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ON DOMESTIC ALGEBRAS OF SEMIREGULAR TYPE

BY

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Dedicated to Otto Kerner on the occasion of his 70th birthday

Abstract. We describe the structure of finite-dimensional algebras of domestic representation type over an algebraically closed field whose Auslander–Reiten quiver consists of generalized standard and semiregular components. Moreover, we prove that this class of algebras contains all special biserial algebras whose Auslander–Reiten quiver consists of semiregular components.

1. Introduction and the main results. Throughout the paper, by an *algebra* we mean a basic, indecomposable finite-dimensional K-algebra over an algebraically closed field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules. It follows from general theory that every algebra A is isomorphic to a bound quiver algebra KQ/I, where $Q = Q_A$ is a finite connected quiver, called the *Gabriel quiver* of A, and I is an admissible ideal in the path algebra KQ of Q over K [3, Chapter II]. Moreover, for $A = KQ/I$, the category $\operatorname{mod} A$ is equivalent to the category $\operatorname{rep}_K(Q, I)$ of finite-dimensional representations of Q over K bound by I [3, Chapter III].

The Jacobson radical rad_A of $\operatorname{mod} A$ is the ideal generated by all non-invertible homomorphisms between modules in $\operatorname{ind} A$, and the infinite radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. By a result of Auslander [5], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, $\operatorname{ind} A$ admits a finite number of pairwise non-isomorphic modules (see also [28] for an alternative proof of this result). On the other hand, if A is of infinite representation type then $(\operatorname{rad}_A^\infty)^2 \neq 0$, by a result proved in [10].

From the remarkable Tame and Wild Theorem of Drozd [16] (see also [13]) the class of finite-dimensional algebras over an algebraically closed field K may be divided into two disjoint classes. The first class, called the

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tame algebras, consists of algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite-dimensional algebras over K. Accordingly, we may realistically hope to classify the indecomposable finite-dimensional modules only for tame algebras. More precisely, a finite-dimensional K-algebra A over an algebraically closed field K is called *tame* if, for any dimension d, there exist a finite number of $K[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, which are free of finite rank as left modules over the polynomial algebra $K[x]$ in one variable, and all but finitely many isomorphism classes of indecomposable right A-modules of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Moreover, let $\mu_A(d)$ be the least number of $K[x]$-A-bimodules satisfying the above condition for d. Then A is said to be *domestic* if there exists a positive integer m such that $\mu_A(d) \leq m$ for any $d \geq 1$ (see [14], [35]). Examples of domestic algebras are tilted algebras of Euclidean type. On the other hand, tubular algebras (in the sense of [36, Section 5]) are tame non-domestic algebras (see [40, Lemma 3.6]).

An important combinatorial and homological invariant of an algebra A is its *Auslander–Reiten quiver* Γ_A whose vertices are the isomorphism classes of modules in $\operatorname{ind} A$, the arrows correspond to the irreducible maps between modules in $\operatorname{ind} A$, and we have the Auslander–Reiten translations $\tau_A = \mathrm{D}\,\mathrm{Tr}$ and $\tau_A^{-1} = \mathrm{Tr}\,\mathrm{D}$ related to almost split sequences in $\operatorname{mod} A$ (see [3, Chapter IV] for details). We do not distinguish between a module in $\operatorname{ind} A$ and the corresponding vertex of Γ_A.

By a *component* of Γ_A we mean a connected component of the translation quiver Γ_A. Following [41], a component $\mathcal{C}$ of Γ_A is said to be *generalized standard* if $\operatorname{rad}_A^{\infty}(X,Y) = 0$ for all modules X and Y in $\mathcal{C}$. A component $\mathcal{C}$ of Γ_A is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module, and *semiregular* if $\mathcal{C}$ does not contain both a projective and an injective module. The shapes of regular and semiregular components of Γ_A have been described by Liu in [30], [31] and Zhang (regular components) in [49].

An algebra A is said to be of *semiregular type* if all components in Γ_A are semiregular. This class of algebras consists of algebras of infinite representation type and contains the following classes of algebras: hereditary algebras of infinite representation type [34], [36], [38], [39], tilted algebras with semiregular connecting components [3], [36], [39], tubular algebras [36], canonical algebras [36], and quasitilted algebras of canonical type [12], [29], [45]. It would be interesting to find a description of all algebras of semiregular type. We refer the reader to the recent article [8] providing a complete description of all algebras of semiregular type for which the cycles of indecomposable finite-dimensional modules are finite.

A fundamental role in the current representation theory is played by quasitilted algebras, introduced by Happel, Reiten and Smalø in [26]. It is the class of algebras of the form $A = \mathrm{End}_{\mathcal{H}}(T)$, where T is a tilting object in a hereditary abelian K-category $\mathcal{H}$. It was shown in [26] that an algebra A is quasitilted if and only if A is of global dimension at most two and each module in $\operatorname{ind} A$ has projective dimension at most one or injective dimension at most one. Besides tilted algebras, which are the endomorphism algebras of tilting modules over hereditary algebras, an important class of quasitilted algebras is formed by the quasitilted algebras of canonical type. Following [29], an algebra A is called *quasitilted of canonical type* provided $A = \mathrm{End}_{\mathcal{H}}(T)$ for a tilting object in a hereditary K-category $\mathcal{H}$ whose derived category $\mathrm{D}^b(\mathcal{H})$ (of bounded complexes over $\mathcal{H}$) is equivalent, as a triangulated category, to the derived category $\mathrm{D}^b(\operatorname{mod} \Lambda)$ of the module category of a canonical algebra Λ (introduced in [36]). Then $\mathrm{D}^b(\operatorname{mod} A)$ and $\mathrm{D}^b(\operatorname{mod} \Lambda)$ are also derived equivalent (see [29, Section 3]). It was shown in [29] that an algebra A is quasitilted of canonical type if and only if A is a semiregular branch enlargement of a concealed canonical algebra. Further, it was shown in [45, Theorem A] that A is a tame quasitilted algebra if and only if A is tame tilted or a tame semiregular branch enlargement of a tame concealed algebra. In particular, this implies that every tame quasitilted algebra is a tilted algebra or a quasitilted algebra of canonical type. Finally, Happel [25] proved that this is the case for an arbitrary quasitilted algebra.

Let C be a *tame concealed algebra*, that is, an algebra of the form $\mathrm{End}_H(T)$, where T is a tilting module from the additive category of the postprojective component of a hereditary algebra H of Euclidean type. Then Γ_C consists of a postprojective component $\mathcal{P}^C$, a preinjective component $\mathcal{Q}^C$, and a family $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes separating $\mathcal{P}^C$ from $\mathcal{Q}^C$ (see [36], [38]). By a *semiregular branch enlargement* of C we mean an algebra of the form

$$B = \begin{bmatrix} D & M & 0 \\ 0 & C & D(N) \\ 0 & 0 & E \end{bmatrix},$$

where

$$B^{(r)} = \begin{bmatrix} D & M \\ 0 & C \end{bmatrix} \quad \text{and} \quad B^{(l)} = \begin{bmatrix} C & D(N) \\ 0 & E \end{bmatrix},$$

with $D(N) = \mathrm{Hom}_K(N, K)$, are respectively a tubular extension of C and a tubular coextension of C in the sense of [36, (4.7)] (see also [39, Chapter XV]), and no tube in $\mathcal{T}^C$ admits both a direct summand of M and a direct summand of N (see [29], [45]). Then B is a quasitilted algebra of canonical type, and $B^{(r)}$ and $B^{(l)}$ are called the *right part* and the *left part* of B, respectively. Moreover, following [45], B is said to be a *tame semiregular branch enlargement* of C if $B^{(r)}$ and $B^{(l)}$ are tame algebras, that is, tilted algebras of Euclidean type or tubular algebras. Finally, B is said to be a *domestic semiregular branch enlargement* of C if $B^{(r)}$ and $B^{(l)}$ are domestic algebras, equivalently, tilted algebras of Euclidean type.

The following theorem is the first main result of the paper.

THEOREM A. *Let A be an algebra. The following statements are equivalent:*

(i) *A is a domestic quasitilted algebra of canonical type.*
(ii) *A is a domestic semiregular branch enlargement of a tame concealed algebra.*
(iii) *A is an algebra of semiregular type and* $(\mathrm{rad}_A^\infty)^3 = 0$.
(iv) *A is an algebra of semiregular type, all components in Γ_A are generalized standard, and all but finitely many of them are stable tubes of rank one.*

A prominent role in the representation theory of tame algebras is played by special biserial algebras introduced in [47]. Recall that an algebra A is called *special biserial* if A is isomorphic to a bound quiver algebra KQ/I, where the bound quiver (Q, I) satisfies the following conditions:

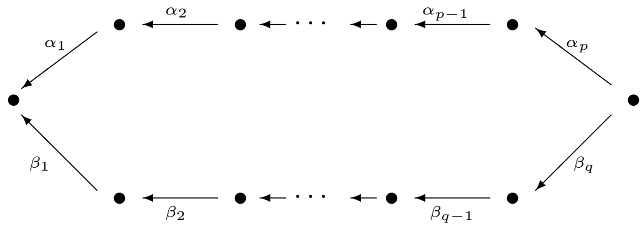
(SB1) The number of arrows in Q with a prescribed source or target is at most two.
(SB2) For any arrow α in Q, there is at most one arrow β and one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I.

Moreover, if the ideal I is generated only by paths of Q, then A is said to be a *string algebra* [9]. Further, following [4], a special biserial algebra $A = KQ/I$ is said to be a *gentle* algebra if the following conditions are satisfied:

(G1) The ideal I is generated by a set of paths in Q of length 2.
(G2) For any arrow α in Q, there is at most one arrow ξ and one arrow η such that $\alpha\xi$ and $\eta\alpha$ belong to I.

Important classes of special biserial algebras are provided by iterated tilted algebras of Dynkin type $\mathbb{A}_n$ [2], iterated tilted algebras of Euclidean type $\widetilde{\mathbb{A}}_n$ [4], biserial algebras of finite representation type [47], algebras whose indecomposable modules all have multiplicity-free composition factors [32], blocks of group algebras with cyclic and dihedral defect groups (see [1], [17]), selfinjective algebras of Dynkin type $\mathbb{A}_n$ and Euclidean type $\widetilde{\mathbb{A}}_n$ (see [46]), biserial selfinjective algebras having simply connected Galois coverings [33], as well as algebras appearing in the Gelfand–Ponomarev classification of singular Harish-Chandra modules over the Lorentz group [22], algebras appearing in the classification of restricted Lie algebras, or more generally infinitesimal group schemes, with tame principal block in odd characteristic (see [19], [20]). We also mention that the special biserial algebras form a distinguished class of tame algebras whose representation theory is rather well understood (see [9], [15], [18], [21], [33], [37], [47], [48]).

For positive integers p and q, we denote by $\widetilde{\mathbb{A}}_{p,q}$ the quiver



and call its path algebra $K\widetilde{\mathbb{A}}_{p,q}$ the *canonical algebra of type* $\widetilde{\mathbb{A}}_{p,q}$. Further, by a *gentle semiregular branch enlargement* of a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ we mean a gentle algebra which is a semiregular branch enlargement of a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_m$. Recall also that, for an algebra A, the invariant $\alpha(A)$ is the largest possible number of indecomposable summands in the middle of any almost split sequence in mod A. It is known that $\alpha(A) \leq 2$ for any string algebra (see [9], [47]).

The following theorem is the second main result of the paper.

THEOREM B. *Let A be an algebra. The following statements are equivalent:*

(i) *A is a special biserial algebra of semiregular type.*
(ii) *A is a quasitilted algebra of canonical type $\widetilde{\mathbb{A}}_{p,q}$ for some positive integers p and q.*
(iii) *A is a gentle semiregular branch enlargement of a hereditary algebra of type $\widetilde{\mathbb{A}}_m$ for some positive integer m.*
(iv) *A satisfies the following conditions:*
  (a) $\alpha(A) \leq 2$.
  (b) *Every component of Γ_A is generalized standard and semiregular.*
  (c) *All but at most four components of Γ_A are stable tubes of rank one.*

The paper is organized as follows. In Section 2 we give a sufficient condition for a gentle algebra to have a non-semiregular Auslander–Reiten component. Section 3 is devoted to presenting some facts on algebras of semiregular type for which all Auslander–Reiten components are generalized standard. These results are used in Section 4, where we give the proof of Theorem A. In Section 5 we consider special biserial algebras of semiregular type and give the proof of Theorem B. The aim of the final Section 6 is to present examples illustrating the necessity of some assumptions of Theorems A and B.

2. Non-semiregular Auslander–Reiten components of gentle algebras. The aim of this section is to give a construction of non-semiregular Auslander–Reiten components of gentle algebras.

Let us recall some basic notions connected with a bound quiver (Q, I) of an algebra $A = KQ/I$. For a given arrow α in (Q, I), by $s(\alpha)$ we denote its source and by $t(\alpha)$ its end. Moreover, by α^{-1} we denote the formal inverse of α and set $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. With each vertex x in (Q, I) we associate a *trivial walk* (*walk of length* 0) denoted by ε_x. By a *non-trivial walk* in (Q, I) we mean a sequence $\omega = \alpha_1 \ldots \alpha_n$, where α_i is an arrow or the inverse of an arrow in (Q, I) and $t(\alpha_i) = s(\alpha_{i+1})$ for any $i \in \{1, \ldots, n-1\}$. In this case we say that ω has *length* n, and we put $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_n)$. A walk $\alpha_1 \ldots \alpha_n$ is called a *path* if all α_i are arrows.

Assume now that $A = KQ/I$ is a string algebra. A *non-zero walk* ω in (Q, I) is a walk which does not contain any subpath v such that v or v^{-1} belongs to I. We note that trivial walks are non-zero walks. Further, by a *zero-path* we mean every element of I, and by a *zero-relation* an element of I for which no proper subpath belongs to I. It is well known that the class of indecomposable modules over a string algebra A divides into two classes: string modules and band modules (see [9], [48]). We study the string modules in $\operatorname{ind} A$, which are modules uniquely determined by non-zero walks in (Q, I). For a non-zero walk ω in (Q, I), we denote by $X(\omega)$ the string module in $\operatorname{ind} A$ attached to it. We refer the reader to [9] and [48] for details.

Following Butler and Ringel [9], we say that a non-zero walk ω in (Q, I) *starts in a deep* or *starts on a peak* if there are no arrows γ such that $\gamma^{-1}\omega$ or $\gamma\omega$ is a non-zero walk, respectively. A non-zero walk ω is said to *end in a deep* or *end on a peak* if there are no arrows δ such that $\omega\delta$ or $\omega\delta^{-1}$ is a non-zero walk, respectively. Moreover, in [9, Section 3], it is shown that for a string algebra $A = KQ/I$ the Auslander–Reiten sequences in $\operatorname{mod} A$ with one middle term have the following form:

$$0 \to X(\delta_1 \ldots \delta_m) \to X(\delta_1 \ldots \delta_m \beta^{-1} \gamma_1 \ldots \gamma_n) \to X(\gamma_1 \ldots \gamma_n) \to 0$$

for a non-zero walk $\delta_1 \ldots \delta_m \beta^{-1} \gamma_1 \ldots \gamma_n$ in (Q, I) which ends in a deep and starts on a peak, where δ_k, γ_l and β are arrows. Observe that, using the above notation, the indecomposable projective module $P(x)$ in $\operatorname{mod} A$ at the vertex x can be described as the string module $X(w^{-1}u)$ for some paths w, u such that $s(w) = s(u) = x$ and $w^{-1}u$ starts and ends in a deep. Dually, the indecomposable injective module $I(x)$ in $\operatorname{mod} A$ at the vertex x is the string module of the form $X(wu^{-1})$, where w, u are paths such that $t(w) = t(u) = x$ and wu^{-1} starts and ends on a peak. The simple module in $\operatorname{mod} A$ at the vertex x will be denoted by $S(x)$.

In [27] Huard and Liu gave a characterization of quasitilted string algebras in terms of non-existence of some special walks in (Q, I). Namely, a walk ω is called a *sequential pair of zero-relations* in (Q, I) if ω contains exactly two zero-relations and they point in the same direction in ω. Clearly, ω is a sequential pair of zero-relations if and only if ω^{-1} is a sequential pair of zero-relations. The following equivalence holds [27, Theorem 2.6].

THEOREM 2.1. *Let $A = KQ/I$ be a string algebra. Then A is quasitilted if and only if there is no sequential pair of zero-relations in (Q, I).*

Let now $A = KQ/I$ be a gentle algebra. Assume ω is a sequential pair of zero-relations in (Q, I) of the form $z_1\omega' z_2$ for a non-zero walk ω' and walks z_1, z_2 of length 2 such that either $z_1, z_2 \in I$ or $z_1^{-1}, z_2^{-1} \in I$. Then we say that ω is a *minimal sequential pair of zero-relations* if, for any arrow γ in (Q, I) and a subwalk ω'' of ω', none of the walks $z_1\omega''\gamma$, $z_1\omega''\gamma^{-1}$, $\gamma\omega'' z_2$, $\gamma^{-1}\omega'' z_2$ forms a sequential pair of zero-relations.

Recall that, for an indecomposable non-projective module Z in a module category $\operatorname{mod} A$, the invariant $\alpha(Z)$ is the number m of indecomposable direct summands in the middle of an Auslander–Reiten sequence

$$0 \to X \to \bigoplus_{i=1}^{m} Y_i \to Z \to 0$$

with the right term Z (see [6], [9], [32], [47], [48] for some results involving this invariant).

The following proposition describes a sufficient condition for a gentle algebra to have a non-semiregular Auslander–Reiten component.

PROPOSITION 2.2. *Let $A = KQ/I$ be a gentle algebra and $\beta_1\beta_2\omega\alpha_1\alpha_2$ be a minimal sequential pair of zero-relations for zero-relations $\beta_1\beta_2$ and $\alpha_1\alpha_2$ in (Q, I). Let M be the indecomposable direct summand of the quotient module $I(t(\alpha_2))/\operatorname{soc} I(t(\alpha_2))$ of the indecomposable injective module $I(t(\alpha_2))$ such that $\operatorname{soc}(M) = S(s(\alpha_2))$, and N be the indecomposable direct summand of the radical $\operatorname{rad} P(s(\beta_1))$ of the indecomposable projective module $P(s(\beta_1))$ with $\operatorname{top}(N) = S(t(\beta_1))$. Then there exists a positive integer m such that $M = \tau_A^m N$ and $\alpha(\tau_A^i N) = 1$ for any $i \in \{0, 1, \dots, m-1\}$.*

Proof. We start with the case when the zero-relations $\alpha_1\alpha_2$ and $\beta_1\beta_2$ in $\beta_1\beta_2\omega\alpha_1\alpha_2$ overlap, that is, we have a path $\beta_1\beta_2\alpha_2$ where $\beta_2 = \alpha_1$ and $\beta_1\beta_2 \in I$ and $\beta_2\alpha_2 \in I$. Consider the indecomposable injective module $I(t(\alpha_2))$ at the vertex $t(\alpha_2)$. Then the quotient module $I(t(\alpha_2))/\operatorname{soc} I(t(\alpha_2))$ contains a unique indecomposable direct summand $M = X(w)$ such that w starts on a peak and $\operatorname{soc}(M) = S(s(\alpha_2))$. Hence, we have the following Auslander–Reiten sequence in $\operatorname{mod} A$:

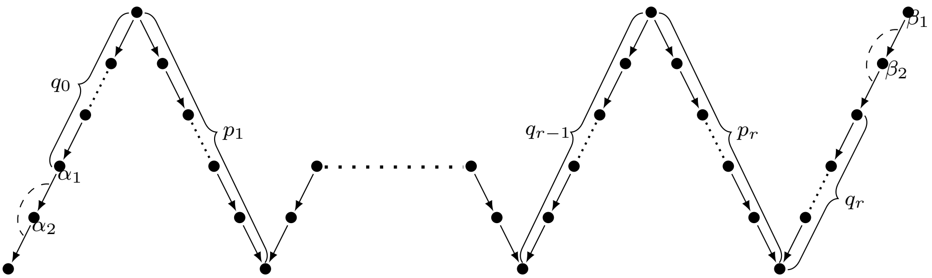
$$0 \to M \to X(w\beta_2^{-1}u) \to N \to 0$$

for $N = X(u)$, where the path u ends in a deep and starts at $s(\beta_2)$. Since $s(\beta_2) = t(\beta_1)$ and u ends in a deep, $X(u)$ is a direct summand of the radical

$\operatorname{rad} P(s(\beta_1))$ of the indecomposable projective module at the vertex $s(\beta_1)$. Clearly, $M = \tau_A N$ and $\alpha(\tau_A N) = 1$.

Assume now $\beta_1\beta_2\omega\alpha_1\alpha_2$ is a minimal sequential pair of zero-relations for some walk ω in (Q, I). For the purpose of this proof we shall consider the formal inverse of $\beta_1\beta_2\omega\alpha_1\alpha_2$, that is, the sequential pair of zero-relations of the form $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$. Observe that $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$ is also a minimal sequential pair of zero-relations in (Q, I).

Let $\omega^{-1} = q_0^{-1}p_1q_1^{-1}\dots p_rq_r^{-1}$ be such that p_i, q_j are paths (q_0, q_r possibly trivial). Then we may visualize $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$ as follows:



Note that each p_i ends in a deep. Indeed, if this is not the case, then there is an arrow γ in (Q, I) such that $p_i\gamma$ is a non-zero path. But then $q_i\gamma$ is a zero-path since $t(p_i) = t(q_i)$ for any $i \in \{1, \dots, r\}$ and A is gentle. Hence, we obtain a contradiction with the minimality of $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$. Analogously, we show that each p_i starts on a peak. Assume now that $q_i = c_{1,i}c_{2,i}\dots c_{n_i,i}$ for some arrows $c_{l,i}$ and $i \in \{0, 1, \dots, r\}$. Again, by the choice of $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$, for $l \in \{2, \dots, n_i - 1\}$, each $c_{l,i}^{-1}$ ends in a deep and starts on a peak. Hence, we have in $\operatorname{mod} A$ Auslander–Reiten sequences

$$0 \to S(t(c_{l,i})) \to X(c_{l,i}^{-1}) \to S(s(c_{l,i})) \to 0. \tag{1}$$

Moreover, using the same arguments for $n_0 > 1$, we find that

$$0 \to S(t(c_{n_0,0})) \to X(c_{n_0,0}^{-1}) \to S(s(c_{n_0,0})) \to 0 \tag{2}$$

is also an Auslander–Reiten sequence in $\operatorname{mod} A$. Analogously, for $n_r > 1$, there is in $\operatorname{mod} A$ an Auslander–Reiten sequence

$$0 \to S(t(c_{1,r})) \to X(c_{1,r}^{-1}) \to S(s(c_{1,r})) \to 0. \tag{3}$$

Further, for any $i \in \{1, \dots, r\}$,

$$0 \to X(p_i) \to X(p_ic_{n_i,i}^{-1}) \to S(s(c_{n_i,i})) \to 0 \tag{4}$$

is an Auslander–Reiten sequence in $\operatorname{mod} A$. Moreover, for $i \in \{0, \dots, r-1\}$, there are in $\operatorname{mod} A$ Auslander–Reiten sequences

$$0 \to S(t(c_{1,i})) \to X(c_{1,i}^{-1}p_{i+1}) \to X(p_{i+1}) \to 0. \tag{5}$$

Since $s(c_{n_i,i}) = t(c_{n_i-1,i})$ and $s(c_{2,i}) = t(c_{1,i})$, using (1)–(5), we conclude that there is a positive integer k such that $S(t(c_{n_0,0})) = \tau_A^k S(s(c_{1,r}))$ and $\alpha(\tau_A^i S(s(c_{1,r}))) = 1$ for $i \in \{0, \dots, k-1\}$. Hence, for all $i \in \{1, \dots, r\}$, the modules $X(p_i)$ are in the same τ_A-orbit in Γ_A.

Consider now the indecomposable injective module $I(t(\alpha_2))$ in $\operatorname{mod} A$ at the vertex $t(\alpha_2)$. Then the quotient module $I(t(\alpha_2))/\operatorname{soc} I(t(\alpha_2))$ has a unique indecomposable direct summand M with $\operatorname{soc}(M) = S(s(\alpha_2))$, and $M = X(w)$ for some path w which starts on a peak and ends in $t(\alpha_1) = s(\alpha_2)$. Then we have in $\operatorname{mod} A$ an Auslander–Reiten sequence

$$(6) \qquad 0 \to M \to X(w\alpha_1^{-1}) \to S(t(c_{n_0,0})) \to 0,$$

since $\tau_A^{-1} M \neq S(t(c_{n_0,0}))$ implies that $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$ is not minimal. Moreover, for the indecomposable projective module $P(s(\beta_1))$ in $\operatorname{mod} A$ at the vertex $s(\beta_1)$, its radical $\operatorname{rad} P(s(\beta_1))$ has a unique indecomposable direct summand N with $\operatorname{top}(N) = S(t(\beta_1))$, and $N = X(u)$ for some path u ending in a deep and starting at $t(\beta_1) = s(\beta_2)$. Again, by the choice of $\alpha_2^{-1}\alpha_1^{-1}\omega^{-1}\beta_2^{-1}\beta_1^{-1}$, β_2^{-1} starts on a peak, and since $S(s(c_{1,r})) = S(t(\beta_2))$, we have in $\operatorname{mod} A$ an Auslander–Reiten sequence

$$(7) \qquad 0 \to S(s(c_{1,r})) \to X(\beta_2^{-1}u) \to N \to 0.$$

Summing up, combining the Auslander–Reiten sequences (1)–(7) and taking $m = k + 2$, we have $M = \tau_A^m N$ and $\alpha(\tau_A^i N) = 1$ for $i \in \{0, 1, \dots, m-1\}$. ■

3. Algebras with generalized standard semiregular components. In this section we prove several results needed in the proof of Theorem A.

PROPOSITION 3.1. *Let A be an algebra such that Γ_A consists of generalized standard semiregular components. Then the following statements hold:*

- (i) *A is a tame algebra.*
- (ii) *Every component in Γ_A has one of the forms: a postprojective component of Euclidean type, a preinjective component of Euclidean type, a ray tube, or a coray tube.*

Proof. (i) Since every component in Γ_A is generalized standard, we have $\operatorname{rad}_A^\infty(X, X) = 0$ for any module X in $\operatorname{ind} A$. Then it follows from [44, Proposition 3.3] that A is a tame algebra.

The statement (ii) follows from (i) and [41, Corollary 3.10]. ■

PROPOSITION 3.2. *Let A be an algebra such that Γ_A consists of generalized standard semiregular components. Moreover, let B be a quotient algebra of A and $\mathcal{T}$ a stable tube of Γ_B. Then the following statements hold:*

- (i) *There is a semiregular tube $\Gamma(\mathcal{T})$ in Γ_A containing all modules of $\mathcal{T}$.*
- (ii) *If $\Gamma(\mathcal{T})$ is a ray tube, then all rays of $\mathcal{T}$ are rays of $\Gamma(\mathcal{T})$.*

(iii) *If $\Gamma(\mathcal{T})$ is a coray tube, then all corays of $\mathcal{T}$ are corays of $\Gamma(\mathcal{T})$.*
(iv) *If $\Gamma(\mathcal{T})$ is a stable tube, then $\Gamma(\mathcal{T}) = \mathcal{T}$.*

Proof. (i) Let r be the rank of $\mathcal{T}$ and $E_1, \ldots, E_r$ the family of modules lying on the mouth of $\mathcal{T}$ such that $\tau_B E_{i+1} = E_i$ for any $i \in \{1, \ldots, r\}$, where we set $E_{r+1} = E_1$. Then we have in $\mathcal{T}$ rays

$$E_i = E_i[1] \to E_i[2] \to \cdots$$

and corays

$$\cdots \to [2]E_i \to [1]E_i = E_i$$

with $E_i[j] = [j]E_{i+j-1}$ for $i \in \{1, \ldots, r\}$, $j \in \mathbb{N}^+ = \{1, 2, \ldots\}$. In particular, we have in $\operatorname{mod} B$ irreducible monomorphisms $f_{i,j} : E_i[j] \to E_i[j+1]$ and irreducible epimorphisms $g_{i,j} : [j+1]E_i \to [j]E_i$ for all $i \in \{1, \ldots, r\}$ and $j \in \mathbb{N}^+$.

Let m be a non-negative integer and $i \in \{1, \ldots, r\}$. Observe that $E_i[mr+1] = [mr+1]E_i$ (see [38, Lemma X.1.4]). Consider the irreducible monomorphisms $f_{i,j} : E_i[j] \to E_i[j+1]$ and the irreducible epimorphisms $g_{i,j} : [j+1]E_i \to [j]E_i$ for all $j \in \{1, \ldots, mr\}$. Then we have the composed monomorphism

$$f_i^{(m)} = f_{i,mr} \ldots f_{i,1} : E_i[1] \to E_i[mr+1]$$

and the composed epimorphism

$$g_i^{(m)} = g_{i,1} \ldots g_{i,mr} : [mr+1]E_i \to [1]E_i,$$

and so a non-zero homomorphism $h_i^{(m)} = f_i^{(m)} g_i^{(m)} \in \operatorname{rad} \operatorname{End}_B(E_i[mr+1])$, because $E_i[mr+1] = [mr+1]E_i$ and $E_i[1] = [1]E_i$. Since $\operatorname{End}_A(E_i[mr+1]) = \operatorname{End}_B(E_i[mr+1])$, we conclude that $h_i^{(m)}$ belongs to $\operatorname{rad}_A(E_i[mr+1], E_i[mr+1]) = \operatorname{rad} \operatorname{End}_A(E_i[mr+1])$. On the other hand, since every component of Γ_A is generalized standard, we have $\operatorname{rad}_A^\infty(E_i[mr+1], E_i[mr+1]) = 0$. Hence $h_i^{(m)} \notin \operatorname{rad}_A^\infty(E_i[mr+1], E_i[mr+1])$. But then we conclude that $f_{i,j} \notin \operatorname{rad}_A^\infty(E_i[j], E_i[j+1])$ and $g_{i,j} \notin \operatorname{rad}_A^\infty([j+1]E_i, [j]E_i)$ for any $j \in \{1, \ldots, mr\}$. This shows that all the modules $E_i[k]$ and $[k]E_i$, for $k \in \{1, \ldots, mr, mr+1\}$, belong to the same component of Γ_A as the module $E_i[1] = [1]E_i$.

Observe also that, for any $i \in \{1, \ldots, r\}$, we have $E_i[2] = [2]E_{i+1}$, because $\tau_B E_{i+1} = E_i$. Hence all the modules $E_i[k]$, for $i \in \{1, \ldots, r\}$, $k \in \{1, \ldots, mr+1\}$, belong to the same component of Γ_A. Therefore, since m is an arbitrary non-negative integer, all the indecomposable modules of the stable tube $\mathcal{T}$ of Γ_B belong to a component $\Gamma(\mathcal{T})$ of Γ_A. Finally, every indecomposable module in $\mathcal{T}$ has infinitely many predecessors and infinitely many successors in $\operatorname{ind} B$, and hence in $\operatorname{ind} A$, and consequently $\Gamma(\mathcal{T})$ is neither a postprojective nor a preinjective component of Γ_A. Hence, applying Proposition 3.1, we infer that $\Gamma(\mathcal{T})$ is a semiregular tube.

(ii) Assume that $\varGamma(\mathcal{T})$ is a ray tube. It is well known that, for any ray

$$X_1 \to X_2 \to \cdots$$

in $\varGamma(\mathcal{T})$, all irreducible homomorphisms $u_k : X_k \to X_{k+1}$, $k \in \mathbb{N}^+$, in $\operatorname{mod} A$ are monomorphisms. Hence, if a module X_t, with $t \geq 2$, of such a ray is an indecomposable B-module of the form $E_i[j]$ for some $i \in \{1, \dots, r\}$ and $j \in \mathbb{N}^+$, then all the modules $X_1, \dots, X_t$ are B-modules and there are irreducible monomorphisms $u_k : X_k \to X_{k+1}$ in $\operatorname{mod} A$, and hence in $\operatorname{mod} B$, and consequently $t = j$ and $X_1 = E_i[1], X_2 = E_i[2], \dots, X_t = X_j = E_i[j]$. This shows that every ray of $\mathcal{T}$ is a ray of $\varGamma(\mathcal{T})$.

The proof of (iii) is dual to the proof of (ii), and uses the fact that the irreducible homomorphisms corresponding to arrows of a coray in a coray tube of $\varGamma_A$ are epimorphisms.

The statement (iv) is an obvious consequence of (ii) and (iii). ∎

Corollary 3.3. *Let A be an algebra such that every component in $\varGamma_A$ is generalized standard and semiregular. Assume A admits a tubular quotient algebra B. Then $\varGamma_A$ contains infinitely many stable tubes of rank at least two.*

Proof. As described by Ringel (see [36, Chapter 5]), we have

$$\varGamma_B = \mathcal{P}(B) \cup \mathcal{T}_0^B \cup \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \Big) \cup \mathcal{T}_\infty^B \cup \mathcal{Q}(B),$$

where $\mathbb{Q}^+$ is the set of positive rational numbers, $\mathcal{P}(B)$ is a postprojective component of Euclidean type, $\mathcal{Q}(B)$ is a preinjective component of Euclidean type, $\mathcal{T}_0^B$ is a family $\mathcal{T}_{0,\lambda}^B$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard ray tubes containing at least one indecomposable projective B-module, $\mathcal{T}_\infty^B$ is a family $\mathcal{T}_{\infty,\lambda}^B$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard coray tubes containing at least one indecomposable injective B-module, and each $\mathcal{T}_q^B$, for $q \in \mathbb{Q}^+$, is a family $\mathcal{T}_{q,\lambda}^B$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard stable tubes of one of the tubular types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. In particular, $\varGamma_B$ contains infinitely many stable tubes of rank at least two. Then the statement follows from Proposition 3.2(i), (iv). ∎

Lemma 3.4. *Let A be an algebra and $\mathcal{C}$ be a semiregular tube in $\varGamma_A$. Then, for all but finitely many indecomposable modules X in $\mathcal{C}$, we have $\dim_K \operatorname{End}_A(X) \geq 2$.*

Proof. We give the proof for a ray tube $\mathcal{C}$, since the case of a coray tube is similar.

Assume that $\mathcal{C}$ is a ray tube and r is the number of rays in $\mathcal{C}$. For a module X in $\mathcal{C}$, we denote by $\varOmega_X$ the maximal sectional path in $\mathcal{C}$ which starts at X and does not lie on a ray, and by $\varSigma_X$ the maximal sectional path in $\mathcal{C}$ which lies on a ray and ends in X. Then, for all but finitely many modules

X in $\mathcal{C}$, Ω_X is a path of length at least r, and hence Ω_X and Σ_X contain a module Z such that the subpath of Ω_X starting at X and ending in Z is of non-zero length. From a result of Bautista and Smalø [7], we know that the composition f of irreducible morphisms lying on a sectional subpath of Ω_X starting at X and ending in Z is non-zero. Denote by g the composition of the irreducible monomorphisms lying on the sectional subpath of Σ_X starting at Z and ending in X. Since g is a monomorphism, gf is non-zero and belongs to $\operatorname{rad} \operatorname{End}_A(X)$. This shows that $\dim_K \operatorname{End}_A(X) \geq 2$. ■

LEMMA 3.5. *Let A be an algebra and Λ a quotient algebra of A. Then for any module M in* $\operatorname{ind} \Lambda$ *the following statements hold:*

(i) *$\tau_\Lambda M$ is a submodule of $\tau_A M$.*
(ii) *$\tau_\Lambda^{-1} M$ is a factor module of $\tau_A^{-1} M$.*

Proof. See [3, Lemma VIII.5.2] and its dual version. ■

PROPOSITION 3.6. *Let A be an algebra such that Γ_A consists of generalized standard semiregular components and all but finitely many of them are stable tubes of rank one. Moreover, let C be a tame concealed quotient algebra of A. Then the following statements hold:*

(i) *There is a postprojective component $\mathcal{P}(C)$ of Euclidean type in Γ_A containing all modules of the unique postprojective component $\mathcal{P}^C$ of Γ_C.*
(ii) *There is a preinjective component $\mathcal{Q}(C)$ of Euclidean type in Γ_A containing all modules of the unique preinjective component $\mathcal{Q}^C$ of Γ_C.*

Proof. It follows from Proposition 3.1 that A is a tame algebra and every component in Γ_A has one of the following forms: a postprojective component of Euclidean type, a preinjective component of Euclidean type, a ray tube, or a coray tube.

(i) Consider now the postprojective component $\mathcal{P}^C$ of Γ_C. Since C is a quotient algebra of A, all indecomposable modules of $\mathcal{P}^C$ are indecomposable modules in $\operatorname{mod} A$. We will show that there exists a postprojective component $\mathcal{P}(C)$ of Γ_A containing all modules of $\mathcal{P}^C$. Suppose that this is not true. Since every postprojective component in Γ_A contains a projective module and is closed under predecessors in $\operatorname{ind} A$, we conclude that all but finitely many indecomposable modules in $\mathcal{P}^C$ do not belong to postprojective components of Γ_A. Further, since every module in a preinjective component of Γ_A has only finitely many successors in $\operatorname{ind} A$, no module from $\mathcal{P}^C$ belongs to a preinjective component of Γ_A. Therefore, all but finitely many modules from $\mathcal{P}^C$ lie in ray or coray tubes of Γ_A. On the other hand, for any module X in $\mathcal{P}^C$ we have $\operatorname{End}_A(X) = \operatorname{End}_C(X) \cong K$. Then applying Lemma 3.4 and the assumption on Γ_A, we conclude that all but finitely many indecomposable modules of $\mathcal{P}^C$ lie on the mouth of stable tubes of rank one of Γ_A (see also [38, Corollary X.2.7]). Summing up, there exists an indecomposable module M in $\mathcal{P}^C$ such that the pairwise non-isomorphic indecomposable modules $\tau_C^{-s}M$, $s \in \mathbb{N}$, lie on the mouth of pairwise different stable tubes of rank one in Γ_A. Take a non-negative integer s. Then $\tau_A^{-1}(\tau_C^{-s}M) = \tau_C^{-s}M$. Now Lemma 3.5(ii) implies that there is an epimorphism $\tau_A^{-1}(\tau_C^{-s}M) \to \tau_C^{-1}(\tau_C^{-s}M)$, and hence an epimorphism $\tau_C^{-s}M \to \tau_C^{-s-1}M$, which is a proper epimorphism, because $\tau_C^{-s}M$ and $\tau^{-s-1}CM$ are non-isomorphic. Therefore, we obtain an infinite sequence of proper epimorphisms

$$M \to \tau_C^{-1}M \to \tau_C^{-2}M \to \cdots,$$

a contradiction. This completes the proof of (i).

The proof of (ii) is similar. ■

Proposition 3.7. *Let A be an algebra such that Γ_A consists of generalized standard semiregular components. Moreover, let C be a tame concealed quotient algebra of A and $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda\in\mathbb{P}_1(K)}$ the family of all stable tubes of Γ_C. Then the following statements hold:*

(i) *For each $\lambda \in \mathbb{P}_1(K)$, there is a unique semiregular tube $\mathcal{T}_\lambda^A(C)$ in Γ_A containing all modules of the stable tube $\mathcal{T}_\lambda^C$.*
(ii) *For any $\lambda \in \mathbb{P}_1(K)$ with $\mathcal{T}_\lambda^A(C)$ being a stable tube, we have $\mathcal{T}_\lambda^A(C) = \mathcal{T}_\lambda^C$.*
(iii) *The quotient algebra $B(C) = A/\mathrm{ann}_A(\mathcal{T}^A(C))$ of A by the annihilator of the family $\mathcal{T}^A(C) = (\mathcal{T}_\lambda^A(C))_{\lambda\in\mathbb{P}_1(K)}$ of semiregular tubes of Γ_A is a tame semiregular branch enlargement of C.*
(iv) *The semiregular tubes $\mathcal{T}_\lambda^A(C)$, $\lambda \in \mathbb{P}_1(K)$, are pairwise orthogonal and generalized standard.*

Proof. It follows from Proposition 3.2(i) that, for each $\lambda \in \mathbb{P}_1(K)$, there exists a semiregular tube $\mathcal{T}_\lambda^A(C)$ in Γ_A containing all modules of $\mathcal{T}_\lambda^C$. Suppose $\mathcal{T}_\lambda^A(C) = \mathcal{T}_\mu^A(C)$ for some $\lambda \neq \mu$ in $\mathbb{P}_1(K)$. Since $\mathcal{T}_\lambda^A(C) = \mathcal{T}_\mu^A(C)$ contains all indecomposable modules of $\mathcal{T}_\lambda^C$ and $\mathcal{T}_\mu^C$, we conclude that there are indecomposable modules X in $\mathcal{T}_\lambda^C$ and Y in $\mathcal{T}_\mu^C$, and sectional paths of irreducible homomorphisms in $\mathrm{mod}\, A$ between indecomposable modules of $\mathcal{T}_\lambda^A(C)$ of the forms

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} X_s = Z,$$

corresponding to the arrows of a coray of $\mathcal{T}_\lambda^A(C)$, and

$$Z = Y_r \xrightarrow{g_r} Y_{r-1} \xrightarrow{g_{r-1}} \cdots \xrightarrow{g_1} Y_0 = Y,$$

corresponding to arrows of a ray in $\mathcal{T}_\lambda^A(C)$. Moreover, $g_1, \dots, g_r$ are monomorphisms if $\mathcal{T}_\lambda^A(C)$ is a ray tube, and $f_1, \dots, f_s$ are epimorphisms if $\mathcal{T}_\lambda^A(C)$ is a coray tube. Since the composition of irreducible homomorphisms forming a sectional path is non-zero [7], we conclude that $g_1 \dots g_r f_s \dots f_1$ is a non-zero homomorphism in $\mathrm{Hom}_A(X,Y) = \mathrm{Hom}_C(X,Y)$, and this contradicts the fact that the stable tubes $\mathcal{T}_\lambda^C$ and $\mathcal{T}_\mu^C$ are orthogonal. Therefore, $\mathcal{T}_\lambda^A(C) \neq \mathcal{T}_\mu^A(C)$ for $\lambda \neq \mu$ in $\mathbb{P}_1(K)$. This shows (i). The statement (ii) follows from Proposition 3.2(iv).

(iii) Applying [39, Theorems XV.3.9, XV.4.3 and XV.4.4], we conclude that there is a branch extension $B(C)^{(r)}$ of C such that $B(C)^{(r)}$ is a quotient algebra of $B(C)$ and all ray tubes of $\mathcal{T}^A(C)$ are ray tubes of $\Gamma_{B(C)^{(r)}}$, and there is a branch coextension $B(C)^{(l)}$ of C such that $B(C)^{(l)}$ is a quotient algebra of $B(C)$ and all coray tubes of $\mathcal{T}^A(C)$ are coray tubes of $\Gamma_{B(C)^{(l)}}$. Since $\mathcal{T}^A(C)$ is a family of semiregular tubes in $\Gamma_{B(C)}$, we find that $B(C)$ is a semiregular branch enlargement of C with the right part $B(C)^{(r)}$ and the left part $B(C)^{(l)}$. Moreover, since A is a tame algebra, by Proposition 3.1 the quotient algebras $B(C)^{(r)}$ and $B(C)^{(l)}$ of A are also tame algebras. Therefore, $B(C)$ is a tame semiregular branch enlargement of C.

The statement (iv) follows from the fact that $B(C) = A/\mathrm{ann}_A(\mathcal{T}^A(C))$ is a semiregular branch enlargement of a tame concealed algebra and from [39, Theorems XV.4.3 and XV.4.4]. ∎

4. Proof of Theorem A. The equivalence (i)⇔(ii) follows from [45, Theorem A, Corollary B] and [29, Theorem 3.4]. Further, (ii)⇔(iii) has been proved in [11, Theorem]. The implication (ii)⇒(iv) is a direct consequence of [45, Theorem A, Corollaries B and C]. Therefore, it remains to prove that (iv) implies (ii).

Assume A satisfies (iv). Proposition 3.1 implies that A is a tame algebra and each component in Γ_A has one of the forms: a postprojective component of Euclidean type, a preinjective component of Euclidean type, a ray tube, or a coray tube. Further, since every component of Γ_A is generalized standard, it follows from [42, Corollary 4.4] that there is a tame concealed algebra C which is a quotient algebra of A. Applying now Propositions 3.6 and 3.7, we conclude that the Auslander–Reiten quiver Γ_A contains:

- a postprojective component $\mathcal{P}^A(C)$ of Euclidean type containing all modules of the postprojective component $\mathcal{P}^C$ of Γ_C;
- a preinjective component $\mathcal{Q}^A(C)$ of Euclidean type containing all modules of the preinjective component $\mathcal{Q}^C$ of Γ_C;
- a family $\mathcal{T}^A(C) = (\mathcal{T}_\lambda^A(C))_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes such that, for any $\lambda \in \mathbb{P}_1(K)$, $\mathcal{T}_\lambda^A(C)$ contains all modules of the stable tube $\mathcal{T}_\lambda^C$ from the family $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda\in\mathbb{P}_1(K)}$ of all stable tubes of Γ_C.

Moreover, the quotient algebra $B(C) = A/\mathrm{ann}_A(\mathcal{T}^A(C))$ is a tame semiregular branch enlargement of the tame concealed algebra C, and clearly $\mathcal{T}^A(C)$ is a family of pairwise orthogonal generalized standard semiregular tubes in $\Gamma_{B(C)}$. Since all but finitely many components of Γ_A are stable tubes of rank one, Corollary 3.3 implies that A, and hence $B(C)$, has no tubular quotient algebra. Therefore, the left part $B(C)^{(l)}$ and the right part $B(C)^{(r)}$ of $B(C)$ are tilted algebras of Euclidean type, and consequently $B(C)$ is a domestic semiregular branch enlargement of C. In particular, we conclude from [29, Theorem 3.4] and [39, Theorems XVII.4.3, XVII.4.4, XVII.5.1 and XVII.5.2] that $\Gamma_{B(C)}$ has a disjoint union decomposition

$$\Gamma_{B(C)} = \mathcal{P}^{B(C)} \cup \mathcal{T}^{B(C)} \cup \mathcal{Q}^{B(C)},$$

where $\mathcal{T}^{B(C)} = \mathcal{T}^A(C)$, $\mathcal{P}^{B(C)}$ is a postprojective component of Euclidean type containing all modules of the postprojective component $\mathcal{P}^C$ of Γ_C, and $\mathcal{Q}^{B(C)}$ is a preinjective component of Euclidean type containing all indecomposable modules of the preinjective component $\mathcal{Q}^C$ of Γ_C. On the other hand, it follows from [39, Theorem XVII.5.1] that $B^{(l)} = A/\mathrm{ann}_A(\mathcal{P}(C))$ is a tilted algebra of Euclidean type having all indecomposable projective modules in the postprojective component $\mathcal{P}(C)$, and hence a domestic branch coextension of a tame concealed algebra $C^{(l)}$. Moreover, $\mathcal{P}(C)$ contains all modules of the postprojective component $\mathcal{P}^{C^{(l)}}$ of $\Gamma_{C^{(l)}}$. Since $\mathcal{P}(C)$ contains all modules of $\mathcal{P}^C$, and the coray tubes of $\mathcal{T}^{B(C)}$ are coray tubes of Γ_A, we conclude that $C^{(l)} = C$, $B^{(l)} = B(C)^{(l)}$, and $\mathcal{P}(C) = \mathcal{P}^{B(C)}$ (see also [43, Propositions 2.2 and 2.3]). Similarly, we prove that $\mathcal{Q}^{B(C)} = \mathcal{Q}(C)$ and $B^{(r)} = A/\mathrm{ann}_A(\mathcal{Q}(C))$ coincides with $B(C)^{(r)}$. Summing up, $\Gamma_{B(C)}$ has a disjoint union decomposition

$$\Gamma_{B(C)} = \mathcal{P}(C) \cup \mathcal{T}^A(C) \cup \mathcal{Q}(C),$$

and hence consists of components of Γ_A.

We claim that $A = B(C)$. Since all indecomposable projective $B(C)$-modules and all indecomposable injective $B(C)$-modules are A-modules, it is enough to show that the quivers Q_A and $Q_{B(C)}$ have the same set of vertices. Clearly, the number of vertices of Q_A is greater than or equal to the number of vertices of $Q_{B(C)}$, because $B(C)$ is a quotient algebra of A. Suppose Q_A has more vertices than $Q_{B(C)}$. Then, since A is an indecomposable algebra, there exist vertices x in Q_A and y in $Q_{B(C)}$, with x not in $Q_{B(C)}$, connected in Q_A by an arrow α.

Assume first that $y = s(\alpha)$ and $x = t(\alpha)$. Then the indecomposable projective A-module $P_A(y)$ at the vertex y has radical $\mathrm{rad}\, P_A(y)$ which is not a $B(C)$-module, because the simple A-module $S_A(x)$ at x is a direct summand of the top of $\mathrm{rad}\, P_A(y)$. But then $P_A(y)$ is not a $B(C)$-module, which contradicts the facts that y belongs to $Q_{B(C)}$ and all indecomposable projective $B(C)$-modules are A-modules.

Assume now that $x = s(\alpha)$ and $y = t(\alpha)$. Then the indecomposable injective A-module $I_A(y)$ at y has socle factor $I_A(y)/\mathrm{soc}\, I_A(y)$ which is not a

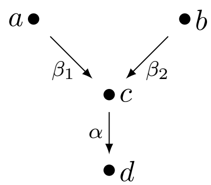
$B(C)$-module, because the simple A-module $S_A(x)$ at x is a direct summand of the socle of $I_A(y)/\operatorname{soc} I_A(y)$. But then $I_A(y)$ is not a $B(C)$-module, which contradicts the fact that all indecomposable injective $B(C)$-modules are A-modules. Therefore, indeed $A = B(C)$, and A is a domestic semiregular branch enlargement of the tame concealed algebra C. This shows that (iv) implies (ii). ■

5. Proof of Theorem B. We start with the following observation.

LEMMA 5.1. *Let A be a special biserial algebra of semiregular type. Then A is a gentle algebra.*

Proof. Let $A = KQ/I$ be a special biserial algebra of semiregular type. Since, for a special biserial algebra which is not a string algebra, there exists at least one indecomposable projective-injective module (see [47], [48]), we may assume that $A = KQ/I$ is a string algebra. Assume now that $\alpha_1\omega\alpha_2$ is a zero-relation for some arrows α_1, α_2 and a non-trivial path ω in Q. Then $X(\omega)$ is a direct summand of the radical $\operatorname{rad} P(s(\alpha_1))$ of the projective module $P(s(\alpha_1))$ and a direct summand of the quotient module $I(t(\alpha_2))/\operatorname{soc} I(t(\alpha_2))$ of the injective module $I(t(\alpha_2))$, and then $P(s(\alpha_1))$ and $I(t(\alpha_2))$ belong to the same component of Γ_A, a contradiction.

Therefore, ω is trivial and I is generated by paths of length 2. Assume that Q contains a subquiver of the form



where α, β_1, β_2 are distinct arrows and $\beta_1\alpha, \beta_2\alpha \in I$. Then $S(c)$ is a direct summand of $I(d)/\operatorname{soc} I(d)$ for the injective module $I(d)$. On the other hand, $S(c)$ is also a direct summand of $\operatorname{rad} P(a)$ or $\operatorname{rad} P(b)$ for the projective modules $P(a)$ and $P(b)$. Indeed, if α is the only arrow in Q which starts at the vertex c, then $S(c)$ is a direct summand of $\operatorname{rad} P(a)$ and $\operatorname{rad} P(b)$. If there exists an arrow $\delta \neq \alpha$ such that $s(\delta) = c$, then the claim is an immediate consequence of the condition (SB2), because then $\beta_1\delta \in I$ or $\beta_2\delta \in I$. Dually, there are no arrows α in Q such that $\alpha\beta_1, \alpha\beta_2 \in I$ for distinct arrows β_1, β_2 such that $t(\alpha) = s(\beta_1) = s(\beta_2)$. Hence (Q, I) satisfies the conditions (G1) and (G2), which means that $A = KQ/I$ is a gentle algebra. ■

PROPOSITION 5.2. *Let A be a special biserial algebra of semiregular type. Then A is a gentle semiregular branch enlargement of a hereditary algebra of type $\widetilde{\mathbb{A}}_m$ for some positive integer m.*

Proof. It follows from Lemma 5.1 that A is a gentle algebra KQ/I. Proposition 2.2 implies that there is no sequential pair of zero-relations in (Q, I). Hence, by Theorem 2.1, A is a quasitilted algebra. Since A is tame, we conclude from [45, Theorem A] that A is a tame tilted algebra or a tame semiregular branch enlargement of a tame concealed algebra.

Assume A is a tilted algebra, and let $A = \mathrm{End}_H(T)$ for a multiplicity-free tilting module T in the module category $\mathrm{mod}\, H$ of a hereditary algebra H. By general theory, the images of the indecomposable injective modules in $\mathrm{mod}\, H$ via the functor $\mathrm{Hom}_H(T, -) : \mathrm{mod}\, H \to \mathrm{mod}\, A$ form a section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_A determined by T (see [3, Proposition VIII.3.5]). Moreover, $\mathcal{C}_T$ is an acyclic and generalized standard component of Γ_A (see [38, Proposition X.3.2]). Further, since $\mathcal{C}_T$ is a semiregular component of Γ_A and A is a tame algebra, applying [41, Corollary 3.10], we conclude that $\mathcal{C}_T$ is either a postprojective component of Euclidean type or a preinjective component of Euclidean type. Moreover, since $\alpha(A) \le 2$, we deduce that Δ_T is a quiver whose underlying graph is $\widetilde{\mathbb{A}}_r$ for some positive integer r. Therefore, A is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_r$, and so a branch coextension or a branch extension of a hereditary algebra H^* of Euclidean type $\widetilde{\mathbb{A}}_s$ for some positive integer $s \le r$ (see [39, Theorem XVII.5.1]).

Summing up, A is a gentle semiregular branch enlargement of a tame concealed algebra C of type $\widetilde{\mathbb{A}}_m$, which is in fact a hereditary algebra of type $\widetilde{\mathbb{A}}_m$ for some positive integer m, by the well known classification of tame concealed algebras (see [38, Section XIV.4]). ∎

We observe that Proposition 5.2 provides the proof of the implication (i)⇒(iii) of Theorem B.

Assume now that A is a gentle semiregular branch enlargement of a hereditary algebra H of type $\widetilde{\mathbb{A}}_m$ for some positive integer m. Then Γ_A has the disjoint union decomposition

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A,$$

where $\mathcal{P}^A$ is a postprojective component of a Euclidean type $\widetilde{\mathbb{A}}_s$, $\mathcal{Q}^A$ is a preinjective component of a Euclidean type $\widetilde{\mathbb{A}}_t$, for some positive integers s and t, $\mathcal{T}^A = (\mathcal{T}^A_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is a family of generalized standard semiregular tubes, and all but at most two tubes in $\mathcal{T}^A$ are stable tubes of rank one. This shows that A satisfies the conditions (a), (b), (c) of (iv). Conversely, assume that A satisfies (a)–(c). Then it follows from Theorem A that A is a domestic semiregular branch enlargement of a tame concealed algebra C and $\alpha(A) \le 2$. Since the left part $A^{(l)}$ and the right part $A^{(r)}$ of A are tilted algebras of Euclidean type, we conclude that Γ_A has the disjoint union decomposition

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A,$$

where $\mathcal{P}^A$ is a postprojective component of a Euclidean type $\Delta^{(l)}$ consisting of indecomposable postprojective $A^{(l)}$-modules, $\mathcal{Q}^A$ is a preinjective component of a Euclidean type $\Delta^{(r)}$ consisting of indecomposable preinjective $A^{(r)}$-modules, and $\mathcal{T}^A = (\mathcal{T}_\lambda^A)_{\lambda\in\mathbb{P}_1(K)}$ is a family of generalized standard semiregular tubes. Further, the assumption $\alpha(A) \le 2$ forces $\Delta^{(l)}$ and $\Delta^{(r)}$ to be Euclidean graphs $\widetilde{\mathbb{A}}_s$ and $\widetilde{\mathbb{A}}_t$ for some positive integers s and t, respectively. Then it follows from the classification of tame concealed algebras (see [38, Section XIV.4]) that C is a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ for a positive integer m. Finally, the assumption (c) implies that at most two stable tubes of Γ_C have been used in the semiregular branch enlargement of C leading to A. Altogether this implies that A is a gentle semiregular branch enlargement of C, which is a hereditary algebra of type $\widetilde{\mathbb{A}}_m$. We conclude that the statements (iii) and (iv) are equivalent.

Clearly, (iii) implies (i). Therefore, it remains to show that (ii) and (iii) are equivalent.

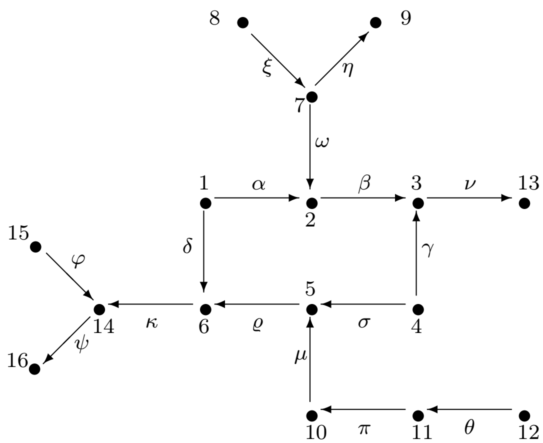
Following [36, (4.1)], a module T in a module category $\operatorname{mod} A$ is said to be *tilting* (respectively, *cotilting*) provided $\operatorname{pd}_A T \le 1$ (respectively, $\operatorname{id}_A T \le 1$), $\operatorname{Ext}^1_A(T,T) = 0$, and T is a direct sum of n pairwise non-isomorphic indecomposable modules, with n being the rank of the Grothendieck group $K_0(A)$ of A. Then two algebras A and B are said to be *tilting-cotilting equivalent* if there exists a sequence of algebras $A = A_0, A_1, \ldots, A_m, A_{m+1} = B$ and a sequence of modules $T^{(i)}$ in $\operatorname{mod} A_i$, with $i \in \{0, 1, \ldots, m\}$, such that $A_{i+1} = \operatorname{End}_{A_i}(T^{(i)})$ and $T^{(i)}$ is either a tilting or a cotilting module. It follows from [23, Corollary 1.7] that, for two tilting-cotilting equivalent algebras A and B, their derived categories $\mathrm{D}^b(\operatorname{mod} A)$ and $\mathrm{D}^b(\operatorname{mod} B)$ are equivalent as triangulated categories. Moreover, by [24, Theorem 4.9], for an algebra A of finite global dimension, $\mathrm{D}^b(\operatorname{mod} A)$ is equivalent, as a triangulated category, to the stable module category $\underline{\operatorname{mod}}\,\widehat{A}$ of the repetitive algebra $\widehat{A}$ of A.

Assume now that A is a quasitilted algebra of canonical type $\widetilde{\mathbb{A}}_{p,q}$ for some positive integers p and q. Then $\mathrm{D}^b(\operatorname{mod} A)$ is equivalent, as a triangulated category, to $\mathrm{D}^b(\operatorname{mod} H_{p,q})$ of the hereditary algebra $H_{p,q} = K\widetilde{\mathbb{A}}_{p,q}$, and consequently the stable module categories $\underline{\operatorname{mod}}\,\widehat{A}$ and $\underline{\operatorname{mod}}\,\widehat{H}_{p,q}$ are equivalent. Applying [4, Theorem (A)] and [29, Theorem 3.4] we conclude that A is a gentle semiregular branch enlargement of a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ for some positive integer m. Hence (ii) implies (iii).

Assume finally that A is a gentle semiregular branch enlargement of a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ for some positive integer m. Then, applying [4, Theorem (A)] again, we conclude that A is tilting-cotilting equivalent to a hereditary algebra $H = K\Delta$ for a quiver Δ whose underlying graph is $\widetilde{\mathbb{A}}_n$ for some positive integer $n \geq m$. In particular, $\mathrm{D}^b(\operatorname{mod} A)$ and $\mathrm{D}^b(\operatorname{mod} H)$ are equivalent as triangulated categories. Consider the hereditary algebra $H_{p,q} = K\widetilde{\mathbb{A}}_{p,q}$, where p is the number of clockwise oriented arrows in Δ and q is the number of counterclockwise oriented arrows in Δ. Then it is well known that H can be obtained from $H_{p,q}$ by a finite sequence of APR-tilting modules (see [3, Sections VI.2, VII.5 and Lemma VIII.1.8]), and hence H and $H_{p,q}$ are tilting-cotilting equivalent. This implies that there exists a triangle equivalence of the derived categories $\mathrm{D}^b(\operatorname{mod} H)$ and $\mathrm{D}^b(\operatorname{mod} H_{p,q})$. Therefore, $\mathrm{D}^b(\operatorname{mod} A)$ and $\mathrm{D}^b(\operatorname{mod} H_{p,q})$ are equivalent as triangulated categories. Since A is a quasitilted algebra of canonical type, we conclude that A is a quasitilted algebra of canonical type $\widetilde{\mathbb{A}}_{p,q}$. This shows that (iii) implies (ii). ∎

6. Examples. The aim of this section is to present some examples illustrating the above considerations.

EXAMPLE 6.1. Let $A = KQ/I$ be the following bound quiver algebra given by the quiver Q:



where I is the ideal in the path algebra KQ generated by the paths $\omega\beta, \xi\eta, \gamma\nu, \mu\varrho, \delta\kappa, \varphi\psi$ of length two. Observe that A is a gentle algebra. Consider the path algebra $H = K\Delta$ of the full subquiver Δ of Q given by the vertices $1, 2, 3, 4, 5, 6$ and the arrows $\alpha, \beta, \gamma, \sigma, \varrho, \delta$. Then H is a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_5$, and the family $\mathcal{T}^H = (\mathcal{T}^H_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes in Γ_H consists of one stable tube $\mathcal{T}^H_0$ of rank two having the string modules $X(\alpha\beta)$ and $X(\sigma\varrho)$ on the mouth, one stable tube $\mathcal{T}_\infty$ of rank four whose mouth consists of the simple modules $S(2)$, $S(5)$ and the string modules $X(\gamma)$ and $X(\delta)$, and the remaining stable tubes $\mathcal{T}^H_\lambda$, $\lambda \in K \setminus \{0\}$, all of rank one. Then

A is a gentle semiregular branch enlargement of H whose left part $A^{(l)}$ and right part $A^{(r)}$ can be described as follows:

- $A^{(l)} = KQ^{(l)}/I^{(l)}$, where $Q^{(l)}$ is the full subquiver of Q given by the vertices $1, 2, 3, 4, 5, 6, 13, 14, 15, 16$ and the arrows $\alpha, \beta, \gamma, \sigma, \varrho, \delta, \nu, \kappa, \varphi, \psi$, and $I^{(l)}$ is the ideal in the path algebra $KQ^{(l)}$ of $Q^{(l)}$ generated by the paths $\gamma\nu, \delta\kappa, \varphi\psi$;
- $A^{(r)} = KQ^{(r)}/I^{(r)}$, where $Q^{(r)}$ is the full subquiver of Q given by the vertices $1, 2, 3, \ldots, 10, 11, 12$ and the arrows $\alpha, \beta, \gamma, \sigma, \varrho, \delta, \omega, \xi, \eta, \mu, \pi, \theta$, and $I^{(r)}$ is the ideal in the path algebra $KQ^{(r)}$ of $Q^{(r)}$ generated by the paths $\omega\beta, \xi\eta, \mu\varrho$.

Then Γ_A has a disjoint union decomposition

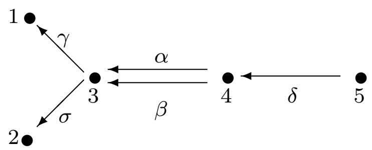
$$\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A,$$

where

- $\mathcal{P}^A$ is a postprojective component of Euclidean type $\widetilde{\mathbb{A}}_9$, containing the indecomposable projective modules $P(1), P(2), P(3), P(4), P(5), P(6), P(13), P(14), P(15), P(16)$;
- $\mathcal{Q}^A$ is a preinjective component of Euclidean type $\widetilde{\mathbb{A}}_{11}$, containing the indecomposable injective modules $I(1), I(2), I(3), I(4), I(5), I(6), I(7), I(8), I(9), I(10), I(11), I(12)$;
- $\mathcal{T}^A = (\mathcal{T}^A_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ is a family of generalized standard semiregular tubes, where $\mathcal{T}^A_\lambda = \mathcal{T}^H_\lambda$ for $\lambda \in K \setminus \{0\}$, $\mathcal{T}^A_0$ is a coray tube with six corays and containing the indecomposable injective modules $I(13), I(14), I(15), I(16)$, and $\mathcal{T}^A_\infty$ is a ray tube with ten rays and containing the indecomposable projective modules $P(7), P(8), P(9), P(10), P(11), P(12)$.

Moreover, applying arguments from [4, Section 2], one can show that A is tilting-cotilting equivalent to the hereditary algebra $K\widetilde{\mathbb{A}}_{10,6}$ and consequently A is a quasitilted algebra of canonical type $\widetilde{\mathbb{A}}_{10,6}$.

EXAMPLE 6.2. Let $A = KQ/I$, where Q is the quiver



and I is the ideal in KQ generated by the paths $\alpha\gamma, \beta\sigma$ and the difference of paths $\delta\alpha - \delta\beta$. A simple checking shows that A is not a special biserial algebra, and hence A is not a gentle algebra. Let H be the Kronecker algebra

$K\Delta$ of the subquiver Δ of Q given by the vertices $3, 4$ and the arrows α, β. Then A is a domestic semiregular branch enlargement of H invoking three pairwise different stable tubes in the family $\mathcal{T}^H = (\mathcal{T}^H_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ of stable tubes of rank one in Γ_H (see [38, Section XI.4] for the description of $\mathcal{T}^H$). We have a disjoint union decomposition

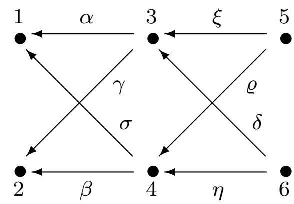
$$\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A,$$

where

- $\mathcal{P}^A$ is a postprojective component of Euclidean type $\widetilde{\mathbb{A}}_3$, containing the indecomposable projective modules $P(1), P(2), P(3), P(4)$;
- $\mathcal{Q}^A$ is a preinjective component of Euclidean type $\widetilde{\mathbb{A}}_2$, containing the indecomposable injective modules $I(3), I(4), I(5)$;
- $\mathcal{T}^A = (\mathcal{T}^A_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ is a family of generalized standard semiregular tubes, where $\mathcal{T}^A_\lambda = \mathcal{T}^H_\lambda$ for $\lambda \in \mathbb{P}_1(K) \setminus \{0, 1, \infty\}$, $\mathcal{T}^A_\infty$ is a coray tube with two corays and containing the indecomposable injective module $I(1)$, $\mathcal{T}^A_0$ is a coray tube with two corays and containing the indecomposable injective module $I(2)$, and $\mathcal{T}^A_1$ is a ray tube with two rays and containing the indecomposable projective module $P(5)$.

We note that $\alpha(A) \leq 2$, every component in Γ_A is semiregular and generalized standard, and all but five components of Γ_A are stable tubes of rank one. This shows that the condition (c) in the statement (iv) of Theorem B is necessary.

EXAMPLE 6.3. Let $a \in K \setminus \{0, 1\}$ and $A^{(a)} = KQ/I^{(a)}$ be the bound quiver algebra given by the following quiver Q:



where $I^{(a)}$ is the ideal in the path algebra KQ of Q generated by the elements $\xi\alpha - \varrho\sigma$, $\xi\gamma - \varrho\beta$, $\delta\alpha - \eta\sigma$, $\delta\gamma - a\eta\beta$. Then $A = A^{(a)}$ is a tubular algebra of tubular type $(2, 2, 2, 2)$, which is a tubular (branch) extension of the hereditary algebra $H_0 = K\Delta_0$ and a tubular (branch) coextension of the hereditary algebra $H_\infty = KQ_\infty$, where Δ_0 is the quiver given by the vertices $1, 2, 3, 4$ and the arrows $\alpha, \beta, \gamma, \sigma$, and Δ_∞ is the quiver given by the vertices $3, 4, 5, 6$ and the arrows $\xi, \eta, \varrho, \delta$ (see [36, Section 5] for the general theory of tubular algebras). Moreover,

$$\Gamma_A = \mathcal{P}^A_0 \cup \mathcal{T}^A_0 \cup \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}^A_q\Big) \cup \mathcal{T}^A_\infty \cup \mathcal{Q}^A_\infty,$$

where $\mathbb{Q}^+$ is the set of positive rational numbers and

- $\mathcal{P}_0^A$ is the postprojective component $\mathcal{P}^{H_0}$ of Γ_{H_0} of type $\widetilde{\mathbb{A}}_3$, containing the indecomposable projective modules $P(1), P(2), P(3), P(4)$;
- $\mathcal{Q}_\infty^A$ is the preinjective component $\mathcal{Q}^{H_\infty}$ of Γ_{H_∞} of type $\widetilde{\mathbb{A}}_3$, containing the indecomposable injective modules $I(3), I(4), I(5), I(6)$;
- $\mathcal{T}_0^A$ is a family $(\mathcal{T}_{0,\lambda}^A)_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard ray tubes, containing two stable tubes of rank two from Γ_{H_0}, two ray tubes with two rays containing the indecomposable projective modules $P(5), P(6)$, and the remaining ray tubes being stable tubes of rank one;
- $\mathcal{T}_\infty^A$ is a family $(\mathcal{T}_{\infty,\lambda}^A)_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard coray tubes, containing two stable tubes of rank two from Γ_{H_∞}, two coray tubes with two corays containing the indecomposable injective modules $I(1), I(2)$, and the remaining coray tubes being stable tubes of rank one;
- for each $q \in \mathbb{Q}^+$, $\mathcal{T}_q^A$ is a family $(\mathcal{T}_{q,\lambda}^A)_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(2,2,2,2)$.

In particular, A is an algebra of semiregular type, with $\alpha(A) \le 2$ and all components in Γ_A generalized standard. On the other hand, for each $q \in \mathbb{Q}^+$, the family $\mathcal{T}_q^A$ contains four stable tubes of rank two. Hence, Γ_A admits infinitely many generalized standard stable tubes of rank two. We note that A is a tame non-domestic quasitilted algebra of canonical type (see [40, Lemma 3.6]). This shows that the requirement in the statement (iv) of Theorem A that all but finitely many components of Γ_A are stable tubes of rank one is necessary.

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