mathcal{S}(E)$, and $\mathcal{S}_\alpha^*(E) := \tau_H^{2\alpha}\mathcal{S}_0^*(E) \subset \mathcal{S}_0^*(E)$ for all $\alpha \geq 0$. Consequently, if there is a path of the form $(*)$ in $\mathcal{Q}^H$, and Y_{t_0} is not in $\mathcal{S}(E)$, then Y_{t_0+1} belongs to $\mathcal{S}(E)$, and hence there are infinitely many integers $n_k \geq 0$, $k \geq 0$, such that $Y_{n_k} \in \mathcal{S}(E)$ for all $k \geq 0$. Assume now that $E = E_0'$ is contained in the stable tube of rank 4. Then, for every $\alpha \geq 0$, the set $\mathcal{S}(E)$ contains $\mathcal{S}_\alpha^*(E) = \tau_H^{4\alpha}\mathcal{S}_0^*(E)$, where

$$\mathcal{S}_0^*(E) = \bigcup_{i=0}^{3}\{\tau_H^i I_a;\ a \in Q^0 \setminus Q^{0,i}\},$$

$Q^{0,0} = \{a_1, a_2, a_3, b_1\}$, $Q^{0,1} = \{a_1, a_2, c\}$, $Q^{0,2} = \{b_1, b_2, b_3, a_1\}$, and $Q^{0,3} = \{b_1, b_2, c\}$. It follows that, if there is a path in $\mathcal{Q}^H$ of the form $(*)$, with $Y_{t_0} = \tau_H^{4\alpha+i} I_a$, $\alpha \geq 0$, $i \in \{0,1,2,3\}$, and Y_{t_0} is not in $\mathcal{S}_\alpha^*(E)$, then either there is an integer $k \geq 1$ such that $Y_{t_0+k} \in \mathcal{S}_\alpha^*(E)$, or there is an integer $k \geq 1$ such that Y_{t_0+k} is in $\mathcal{S}_{\alpha+1}^*(E)$.

Finally, assume that H is of type $\tilde{\mathbb{E}}_8$. Observe that there is a sincere module both in the stable tube of rank 2 (the module E_1'') and in the stable tube of rank 5 (the module E_1). Thus, we may assume that $E = E_0'$ is a module in the (unique) stable tube of rank 3. Then we deduce from [12, 6. Tables] that the set

$$\mathcal{S}_0^*(E) = \bigcup_{i=0}^{2}\{\tau_H^i I_a;\ a \in Q^0 \setminus Q^{0,i}\},$$

where $Q^{0,0} = \{a_1, a_2\}$, $Q^{0,1} = \{a_1\}$, and $Q^{0,2} = \{b_1\}$, is contained in $\mathcal{S}(E)$, and $\mathcal{S}_\alpha^*(E) := \tau_H^{3\alpha}\mathcal{S}_0^*(E) \subset \mathcal{S}(E)$ for all $\alpha \geq 0$. Hence, if there is a path in $\mathcal{Q}^H$ of the form $(*)$, and the module $Y_{t_0} = \tau_H^{3\alpha+i} I_a$ with $i \in \{0,1,2\}$ and $\alpha \geq 0$ is not contained in $\mathcal{S}_\alpha^*(E)$, then $Y_{t_0+1} \in \mathcal{S}_\alpha^*(E)$ or $Y_{t_0+2} \in \mathcal{S}_\alpha^*(E)$, and the claim follows.

(4) In the last step, we consider the types $\tilde{\mathbb{F}}_{41}$ and $\tilde{\mathbb{F}}_{42}$. First, let H be of type $\tilde{\mathbb{F}}_{41}$. Then we may assume that $E = E_0'$ lies in a stable tube of rank 2, because the unique stable tube of rank 3 contains the sincere module E_1. Further, we calculate that

$$\mathcal{S}_0^*(E) = \{I_a;\ a \in Q^0 \setminus \{a_1\}\} \cup \{\tau_H I_a;\ a \in Q^0 \setminus \{b\}\}$$

is contained in $\mathcal{S}(E)$. Clearly, $\mathcal{S}_\alpha^*(E) = \tau_H^{2\alpha}\mathcal{S}_0^*(E)$ is also contained in $\mathcal{S}(E)$. Therefore, if the module $Y = \tau_H^{2\alpha+i} I_a$, $i \in \{0,1\}$, is not in $\mathcal{S}_\alpha^*(E)$, then $a = a_1$ and $i = 0$ or $i = 1$ and $a = b$, and in both cases every direct predecessor of

Y is in $\mathcal{S}^*_\alpha(E) \cup \mathcal{S}^*_{\alpha+1}(E)$. This shows that the claim holds true in this case. Assume finally that H is of type $\tilde{\mathbb{F}}_{42}$. Then the unique stable tube of rank 2 admits the sincere mouth module E'_1. Let $E = E_0$ be a module lying on the mouth of the remaining (nonhomogeneous) stable tube of rank 3. Then, as before, the set

$$\mathcal{S}^*_0(E) = \bigcup_{i=0}^{2} \{\tau_H^i I_a;\ a \in Q^0 \setminus Q^{0,i}\},$$

where $Q^{0,0} = \{a_1\}$, $Q^{0,1} = \{b_1\}$, and $Q^{0,2} = \{a_1, a_2\}$, is a subset of $\mathcal{S}(E)$, as well as $\mathcal{S}^*_\alpha(E) := \tau_H^{3\alpha}\mathcal{S}^*_0(E)$ is contained in $\mathcal{S}(E)$ for every $\alpha \geq 0$. Moreover, a direct observation shows that, if there is a path in $\mathcal{Q}^H$ of the form $(*)$ with a module $Y_{t_0} = \tau_H^{3\alpha+i} I_a$, $i \in \{0,1,2\}$, not lying in $\mathcal{S}^*_\alpha(E)$, then there is $k \in \{1,2,3\}$ such that $Y_{t_0+k} \in \mathcal{S}^*_\alpha(E) \cup \mathcal{S}^*_{\alpha+1}(E)$. ∎

For a hereditary algebra H of type $\tilde{\mathbb{A}}_m$, we have the following slightly different result.

LEMMA 6.2. *Let H be a hereditary algebra of Euclidean type $\tilde{\mathbb{A}}_m$, $m \geq 2$, with valued quiver Q_H oriented canonically* (*as in* [12]), *and let E be a module lying on the mouth of a stable tube $\mathcal{T}$ of Γ_H. Then, for every module Y from the preinjective component $\mathcal{Q}^H$ of Γ_H, there is an infinite sectional path*

$$\cdots \to Y_1 \to Y_0 = Y$$

in $\mathcal{Q}^H$ such that:

(i) *There exists a sequence $n_0 < n_1 < \cdots$ of non-negative integers such that* $\mathrm{Hom}_H(E, Y_{n_k}) \neq 0$, *for all $k \geq 0$.*
(ii) *For every non-negative integer c, there exists a sequence $n_0^c < n_1^c < \cdots$ of non-negative integers such that* $\mathrm{Hom}_H(E, \tau_H^c Y_{n_k^c}) \neq 0$ *for all $k \geq 0$.*

Proof. We use the notations introduced in Lemma 6.1 (we also stick to the notations for the vertices of Q_H used in [12, 6. Tables]). As above, we may assume that E belongs to one of the stable tubes of Γ_H of rank ≥ 2, because otherwise E is a sincere module lying on the mouth of a homogeneous stable tube, hence $\mathrm{Hom}_B(E, Q) \neq 0$ for all modules Q from $\mathcal{Q}^H$.

(1) It follows from [12, 6. Tables] that

$$\mathcal{S}(E_0) = \bigcup_{\alpha=0}^{\infty} \mathcal{S}^*_\alpha(E_0),$$

where $\mathcal{S}^*_\alpha(E_0) = \tau_H^{(p+1)\alpha}\mathcal{S}^*_0(E_0)$, and

$$\mathcal{S}^*_0(E_0) = \{I_{c_p}, \tau_H I_{c_{p-1}}, \ldots, \tau_H^{p-1} I_{c_1}\} \cup \{\tau_H^p I_a;\ a \in \{d_1, \ldots, d_q, a, b\}\}.$$

Moreover, observe that there is a sectional path in $\mathcal{Q}^H$ of the form

$$\cdots \to Q_1 \to Q_0$$

such that $\mathcal{S}(E_0) = \{Q_i\}_{i\geq 0}$. Similarly, for every $t \in \{0,\ldots,p\}$, the set $\mathcal{S}(\tau_H^{-t}E_0) = \tau_H^{-t}\mathcal{S}(E_0)$ is formed by the modules $\tau_H^{-t}Q_n$, $n \geq 0$, lying on a sectional path in $\mathcal{Q}^H$. Consequently, $\mathcal{Q}^H$, viewed as a set, has the disjoint union decomposition

$$\mathcal{Q}^H = \mathcal{S}(E_0) \cup \mathcal{S}(\tau_H^{-1}E_0) \cup \cdots \cup \mathcal{S}(\tau_H^{-p}E_0).$$

In the same manner, it has the disjoint union decomposition

$$\mathcal{Q}^H = \mathcal{S}(E_0') \cup \mathcal{S}(\tau_H^{-1}E_0') \cup \cdots \cup \mathcal{S}(\tau_H^{-q}E_0').$$

(2) Now, observe that $\mathcal{S}(\tau_H^{-t_1}E_0) \cap \mathcal{S}(\tau_H^{-t_2}E_0')$ is an infinite set for every $t_1 \in \{0,\ldots,p\}$ and $t_2 \in \{0,\ldots,q\}$. Indeed, a module X in $\mathcal{Q}^H$ belongs to $\mathcal{S}(\tau_H^{-t_1}E_0)$ if and only if X is a non-zero module lying on the following path in $\mathcal{Q}^H$:

$$\cdots \to \tau_H^{2(p+1)}Z_0 \to \cdots \to \tau_H^{p+1}Z_0 \to \tau_H^{-t_1+p}I_b \to \tau_H^{-t_1+p}d_q \to \cdots \to \tau_H^{-t_1+p}I_{d_1}$$
$$\to \tau_H^{-t_1+p}I_a \to \tau_H^{-t_1+p-1}I_{c_1} \to \cdots \to \tau_H^{-t_1+1}I_{c_{p-1}} \to \tau_H^{-t_1}I_{c_p} = Z_0.$$

Applying [12, 6. Tables] again, we infer that a module X in $\mathcal{Q}^H$ belongs to $\mathcal{S}(\tau_H^{-t_2}E_0')$ if and only if X is a non-zero module lying on the following sectional path in $\mathcal{Q}^H$:

$$\cdots \to \tau_H^{2(q+1)}Z_0' \to \cdots \to \tau_H^{q+1}Z_0' \to \tau_H^{-t_2+q}I_b \to \tau_H^{-t_2+q}Ic_p \to \cdots \to \tau_H^{-t_2+q}I_{c_1}$$
$$\to \tau_H^{-t_2+q}I_a \to \tau_H^{-t_2+q-1}I_{d_1} \to \cdots \to \tau_H^{-t_2+1}I_{d_{q-1}} \to \tau_H^{-t_2}I_{d_q} = Z_0'.$$

Further, observe that, for every integer $\alpha \geq 1$, there are an integer $\beta \geq 0$ and $r \in \{0,\ldots,q\}$ such that $\alpha(p+1)+p+t_2-t_1 = \beta(q+1)+r$, and hence $\alpha(p+1)-t_1+p = \beta(q+1)-t_2+r$. Consequently, the module $X_\alpha = \tau_H^{\alpha(p+1)-t_1+p}I_{d-r} = \tau_H^{\beta(q+1)-t_2+r}I_{d-r}$ is contained in $\mathcal{S}(\tau_H^{-t_1}E_0) \cap \mathcal{S}(\tau_H^{-t_2}E_0')$ for every integer $\alpha \geq 1$, and we are done.

Finally, consider the module E lying on the mouth of a stable tube. We assume that the rank of $\mathcal{T}$ is p (similar arguments provide the claim if $\mathcal{T}$ is a stable tube of rank q). Then $E \cong \tau_H^{-t_1}E_0$ for some $t_1 \in \{0,\ldots,p\}$. Using (1), we conclude that there is $t_2 \in \{0,\ldots,q\}$ such that $Y \in \mathcal{S}(\tau_H^{-t_2}E_0')$ and there exists a sectional path in $\mathcal{Q}^H$ of the form required in (i). Moreover, $Y_n \in \mathcal{S}(\tau_H^{-t_2}E_0')$ for all $n \geq 0$. Applying the arguments in (2), we deduce that there is a sequence $n_0 < n_1 < \cdots$ of non-negative integers such that $Y_{n_k} \in \mathcal{S}(\tau_H^{-t_1}E_0) \cap \mathcal{S}(\tau_H^{-t_2}E_0')$ for all $k \geq 0$, and consequently $\mathrm{Hom}_H(E, Y_{n_k}) \neq 0$ for every $k \geq 0$. Hence (ii) also holds. For the proof of (iii), it is sufficient to use the fact that $\mathcal{S}(E) \cap \mathcal{S}(\tau_H^{-t_2+c}E_0')$ is an infinite set. ∎

Our next aim is to prove the following lemma, essential to the proof of the main theorem in the non-semiregular case (see Theorem 6.4 below).

LEMMA 6.3. *Let B be a tilted algebra of Euclidean type such that Γ_B admits an infinite preinjective connecting component. Moreover, assume that there are modules M and R in* $\operatorname{ind} B$ *satisfying the following conditions:*

(i) *M lies on the mouth of a stable tube of Γ_B.*
(ii) *R is contained in the preinjective component $\mathcal{Q}^B$ of Γ_B.*

Then there are infinitely many pairwise non-isomorphic indecomposable modules Z_n in $\mathcal{Q}^B$, $n \in \mathbb{N}$, such that

$$\operatorname{Hom}_B(M, \tau_B Z_n) \neq 0 \quad \text{and} \quad \operatorname{Hom}_B(\tau_B^{-1} Z_n, R) \neq 0,$$

for all $n \geq 0$.

Proof. Let H be a hereditary algebra of Euclidean type and T be a tilting module in $\operatorname{mod} H$ such that $B \cong \operatorname{End}_H(T)$. Using the assumptions on Γ_B, we infer that $T = T^{pp} \oplus T^{rg}$, where T^{pp} (respectively, T^{rg}) is in $\operatorname{add}(\mathcal{P}^H)$ (respectively, $\operatorname{add}(\mathcal{T}^H)$), the family $\mathcal{T}^B$ (of all semiregular tubes of Γ_B) does not admit a coray tube containing an injective module, and the connecting component $\mathcal{C}_T = \mathcal{Q}^B$ determined by T contains all indecomposable injective B-modules. We denote by Σ the section in $\mathcal{Q}^H$ formed by all indecomposable injective H-modules, and by Δ the associated section in $\mathcal{Q}^B$ formed by all modules of the form $\operatorname{Hom}_H(T, I)$ with I in Σ. Moreover, since M lies on the mouth of a stable tube of Γ_B, there is a stable tube of Γ_H without modules from $\operatorname{add}(T)$ and containing a mouth module E such that $\operatorname{Hom}_H(T, E) \cong M$ as B-modules. Note also that we may assume that the valued quiver $Q_H \cong \Delta^{\mathrm{op}}$ of H is oriented canonically (as in [12, 6. Tables]). Indeed, since $\mathcal{Q}^B$ is an acyclic and generalized standard component of Γ_B with section Δ, the component $\mathcal{Q}^B$ of Γ_B admits a section Δ' with the same number of vertices as Δ such that Δ' is oriented canonically, Δ and Δ' are of the same Euclidean type, and $\operatorname{Hom}_B(U_0, \tau_B U_1) = 0$ for all modules U_0, U_1 from Δ' (see also [37, Theorem 2]). Therefore, using the Liu–Skowroński criterion [22], [37] (see also [1, Theorem VIII.5.6]), we find that the direct sum U_B of all modules lying on Δ' is a tilting B-module, the algebra $H' = \operatorname{End}_B(U_B)$ is a hereditary algebra with $Q_{H'} \cong (\Delta')^{\mathrm{op}}$ oriented canonically, and $B \cong \operatorname{End}_{H'}(T')$, where $T' = T'_{H'} = D({}_{H'}U)$ is a tilting module in $\operatorname{mod} H'$. Recall that there is an induced splitting torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$, where the torsion part $\mathcal{X}(T)$ and torsion-free part $\mathcal{Y}(T)$ of $\operatorname{mod} B$ are defined as follows: $\mathcal{X}(T) = \{X \in \operatorname{mod} B;\ X \otimes_B T = 0\}$ and $\mathcal{Y}(T) = \{Y \in \operatorname{mod} B;\ \operatorname{Ext}^1_B(Y, D(T)) = 0\}$. Moreover, every indecomposable module in $\mathcal{X}(T)$ (respectively, in $\mathcal{Y}(T)$) is isomorphic to a module of the form $\operatorname{Ext}^1_H(T, M)$ (respectively, $\operatorname{Hom}_H(T, M)$) with M a module in $\operatorname{ind} H$ such that $\operatorname{Hom}_H(T, M) = 0$ (respectively, $\operatorname{Ext}^1_H(T, M) = 0$). Note also that every almost split sequence in $\operatorname{mod} B$ is contained entirely in $\mathcal{X}(T)$ or entirely in $\mathcal{Y}(T)$, or it is a connecting sequence, that is, an almost split sequence in $\operatorname{mod} B$ of the form

$$0 \to \operatorname{Hom}_H(T, I_j) \to E \to \operatorname{Ext}^1_H(T, P_j) \to 0,$$

where $E = \operatorname{Hom}_H(T, I_j/\operatorname{soc}(I_j)) \oplus \operatorname{Ext}^1_H(T, \operatorname{rad} P_j)$ and P_j is not in $\operatorname{add}(T)$.

Clearly, there is a projective module P in $\mathcal{P}^B \cup \mathcal{T}^B$ such that $\operatorname{Hom}_B(P, R) \neq 0$. By [40, Lemma 2.1], there is an infinite path (Ω) in $\mathcal{C}_T$ of the form

$$(\Omega): \ \cdots \to V_1 \to V_0 = R,$$

with $\operatorname{Hom}_B(V_n, R) \neq 0$ for all $n \geq 0$. Since $\mathcal{C}_T$ is a preinjective component of Γ_B, we conclude that there is a path in $\mathcal{C}_T$ of the form

$$\cdots \to V'_1 \to V'_0,$$

where $V'_n = \tau_B V_n$ for any $n \geq 0$, and so $\operatorname{Hom}_B(\tau_B^{-1} V'_n, R) = \operatorname{Hom}_B(V_n, R) \neq 0$ for each $n \geq 0$. Now, consider the path in $\mathcal{C}_T$ of the form

$$\cdots \to V''_1 \to V''_0,$$

where $V''_n = \tau_B V'_n = \tau_B^2 V_n$ for all $n \geq 0$. It is clear that there is an integer $m_0 \geq 0$ such that V''_n is a predecessor of Δ in $\mathcal{C}_T$ for all $n \geq m_0$. Therefore, there exists a path in $\mathcal{Q}^H$ of the form

$$(*) \qquad \cdots \to Y_{m_0+1} \to Y_{m_0}$$

such that $\operatorname{Hom}_H(T, Y_n) \cong V''_n$ for every $n \geq m_0$. Hence, if Q_H is a tree, then Lemma 6.1 implies that there is an infinite sequence $m_0 \leq n_0 < n_1 < \cdots$ of integers such that $\operatorname{Hom}_H(E, Y_{n_k}) \neq 0$ for all $k \geq 0$, and consequently $\operatorname{Hom}_B(M, V''_{n_k}) \cong \operatorname{Hom}_H(E, Y_{n_k}) \neq 0$, by the Brenner–Buttler theorem (see [1, Theorem VI.3.8]). Thus, in this case, there are infinitely many pairwise non-isomorphic modules $Z_k = V'_{n_k}$, $k \geq 0$, with the required properties.

Now, assume that H is of Euclidean type $\tilde{\mathbb{A}}_m$, $m \geq 1$. First, let R be a module from the torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ of $\mathcal{C}_T$. Then $R \cong \operatorname{Hom}_H(T, Y)$ for a module Y in $\mathcal{Q}^H$. Hence, Lemma 6.2 implies that there is a sectional path in $\mathcal{Q}^H$ of the form

$$\cdots \to Y_1 \to Y_0 = Y$$

such that there exists a sequence $n_0 < n_1 < \cdots$ of non-negative integers, with $\operatorname{Hom}_H(E, \tau_H^2 Y_{n_k}) \neq 0$ for all $k \geq 0$. Therefore, we have a sectional path (Ω) in $\mathcal{Q}^B$ of the form

$$(\Omega): \ \cdots \to V_1 \to V_0 = R,$$

where $V_n \cong \operatorname{Hom}_H(T, Y_n)$ for every $n \geq 0$, and $\operatorname{Hom}_B(M, \tau_B^2 V_{n_k}) \neq 0$ for every $k \geq 0$. Consequently, there are infinitely many pairwise non-isomorphic modules $Z_k = \tau_B V_{n_k}$, $k \geq 0$, in $\mathcal{Q}^B$ such that $\operatorname{Hom}_B(\tau_B^{-1} Z_k, R) = \operatorname{Hom}_B(V_{n_k}, R) \neq 0$ and $\operatorname{Hom}_B(M, \tau_B Z_k) = \operatorname{Hom}_B(M, \tau_B^2 V_{n_k}) \neq 0$ for all $k \geq 0$.

Now, we consider the last case, where R belongs to the torsion part $\mathcal{X}(T) \cap \mathcal{Q}^B$ of $\mathcal{Q}^B$. Then there is a torsion-free module F in $\mathcal{P}^H$ such that $R \cong \mathrm{Ext}^1_H(T, F)$. Observe also that all irreducible homomorphisms between indecomposable modules in the postprojective component $\mathcal{P}^H$ of Γ_H are irreducible monomorphisms, and $\mathcal{P}^H$ is a generalized standard component of Γ_H. Hence a module F in $\mathcal{P}^H$ belongs to the torsion free part $\mathcal{F}(T)$ of $\operatorname{mod} H$ if and only if F is not a successor in $\mathcal{P}^H$ of an indecomposable direct summand of T^{pp}, and consequently $\mathcal{F}(T) \cap \operatorname{ind} H$ is a full subcategory of $\operatorname{ind} H$, closed under predecessors in $\operatorname{ind} H$. Therefore, applying Lemma 6.2 and its dual, we conclude that the following statements hold:

- There is a sectional path in $\mathcal{P}^H$ of the form
$$P_j = F_0 \to F_1 \to \cdots \to F_t = F,$$
with $j \in Q^0$, P_j not in $\operatorname{add}(T)$, and such that $\mathrm{Hom}_H(\tau_H^{-p} F_s, E) \neq 0$ for an integer $p \geq 0$ and every $s \in \{0, \ldots, t\}$.
- $\mathcal{Q}^H$ admits a sectional path of the form
$$\cdots \to Y_1 \to Y_0 = I_i$$
such that there is a sequence $n_0 < n_1 < \cdots$ of non-negative integers with $\mathrm{Hom}_H(E, \tau_H^2 Y_{n_k}) \neq 0$ for every $k \geq 0$, and there is an irreducible homomorphism in $\operatorname{mod} B$ of the form $\mathrm{Hom}_H(T, I_i) \to \mathrm{Ext}^1_H(T, P_j)$ with I_i an indecomposable direct summand of the injective module $I_j/\operatorname{soc}(I_j)$.
- The modules I_j and Y_1 are non-isomorphic modules in $\operatorname{mod} H$.

It follows that there is a sectional path in $\mathcal{Q}^B$ of the form
$$\cdots \to V_1 \to V_0 \to W_0 \to \cdots \to W_t = R,$$
where $V_n \cong \mathrm{Hom}_H(T, Y_n)$ for any $n \geq 0$ and $W_s \cong \mathrm{Ext}^1_H(T, F_s)$ for all $s \in \{0, \ldots, t\}$. As before, the modules $Z_k = \tau_B V_{n_k}$ for $k \geq 0$ have the required properties, and the proof is now finished. ∎

The following theorem completes the proof of Theorem 1.2.

THEOREM 6.4. *Let A be a cycle-finite algebra such that there exists a non-semiregular component $\mathcal{C}$ in Γ_A. Then the following statements are equivalent:*

(a) *For all but finitely many isomorphism classes of modules X in* $\operatorname{ind} A$*, we have* $\operatorname{pd}_A X \leq 1$ *or* $\operatorname{id}_A X \leq 1$.
(b) *A is a generalized double tilted algebra.*

Proof. The implication (ii)⇒(i) is a consequence of the main result of [47] (see also [31, Theorem 3.4]).

To prove (i)⇒(ii), we may assume that A is not of finite representation type, because otherwise A is a generalized double tilted algebra with finite connecting component (see [31, Section 3]), and there is nothing to prove.

From now on, we assume that A satisfies (i). We will show that A is a generalized double tilted algebra.

(1) Let $\mathcal{C}$ be a non-semiregular component of Γ_A. By Proposition 3.1, every connected component of the cyclic part ${}_c\mathcal{C}$ is finite. Moreover, every finite cyclic component of Γ_A contains both a projective module and an injective module (see [24, Corollary 2.6]). Therefore, $\mathcal{C}$ is an almost acyclic component of Γ_A. Hence, applying [31, Theorem 2.5], we infer that $\mathcal{C}$ admits a multisection Δ. Recall that Δ is a full connected valued subquiver of $\mathcal{C}$ satysfying the following conditions (see [31, Section 2]):

(a) Δ is almost acyclic.
(b) Δ is convex in $\mathcal{C}$.
(c) For each τ_A-orbit $\mathcal{O}$ in $\mathcal{C}$, we have $1 \leq |\Delta \cap \mathcal{O}| < \infty$.
(d) For all but finitely many τ_A-orbits $\mathcal{O}$ in $\mathcal{C}$, we have $|\Delta \cap \mathcal{O}| = 1$.
(e) No proper full valued subquiver of Δ satisfies (a)–(d).

Following Reiten and Skowroński [31], we also consider the following full valued subquivers of $\mathcal{C}$:

- $\Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1}\Delta''_l$, where Δ'_l is a full valued subquiver of Δ containing all modules $X \in \Delta$ such that there is a non-sectional path $X \to \cdots \to P$ with P a projective module and $\Delta''_l = \{X \in \Delta'_l;\ \tau_A^{-1}X \notin \Delta'_l\}$;
- $\Delta_l = (\Delta \setminus \Delta'_r) \cup \tau_A\Delta''_r$, where Δ'_r is a subquiver of Δ containing all modules $X \in \Delta$ such that there is a non-sectional path $I \to \cdots \to X$ with I an injective module and $\Delta''_r = \{X \in \Delta'_r;\ \tau_A X \notin \Delta'_r\}$;
- $\Delta_c = \Delta'_l \cap \Delta'_r$.

Further, using [31, Proposition 2.4], we infer that every oriented cycle in $\mathcal{C}$ lies entirely in Δ_c, and $\mathcal{C}$ has the disjoint union decomposition

$$\mathcal{C} = \mathcal{C}_l \cup \Delta_c \cup \mathcal{C}_r,$$

where $\mathcal{C}_l$ (respectively, $\mathcal{C}_r$) is the full valued translation subquiver of $\mathcal{C}$ formed by all predecessors of Δ_l in $\mathcal{C}$ (respectively, all successors of Δ_r in $\mathcal{C}$). Note that $\mathcal{C}_l$ or $\mathcal{C}_r$ is an infinite full valued subquiver of $\mathcal{C}$, because A is assumed to be indecomposable and of infinite representation type.

Assume that $\mathcal{C}_l$ is infinite. Let $\mathcal{D}_l$ be the full valued translation subquiver of the left stable part ${}_l\mathcal{C}_l$ of $\mathcal{C}_l$ formed by all modules X in $\mathcal{C}_l$ such that X is a predecessor of a projective module in $\mathcal{C}$ and every predecessor of X in $\mathcal{C}$ is in ${}_l\mathcal{C}_l$. Clearly, $\mathcal{D}_l$ is then a non-empty and acyclic left stable full valued translation subquiver of $\mathcal{C}_l$, closed under predecessors in $\mathcal{C}$. Assume that $\mathcal{D}_l$ has a decomposition $\mathcal{D}_l = \mathcal{D}_l^1 \cup \cdots \cup \mathcal{D}_l^p$ into a disjoint union of connected full (valued) translation subquivers. Then, using [23, Theorem 2.2], we infer that, for every $i \in \{1, \dots, p\}$, there exist a hereditary algebra H_i of Euclidean type and a tilting module T_i in $\operatorname{mod} H_i$ without non-zero preinjective direct summands such that the tilted algebra $B_i = \operatorname{End}_H(T_i)$ is a factor algebra of A and the torsion-free part $\mathcal{Y}(T_i) \cap \mathcal{C}_{T_i}$ of the connecting component $\mathcal{C}_{T_i}$ of Γ_{B_i} is a full valued translation subquiver of $\mathcal{D}_l^i$ (closed under predecessors in $\mathcal{C}$). Further, every A-module in $\mathcal{D}_l^i$ is a B_i-module and hence lies in the preinjective connecting component $\mathcal{C}_{T_i} = \mathcal{Q}^{B_i}$ of Γ_{B_i}. Moreover, the product algebra $B = B_1 \times \dots \times B_p$ is a quotient algebra of A. In particular, for every $i \in \{1, \dots, p\}$, there are a module R_i in $\mathcal{D}_l^i$ (lying in $\mathcal{Q}^{B_i}$) and an irreducible monomorphism $R_i \to P^{(i)}$ with $P^{(i)}$ a projective module in $\mathcal{C}$. Note also that $\mathcal{D}_l$ admits at most finitely many τ_A-orbits, because $\mathcal{D}_l$ is acyclic and hence consists only of directing modules (see [29] and [39]).

(2) Consider now the family $\mathcal{T}^{B_i} = (\mathcal{T}_\lambda^{B_i})_{\lambda \in \Lambda_i}$ of all semiregular tubes of Γ_{B_i} for $i \in \{1, \dots, p\}$. Then the disjoint union $\mathcal{T}^{B_1} \cup \dots \cup \mathcal{T}^{B_p}$ is the family $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda \in \Lambda}$, with $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_p$, of all (pairwise orthogonal) semiregular tubes of Γ_B and $B = \operatorname{supp}(\mathcal{T}^B)$. Note also that $\mathcal{T}^B$ contains only ray tubes. Moreover, observe that B is a convex subcategory of A. Indeed, for every $i \in \{1, \dots, p\}$, the algebra B_i is a tubular extension of a tame concealed algebra C_i, hence [7, Theorem 1.5] shows that B_i is a convex subcategory of A. Consequently, so is B.

Moreover, for any two modules X and Y lying in the cyclic part ${}_c\mathcal{T}_\lambda^B$ of a ray tube $\mathcal{T}_\lambda^B$, $\lambda \in \Lambda$, there exists a cycle of irreducible homomorphisms in $\operatorname{mod} B$ passing through X and Y. Since A is a cycle-finite algebra, we deduce that X and Y lie on a common cycle of irreducible homomorphisms in $\operatorname{mod} A$, and hence there is a component $\mathcal{T}_\lambda^A$ of Γ_A containing all modules from ${}_c\mathcal{T}_\lambda^B$. Further, ${}_c\mathcal{T}_\lambda^B$ is infinite, hence the cyclic part ${}_c\mathcal{T}_\lambda^A$ of $\mathcal{T}_\lambda^A$ is infinite, and consequently, by Proposition 3.1, $\mathcal{T}_\lambda^A$ is a ray tube or a coray tube. Note also that $\mathcal{T}_\lambda^A \neq \mathcal{T}_\mu^A$ for any $\lambda \neq \mu$ in Λ, and $\mathcal{T}_\lambda^A = \mathcal{T}_\lambda^B$ for all but finitely many λ in Λ (see the proof of [7, Theorem 4.1]). Denote by $\mathcal{T}^A = \mathcal{T}^A(B)$ the family $(\mathcal{T}_\lambda^A)_{\lambda \in \Lambda}$ of semiregular tubes of Γ_A.

We claim that $\mathcal{T}^A$ has no coray tubes containing injective modules. Suppose to the contrary that a coray tube $\mathcal{T}_{\lambda_0}^A$, $\lambda_0 \in \Lambda$, of $\mathcal{T}^A$ contains an injective module. Then the ray tube $\mathcal{T}_{\lambda_0}^B$ is a stable tube of Γ_B, by [42, Proposition 2.3], and hence there exists a module M lying on the mouth of $\mathcal{T}_{\lambda_0}^B$ and an irreducible epimorphism $I \to M$ in $\operatorname{mod} A$ with I an injective A-module. Therefore, if $\lambda_0 \in \Lambda_i$, then, using Lemma 6.3, we conclude that there are infinitely many pairwise non-isomorphic indecomposable modules Y_n, $n \geq 0$, in $\mathcal{Q}^{B_i}$ such that

$$\operatorname{Hom}_B(M, \tau_B Y_n) \neq 0 \quad \text{and} \quad \operatorname{Hom}_B(\tau_B^{-1} Y_n, R_i) \neq 0,$$

for all $n \geq 0$. But then $\operatorname{Hom}_A(I, \tau_A Y_n) \neq 0$ and $\operatorname{Hom}_A(\tau_A^{-1} Y_n, P^{(i)}) \neq 0$, for every $n \geq 0$, hence, by [1, Lemma IV.2.7], there are infinitely many pairwise non-isomorphic indecomposable modules Y_n in $\mathcal{C}$ with $\mathrm{pd}_A Y_n \geq 2$ and $\mathrm{id}_A Y_n \geq 2$, a contradiction. Thus indeed, the family $\mathcal{T}^A$ does not admit a coray tube which is not a stable tube.

(3) Now, we show that $\mathcal{T}^A = \mathcal{T}^B$. First, observe that, by [42, Proposition 2.3], for any non-regular tube $\mathcal{T}^B_\lambda$, all rays of $\mathcal{T}^B_\lambda$ are complete rays of $\mathcal{T}^A_\lambda$. Since all tubes of $\mathcal{T}^A$ are pairwise orthogonal and generalized standard components of Γ_A, there is a factor algebra $A' = A/\mathrm{ann}(\mathcal{T}^A)$ of A such that A' is tubular extension of B and $\mathcal{T}^A$ is obtained from $\mathcal{T}^B$ by a finite (possibly zero) number of ray insertions.

Suppose now that $\mathcal{T}^A \neq \mathcal{T}^B$. Then B is also a convex subcategory of A' and there is a decomposition $A' = P \oplus Q$ such that P is a direct sum of projective B-modules in $\mathrm{mod}\, A'$. It follows that $P = B$, $\mathrm{Hom}_{A'}(Q, B) = 0$, and there is an isomorphism of K-algebras

$$A' \cong \begin{bmatrix} F & U \\ 0 & B \end{bmatrix},$$

where $F = \mathrm{End}_{A'}(Q)$ and U is a non-zero F-B-bimodule with U_B in $\mathrm{add}(\mathcal{T}^B)$. In particular, there is a module X in $\mathcal{D}_l$ such that $\mathrm{Hom}_B(U, X) \neq 0$ and there is an almost split sequence in $\mathrm{mod}\, B$ of the form

$$0 \to X \to Y \to Z \to 0,$$

which is also an almost split sequence in $\mathrm{mod}\, A$. Since X, Y and Z are A'-modules, the above sequence is an almost split sequence in $\mathrm{mod}\, A'$. But this is impossible, because $\mathrm{Hom}_B(U, X) \neq 0$ (see [32, (2.5)], [36, Theorem XV.1.6] and [49, Lemma 5.6]). Consequently, $\mathcal{T}^A = \mathcal{T}^B$, that is, the family $\mathcal{T}^B$ of all semiregular tubes of Γ_B is a family of components of Γ_A.

Summarizing, we have proved in (1)–(3) that there is a factor tilted algebra $B = B(\mathcal{C})$ of A such that $B = B_1 \times \cdots \times B_p$ is a product of indecomposable tilted algebras of Euclidean type, the torsion-free part $\mathcal{Q}^{B_i} \cap \mathcal{Y}(T)$ of $\mathcal{Q}^{B_i}$ is a full valued translation subquiver of $\mathcal{C}_l$ for every $i \in \{1, \ldots, p\}$, and the family $\mathcal{T}^{B(\mathcal{C})}$ of all semiregular tubes of Γ_B is a family of components of Γ_A. Using dual arguments, we infer that, if $\mathcal{C}_r$ is infinite, then there is a factor algebra $B' = B'(\mathcal{C}) = B'_1 \times \cdots \times B'_q$ of A such that, for every $j \in \{1, \ldots, q\}$, there is a hereditary algebra H'_j of Euclidean type and a tilting module T'_j without postprojective direct summands such that we have an isomorphism of K-algebras $B'_j \cong \mathrm{End}_{H'_j}(T'_j)$, and the torsion part $\mathcal{P}^{B'_j} \cap \mathcal{X}(T'_j)$ of $\mathcal{P}^{B'_j}$ is a full valued translation subquiver of $\mathcal{C}_r$ for every $j \in \{1, \ldots, q\}$. Moreover, the family $\mathcal{T}^{B'(\mathcal{C})} = (\mathcal{T}^{B'}_\lambda)_{\lambda \in \Lambda'} = (\mathcal{T}^A_\lambda)_{\lambda \in \Lambda'}$ of all coray tubes of $\Gamma_{B'}$ is a family of components of Γ_A.

(4) In the last part of the proof we show that A is in fact a generalized double tilted algebra. Denote by $\mathcal{P}^{B(\mathcal{C})}$ (respectively, $\mathcal{Q}^{B'(\mathcal{C})}$) the family

$\mathcal{P}^{B_1} \cup \cdots \cup \mathcal{P}^{B_p}$ of all postprojective components of $\Gamma_{B(\mathcal{C})}$ (respectively, the family $\mathcal{Q}^{B'_1} \cup \cdots \cup \mathcal{Q}^{B'_q}$ of all preinjective components of $\Gamma_{B'(\mathcal{C})}$).

Consider the two-sided ideal $I = \operatorname{ann}(\mathcal{T}^{B(\mathcal{C})} \cup \mathcal{C}) = \operatorname{ann}(\mathcal{T}^{B(\mathcal{C})} \cup \mathcal{C} \cup \mathcal{T}^{B'(\mathcal{C})}) = \operatorname{ann}(\mathcal{C} \cup \mathcal{T}^{B'(\mathcal{C})})$ of A and the quotient algebra $A(\mathcal{C}) = A/I$. Then $\mathcal{T}^{B(\mathcal{C})} \cup \mathcal{C} \cup \mathcal{T}^{B'(\mathcal{C})}$ is a faithful family of components of $\Gamma_{A(\mathcal{C})}$. Moreover, all projective modules in $\operatorname{mod} A(\mathcal{C})$ are contained in $\mathcal{P}^{B(\mathcal{C})} \cup \mathcal{T}^{B(\mathcal{C})} \cup \mathcal{C}$. Dually, each injective module in $\operatorname{mod} A(\mathcal{C})$ belongs to $\mathcal{C} \cup \mathcal{T}^{B'(\mathcal{C})} \cup \mathcal{Q}^{B'(\mathcal{C})}$. In particular, all projective (respectively, injective) $A(\mathcal{C})$-modules are projective (respectively, injective) A-modules, and hence the valued quiver $Q_{A(\mathcal{C})}$ can be treated as a (full) valued subquiver of Q_A.

We claim that $Q_{A(\mathcal{C})} = Q_A$. Suppose otherwise. Then, as Q_A is connected, there are a vertex i_0 in Q_A not lying in $Q_{A(\mathcal{C})}$ and a vertex j_0 in $Q_{A(\mathcal{C})}$ such that there is either an arrow $i_0 \to j_0$ in Q_A or an arrow $j_0 \to i_0$ in Q_A. Suppose the latter. Then there is a homomorphism $f_0 : P_{i_0} \to P_{j_0}$ in $\operatorname{mod} A$, where $P_{j_0} = e_{j_0}A$ and $P_{i_0} = e_{i_0}A$ are indecomposable projective A-modules corresponding to the vertices j_0 and i_0, respectively, and f_0 is given by an element $a_0 \in e_{j_0}(\operatorname{rad} A)e_{i_0} \setminus e_{j_0}(\operatorname{rad} A)^2 e_{i_0}$. Since P_{j_0} and P_{i_0} are non-isomorphic indecomposable projective modules, f_0 is not an epimorphism, and hence $\operatorname{Im} f_0$ is a submodule of $\operatorname{rad} P_{j_0}$. Then the projectivity of P_{i_0} implies that in $\operatorname{mod} A$ there exists a commutative diagram

$$\begin{array}{ccc} & & P_{i_0} \\ & \overset{g_0}{\swarrow} & \downarrow{\scriptstyle \bar{f}_0} \\ P(\operatorname{rad} P_{j_0}) & \xrightarrow{\ \pi\ } & \operatorname{rad} P_{j_0} \end{array}$$

with $P(\operatorname{rad} P_{j_0})$ a projective cover of $\operatorname{rad} P_{j_0}$ in $\operatorname{mod} A(\mathcal{C})$ and $f_0 = u\bar{f}_0$, where $u : \operatorname{rad} P_{j_0} \to P_{j_0}$ is the canonical inclusion. Observe that, if P_i is a direct summand of $P(\operatorname{rad} P_{j_0})$, then i is in $Q_{A(\mathcal{C})}$ and so g_0 is a homomorphism in rad_A. Moreover, the homomorphism $h_0 = u\pi$ is in $\operatorname{rad}_A(P(\operatorname{rad} P_{j_0}), P_{j_0})$. But then $f_0 = h_0 g_0$ implies that a_0 is in $e_{j_0}(\operatorname{rad} A)^2 e_{i_0}$, a contradiction. Summing up, we have proved that, if there is an arrow $j \to i$ in Q_A with j in $Q_{A(\mathcal{C})}$, then i also belongs to $Q_{A(\mathcal{C})}$. Similarly, using injective modules, we prove that, if there is an arrow $i \to j$ in Q_A with j in $Q_{A(\mathcal{C})}$, then i also belongs to $Q_{A(\mathcal{C})}$. Consequently, we get the required equality $Q_{A(\mathcal{C})} = Q_A$.

Hence, all indecomposable projective (respectively, injective) modules in $\operatorname{mod} A$ are in fact indecomposable projective (respectively, injective) modules in $\operatorname{mod} A(\mathcal{C})$, and so are contained in $\mathcal{P}^{B(\mathcal{C})} \cup \mathcal{T}^{B(\mathcal{C})} \cup \mathcal{C} \cup \mathcal{T}^{B'(\mathcal{C})} \cup \mathcal{Q}^{B'(\mathcal{C})}$. Moreover, $\mathcal{C}$ is a faithful component of Γ_A, $\Gamma_A = \Gamma_{A(\mathcal{C})} = \mathcal{P}^{B(\mathcal{C})} \cup \mathcal{T}^{B(\mathcal{C})} \cup \mathcal{C} \cup \mathcal{T}^{B'(\mathcal{C})} \cup \mathcal{Q}^{B'(\mathcal{C})}$, and $A = A(\mathcal{C})$. In particular, we may consider the decomposition $A = P \oplus P'$ of A_A into a direct sum of projective A-modules, where P (respectively, P') is the direct sum of all indecomposable projective modules in $\operatorname{mod} A$ lying in $\mathcal{P}^{B(\mathcal{C})} \cup \mathcal{T}^{B(\mathcal{C})}$ (respectively, in $\mathcal{C}$). Further, observe that $\operatorname{Hom}_A(P', P) = 0$. Therefore, $A = A(\mathcal{C})$ is isomorphic to a K-algebra of triangular matrix form

$$\begin{bmatrix} D & V \\ 0 & B \end{bmatrix},$$

where $D = \operatorname{End}_A(P')$ and $V = \operatorname{Hom}_A(P, P')$ is a D-B-bimodule with V_B lying in $\mathcal{C}$. Moreover, $\operatorname{mod} A$ can be identified with the category whose objects are triples $Y = (Y_0, Y_1, \varphi)$, where $Y_1 \in \operatorname{mod} B$, $Y_0 \in \operatorname{mod} D$, and $\varphi : Y_0 \to \operatorname{Hom}_B(V, Y_1)$ is a D-homomorphism, and a morphism $h : (Y_0, Y_1, \varphi) \to (X_0, X_1, \psi)$ of such triples is a pair (h_0, h_1) such that $h_0 \in \operatorname{Hom}_D(Y_0, X_0)$, $h_1 \in \operatorname{Hom}_B(Y_1, X_1)$, and $\operatorname{Hom}_B(V, h_1)\varphi = \psi h_0$.

Consider now an arbitrary module X from $\mathcal{D}_l$. Then X is a module in $\operatorname{mod} B$, hence $X \cong (X_0, X_1, \psi)$, where $X_0 = 0$ and $\psi = 0$. Now, let $Y \cong (Y_0, Y_1, \varphi)$ be a predecessor of X in $\operatorname{ind} A$. Then there is a pair (h_0, h_1) of homomorphisms such that $h_0 \in \operatorname{Hom}_D(Y_0, X_0)$, $h_1 \in \operatorname{Hom}_B(Y_1, X_1)$, and $\operatorname{Hom}_B(V, h_1)\varphi = \psi h_0$. In particular, $h_0 = 0$.

We claim that Y is in $\operatorname{ind} B$. First, observe that $\varphi = 0$. Indeed, if this is not the case, then applying Lemma 2.1, we conclude that, for every indecomposable direct summand Z of Y_1, we have $\operatorname{Hom}_B(V, Z) \neq 0$. Consequently, Z is a successor in $\operatorname{ind} B$ of an indecomposable direct summand of $\operatorname{rad} P'$, a contradiction, because Z is a predecessor of $X \cong X_1$ lying in $\mathcal{D}_l$. Therefore, indeed $\varphi = 0$.

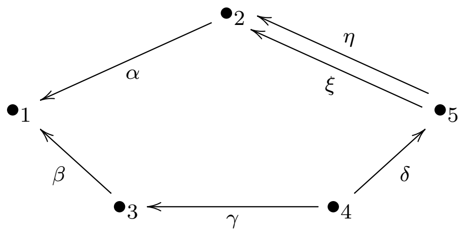
It follows that there is an isomorphism $Y \cong Y_0 \oplus Y_1$ in $\operatorname{mod} A$, hence because Y is in $\operatorname{ind} A$, we conclude that $Y \cong Y_0$ or $Y \cong Y_1$. In the first case, $Y_1 = 0$, hence $h_1 = 0$, and $h = 0$, a contradiction. Thus $Y \cong Y_1$, and we are done. Summing up, we have proved that every predecessor in $\operatorname{ind} A$ of a module X from $\mathcal{D}_l$ is a predecessor of X in $\operatorname{ind} B$.

Finally, we will prove that $\mathcal{C}$ is a generalized standard component. Suppose to the contrary that $\operatorname{rad}_A^\infty(M, N) \neq 0$ for some indecomposable modules M and N in $\mathcal{C}$. Then, applying [40, Lemma 2.1], we deduce that there is an infinite path $\cdots \to N_1 \to N_0 = N$ in $\mathcal{C}$ such that $\operatorname{rad}_A^\infty(M, N_k) \neq 0$ for all $k \geq 0$. Since $\mathcal{C}$ is almost acyclic, there exists an integer $k_0 \geq 0$ such that N_{k_0} lies in $\mathcal{D}_l$, and consequently M is also in $\mathcal{D}_l$. But then we obtain a contradiction, because M and N_{k_0} are indecomposable modules lying in the preinjective component of Γ_B which is generalized standard.

Thus, we have proved that $\mathcal{C}$ is an almost acyclic, faithful, and generalized standard component of Γ_A, and hence A is a generalized double tilted algebra, by [31, Theorem 3.1]. The proof is now complete. ∎

7. An example. We give here an example illustrating the relevance of the homological assumption in Theorem 6.4.

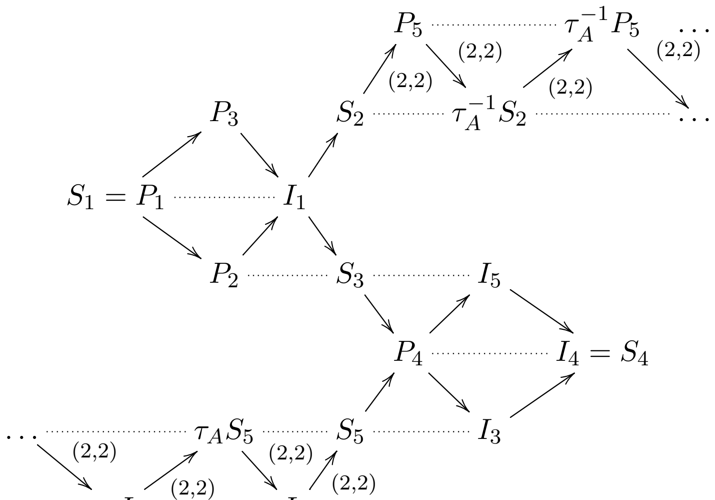
EXAMPLE 7.1. Let $A = KQ/I$, where K is an algebraically closed field, Q is a quiver of the form



and I is an admissible ideal in the path algebra KQ, generated by all paths in Q of length 2. Then $\dim_K A = 11$, and the equivalence of categories $\operatorname{mod} A \cong \operatorname{rep}_K(Q, I)$ (see [1, Theorem III.1.6]) yields the disjoint union decomposition

$$\Gamma_A = \mathcal{C} \cup \bigcup_{\lambda \in \mathbb{P}_1(K)} \mathcal{T}_\lambda^H,$$

where H is the path algebra $K\Sigma$ of the Kronecker subquiver Σ of Q given by the vertices 2, 5 and the arrows ξ, η, $\mathcal{T}^H = (\mathcal{T}_\lambda^H)_{\lambda \in \mathbb{P}_1(K)}$ is the family of all stable tubes of rank 1 in Γ_H (see [35, Section XI.4]), and $\mathcal{C}$ is a component of Γ_A of the shape



where S_i, P_i, I_i denote the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ corresponding to the vertex $i \in \{1, 2, 3, 4, 5\}$ of Q. Observe that A is a cycle-finite algebra, because $\mathcal{T}^H$ is a family of pairwise orthogonal stable tubes and every module in $\mathcal{C}$ has finitely many predecessors or finitely many successors in $\mathcal{C}$, equivalently, in $\operatorname{ind} A$. Moreover, the postprojective component

$\mathcal{P}^H$ of Γ_H is a full valued translation subquiver of $\mathcal{C}$ closed under successors. Dually, the preinjective component $\mathcal{Q}^H$ of Γ_H is a full valued translation subquiver of $\mathcal{C}$ closed under predecessors. Note also that in this case $B(\mathcal{C}) = B'(\mathcal{C}) = H$.

Moreover, $\mathcal{C}$ is an acyclic component of Γ_A containing all indecomposable projective A-modules, hence $\mathcal{C}$ is a faithful and (almost) acyclic component of Γ_A. But $\mathcal{C}$ is not a generalized standard component of Γ_A. Indeed, as $\operatorname{rad}_A(P_5, S_5) = \operatorname{Hom}_A(P_5, S_5) \neq 0$ and there is no path in $\mathcal{C}$ from P_5 to S_5, we infer that $\operatorname{rad}_A^{\infty}(P_5, S_5) \neq 0$.

Summing up, A is a cycle-finite algebra and Γ_A admits a component $\mathcal{C}$ which is faithful and almost acyclic, but not generalized standard. Moreover, the homological condition, imposed in Theorem 6.4, is not satisfied here. We claim that, for every successor Z of $\tau_A^{-1} S_2$ in $\mathcal{C}$, we have $\operatorname{pd}_A Z \geq 2$ and $\operatorname{id}_A Z \geq 2$. Namely, consider a successor Z of the module $\tau_A^{-1} S_2$ in $\mathcal{C}$. Then Z is a module in $\mathcal{P}^H$ with $\tau_A Z = \tau_H Z$ and $\tau_A^{-1} Z = \tau_H^{-1} Z \neq S_2$ in $\mathcal{P}^H$, and hence $\operatorname{Hom}_A(S_2, \tau_A Z) \neq 0$ and $\operatorname{Hom}_A(\tau_A^{-1} Z, S_5) \neq 0$. Further, there are an epimorphism $I_1 \to S_2$ and a monomorphism $S_5 \to P_4$, thus $\operatorname{Hom}_A(I_1, \tau_A Z) \neq 0$ and $\operatorname{Hom}_A(\tau_A^{-1} Z, P_4) \neq 0$. Consequently, [1, Lemma IV.2.7] implies that $\operatorname{pd}_A Z \geq 2$ and $\operatorname{id}_A Z \geq 2$.

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