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ON SEMI-RIEMANNIAN MANIFOLDS SATISFYING SOME CONFORMALLY INVARIANT CURVATURE CONDITION

BY

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Dedicated to Professor Rolf Sulanke on his eighty-third birthday

Abstract. We investigate semi-Riemannian manifolds with pseudosymmetric Weyl curvature tensor satisfying some additional condition imposed on their curvature tensor. Among other things we prove that the so-called Roter type equation holds on such manifolds. We present applications of our results to hypersurfaces in semi-Riemannian space forms, as well as to 4-dimensional warped products.

1. Introduction. We denote by ∇, R, S, C and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor and the scalar curvature of an n-dimensional semi-Riemannian manifold (M,g), respectively. The manifold (M,g), $n \geq 3$, is said to be an *Einstein manifold* ([B]) if $S = (\kappa/n)g$ on M. Einstein manifolds form a subclass of the class of quasi-Einstein manifolds. The manifold (M,g), $n \geq 3$, is called a *quasi-Einstein manifold* if at every point $x \in M$ we have $\operatorname{rank}(S - \alpha g) \leq 1$, for some $\alpha \in \mathbb{R}$, which is equivalent to $(S - \alpha g) \wedge (S - \alpha g) = 0$, i.e.

$$\frac{1}{2} S \wedge S - \alpha g \wedge S + \alpha^2 G = 0, \qquad G = \frac{1}{2} g \wedge g. \tag{1.1}$$

For precise definitions of the symbols used here, we refer to Section 2 (see also [BDG] and [DG3]). If at every point of M we have $\operatorname{rank} S \leq 1$ then (M,g) is called *Ricci-simple* ([DRV]). Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of conformally flat spaces. Such manifolds were investigated by several autors. Quasi-Einstein manifolds, in particular quasi-Einstein hypersurfaces, were studied e.g. in [DG4], [DH4]–[DH7] and [G3]; see also [DG3] and references therein.

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It is well-known that the semi-Riemannian manifold (M, g), $n \geq 4$, is conformally flat if and only if $C = 0$ everywhere in M. The last equation yields

$$R = \frac{1}{n-2} g \wedge S - \frac{\kappa}{(n-2)(n-1)} G. \tag{1.2}$$

The Robertson–Walker spacetimes, and more generally, warped products of a 1-dimensional manifold and an $(n-1)$-dimensional space of constant curvature, $n \geq 4$, are conformally flat quasi-Einstein manifolds. Thus, by (1.1) and (1.2), the curvature tensor R of such manifolds is expressed by

$$R = \frac{1}{2} S \wedge S + \left(\frac{1}{n-2} - \alpha \right) g \wedge S + \left(\alpha^2 - \frac{\kappa}{(n-2)(n-1)} \right) G.$$

There are also non-conformally flat and non-quasi-Einstein manifolds satisfying an equation of this kind. Namely, as explained in Remark 4.1 of this paper, the curvature tensor R of some semi-Riemannian manifolds is a linear combination of the tensors $S \wedge S$, $g \wedge S$ and G, i.e.

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G, \tag{1.3}$$

where ϕ, μ and η are some functions. More precisely, the curvature tensor R of some manifolds (M, g), $n \geq 4$, satisfies (1.3) on a specific open set $\mathcal{U}$ defined in Remark 4.1. The manifold (M, g), $n \geq 4$, is said to be a *Roter type manifold* ([D4], [G2]) if (1.3) holds on the set $\mathcal{U}$ just mentioned, and $\mathcal{U}$ is a non-empty set. Condition (1.3) is called the *Roter type equation* ([D4]). We refer to [D4], [DH5], [DK], [DP], [DS], [G1] and [K2] for results on Roter type manifolds. In particular, examples of warped products satisfying (1.3) are given in [DK], [DP], [DS], [K1] and [K2].

In Section 3 we consider generalized curvature tensors B having some additional properties. The main result (Theorem 3.3) states that under some conditions imposed on a generalized curvature tensor B, its Weyl tensor $\mathrm{Weyl}(B)$, and Ricci tensor $\mathrm{Ric}(B)$ (see (3.5)–(3.7)), the tensor B is a linear combination of $\mathrm{Ric}(B) \wedge \mathrm{Ric}(B)$, $g \wedge \mathrm{Ric}(B)$ and G, i.e. B satisfies the Roter type equation (3.26), a special case of which is (1.3).

It should be pointed out that (3.26) is known to imply further interesting relations ([K2, Proposition 4.1]). As a result we also have a family of such relations when the curvature tensor R of a manifold (M, g), $n \geq 4$, satisfies (1.3). Among those relations there is a condition which is invariant under conformal deformations of g:

$$C \cdot C = L_C Q(g, C), \tag{1.4}$$

for some function L_C, with $C \cdot C$ and $Q(g, C)$ defined as in Section 2 below. A manifold (M, g), $n \geq 4$, satisfying (1.4) is called a *manifold with* *pseudosymmetric Weyl tensor*. Results on manifolds (in particular, hypersurfaces in space forms) satisfying (1.4) appear e.g. in [ACD], [DD], [DDV], [DY1]–[DY3], [MAD] and [S1]. We also mention that every Chen type ideal submanifold and every Wintgen submanifold satisfy (1.4) (see, e.g., [DS1], [DS2], [DG6]).

Section 4 contains results on manifolds with pseudosymmetric Weyl tensor. Our main result (Theorem 4.4) states that under some additional assumptions the Roter type equation (1.3) is satisfied on such manifolds. We present some applications of that theorem to warped product manifolds (Corollaries 4.5–4.7), as well as to hypersurfaces in semi-Riemannian space forms (Theorem 4.8, Corollary 4.9).

2. Preliminaries. Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold and let ∇ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$
$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M,g) are defined by $S(X,Y) = \operatorname{tr}\{Z \to \mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y) = S(X,Y)$ and $\kappa = \operatorname{tr}\mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right)Z.$$

Now the $(0,4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$
$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3, X_4),$$
$$C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3, X_4),$$

where $X_1, X_2, \ldots \in \Xi(M)$. We define the following subsets of M:

$$\mathcal{U}_R = \left\{x \in M \;\middle|\; R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\right\},$$
$$\mathcal{U}_S = \left\{x \in M \;\middle|\; S - \frac{\kappa}{n} g \neq 0 \text{ at } x\right\},$$
$$\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}.$$

We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$.

Let $\mathcal{B}$ be a tensor field sending any $X, Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X,Y)$, and let B be a $(0,4)$-tensor associated with $\mathcal{B}$ by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \tag{2.1}$$

The tensor B is said to be a *generalized curvature tensor* if the following conditions are fulfilled:

$$B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2)$$
$$B(X_1, X_2, X_3, X_4) + B(X_3, X_1, X_2, X_4) + B(X_2, X_3, X_1, X_4) = 0.$$

For $\mathcal{B}$ as above, let B be again defined by (2.1). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y) \cdot f = 0$ for any smooth function f on M. For a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0, k+2)$-tensor $B \cdot T$ by

$$\begin{aligned}(B \cdot T)(X_1, \ldots, X_k, X, Y) &= (\mathcal{B}(X,Y) \cdot T)(X_1, \ldots, X_k)\\ &= -T(\mathcal{B}(X,Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{B}(X,Y)X_k).\end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor then we define the $(0, k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned}Q(A,T)(X_1, \ldots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \ldots, X_k)\\ &= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).\end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B \cdot B$ and $Q(A,B)$. Setting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(g,S)$. The tensor $Q(A,T)$ is called the *Tachibana tensor of the tensors* A and T, or briefly the Tachibana tensor (see, e.g., [DG7]). We mention that in some papers the tensor $Q(g,R)$ is called the Tachibana tensor ([HV1], [JH1], [JH2], [PTV]).

For a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k \geq 2$, we define their Kulkarni–Nomizu product $E \wedge T$ by ([DG2])

$$\begin{aligned}&(E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \ldots, Y_k)\\ &\quad = E(X_1, X_4)T(X_2, X_3, Y_3, \ldots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \ldots, Y_k)\\ &\qquad - E(X_1, X_3)T(X_2, X_4, Y_3, \ldots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \ldots, Y_k).\end{aligned}$$

The following tensors are generalized curvature tensors: R, C, G and $E \wedge F$, where E and F are symmetric $(0,2)$-tensors. For a symmetric $(0,2)$-tensor E we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E} = \frac{1}{2} E \wedge E$. In particular, we have $\overline{g} = G = \frac{1}{2} g \wedge g$. Now we can express the Weyl tensor by

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G. \tag{2.2}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_xM at a point $x \in M$ of a semi-Riemannian manifold (M, g), $n \geq 3$, and let

$$g(e_j, e_k) = \varepsilon_j \delta_{jk}, \qquad \varepsilon_j = \pm 1, \qquad j, k \in \{1, \ldots, n\}.$$

For a generalized curvature tensor B on M we denote by $\operatorname{Ric}(B)$, $\kappa(B)$ and $\operatorname{Weyl}(B)$ its scalar curvature, Ricci tensor and Weyl tensor, respectively. We have

$$\begin{aligned} \operatorname{Ric}(B)(X, Y) &= \sum_{j=1}^{n} \varepsilon_j B(e_j, X, Y, e_j), \\ \kappa(B) &= \sum_{j=1}^{n} \varepsilon_j \operatorname{Ric}(B)(e_j, e_j), \\ \operatorname{Weyl}(B) &= B - \frac{1}{n-2} g \wedge \operatorname{Ric}(B) + \frac{\kappa(B)}{(n-2)(n-1)} G. \end{aligned} \tag{2.3}$$

We denote by $\mathcal{U}_{\operatorname{Ric}(B)}$, respectively, $\mathcal{U}_{\operatorname{Weyl}(B)}$, the set of all points at which the tensor $\operatorname{Ric}(B)$ is not proportional to g, respectively, the tensor $\operatorname{Weyl}(B)$ is non-zero.

Let B_{hijk}, T_{hijk}, and A_{ij} be the local components of generalized curvature tensors B and T and a symmetric $(0,2)$-tensor A on M, respectively, where $h, i, j, k, l, m, p, q \in \{1, \ldots, n\}$. The local components $(B \cdot T)_{hijklm}$ and $Q(A, T)_{hijklm}$ of the tensors $B \cdot T$, $Q(A, T)$, $B \cdot A$ and $Q(g, A)$ are

$$\begin{aligned} (B \cdot T)_{hijklm} &= g^{pq}(T_{pijk} B_{qhlm} + T_{hpjk} B_{qilm} + T_{hipk} B_{qjlm} + T_{hijp} B_{qklm}), \\ Q(A, T)_{hijklm} &= A_{hl} T_{mijk} + A_{il} T_{hmjk} + A_{jl} T_{himk} + A_{kl} T_{hijm} \\ &\quad - A_{hm} T_{lijk} - A_{im} T_{hljk} - A_{jm} T_{hilk} - A_{km} T_{hijl}, \\ (B \cdot A)_{hklm} &= g^{pq}(A_{pk} B_{qhlm} + A_{ph} B_{qklm}), \\ Q(g, A)_{hklm} &= g_{hl} A_{km} + g_{kl} A_{hm} - g_{hm} A_{kl} - g_{km} A_{hl}. \end{aligned}$$

From the last equation, by contraction with g^{ij} and g^{hm}, we obtain

$$\begin{aligned} g^{rs} Q(A, T)_{hrsklm} &= A_l^s T_{skhm} - A_l^s T_{shmk} - A_m^s T_{skhl} + A_m^s T_{shlk} \\ &\quad + Q(A, \operatorname{Ric}(T))_{hklm}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} g^{rs} Q(A, T)_{rijkls} &= -A_i^s T_{sljk} + A_l^s T_{sijk} + A_j^s T_{sikl} + A_k^s T_{silj} \\ &\quad + A_{lk} \operatorname{Ric}(T)_{ij} - A_{jl} \operatorname{Ric}(T)_{ik} - g^{rs} A_{rs} T_{lijk}, \end{aligned} \tag{2.5}$$

$$g^{rs} Q(g, A)_{rkls} = \operatorname{tr}(A) g_{kl} - n A_{kl}. \tag{2.6}$$

[DG7, Lemma 2.1 and Proposition 2.1] yield

Proposition 2.1. *Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g), $n \geq 4$. If on $\mathcal{U}_{\operatorname{Weyl}(T)} \subset M$ we have*

$$T \cdot T - Q(\operatorname{Ric}(T), T) = L_1 Q(g, \operatorname{Weyl}(T))$$

for some function L_1 on $\mathcal{U}_{\mathrm{Weyl}(T)}$, then at every point of this set we have

$$\mathrm{Ric}(T)_h^s T_{sklm} + \mathrm{Ric}(T)_l^s T_{skmh} + \mathrm{Ric}(T)_m^s T_{skhl} = 0, \tag{2.7}$$

where $\mathrm{Ric}(T)_{ij}$ *and* T_{hijk} *are the local components of* $\mathrm{Ric}(T)$ *and* T*, respectively.*

For a symmetric $(0,2)$-tensor A we define the endomorphism $\mathcal{A}$ and the tensor A^2 by $g(\mathcal{A}X, Y) = A(X,Y)$ and $A^2(X,Y) = A(\mathcal{A}X, Y)$, respectively.

LEMMA 2.2. *Let E_1, E_2 and F be symmetric* $(0,2)$*-tensors at a point x of a semi-Riemannian manifold (M,g), $n \geq 3$.*

(i) ([DG1], [DG2]) *At x we have*

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) + Q(F, E_1 \wedge E_2) = 0.$$

In particular, if $E = E_1 = E_2$ then at x we have

$$E \wedge Q(E,F) = -Q(F, \overline{E}). \tag{2.8}$$

Moreover (*see, e.g.,* [DG4, Section 3])

$$Q(E, E \wedge F) = -Q(F, \overline{E}). \tag{2.9}$$

(ii) [K2, Lemma 3.2] *At x we have*

$$\begin{aligned} &G \cdot F = Q(g,F), \qquad (g \wedge F) \cdot F = Q(g, F^2), \\ &-(g \wedge F) \cdot (g \wedge F) = Q(F^2, G). \end{aligned} \tag{2.10}$$

Moreover, if B is a generalized curvature tensor then

$$G \cdot B = Q(g,B). \tag{2.11}$$

(iii) (*see, e.g.,* [DY5, Lemma 2.4(iii)]) *At x we have*

$$Q(E_1, E_2 \wedge F) + Q(E_2, F \wedge E_1) + Q(F, E_1 \wedge E_2) = 0.$$

REMARK 2.3. Let (M,g), $n \geq 3$, be a semi-Riemannian manifold. Let A, E and F be symmetric $(0,2)$-tensors and B and T generalized curvature tensors at a point $x \in M$.

(i) [G, Lemma 2] states that if $R \cdot E = 0$ and $R \cdot A = Q(g,F)$ at x then $\big(E - \frac{\mathrm{tr}(E)}{n} g\big)\big(F - \frac{\mathrm{tr}(F)}{n} g\big) = 0$ at this point. The same proof shows that R can be replaced by T.

(ii) By [G, Theorem 1], if $R \cdot B = 0$ and $R \cdot A = Q(g,F)$ at x then $\big(F - \frac{\mathrm{tr}(F)}{n} g\big)\big(B - \frac{\kappa(B)}{(n-1)n} G\big) = 0$ at this point. Again, R can be replaced by T.

(iii) By [DH2, Lemma 1], if $R \cdot E = \alpha Q(g,E)$, $\alpha \in \mathbb{R}$, and $R \cdot A = Q(g,F)$ at x then $\big(F - \alpha A - \frac{\mathrm{tr}(F - \alpha A)}{n} g\big)\big(E - \frac{\mathrm{tr}(E)}{n} g\big) = 0$ at this point. Here too, R can be replaced by T.

(iv) By [DH2, Lemma 2], if $R\cdot B = \alpha Q(g, B)$, $\alpha \in \mathbb{R}$, and $R\cdot A = Q(g, F)$ at x then $\big(F - \alpha A - \frac{\mathrm{tr}(F-\alpha A)}{n} g\big)\big(B - \frac{\kappa(B)}{(n-1)n} G\big) = 0$ at this point, and again we observe that R can be replaced by T.

Proposition 2.4 (see, e.g., [DH3, Lemma 3.4]). *Let (M, g), $n \geq 3$, be a semi-Riemannian manifold. Let a non-zero symmetric $(0, 2)$-tensor E and a generalized curvature tensor T, defined at $x \in M$, satisfy $Q(E, T) = 0$ at x. In addition, let Y be a vector at x such that the scalar $\rho = w(Y)$ is non-zero, where w is the covector defined by $w(X) = E(X, Y)$, $X \in T_xM$. Then either*

(i) *$E - \rho w \otimes w \neq 0$ and $T = \lambda E \wedge E$, $\lambda \in \mathbb{R}$, or*
(ii) *$E = \rho w \otimes w$ and*

$$\begin{aligned}w(X)T(Y, Z, X_1, X_2) &+ w(Y)T(Z, X, X_1, X_2)\\ &+ w(Z)T(X, Y, X_1, X_2) = 0, \quad X, Y, Z, X_1, X_2 \in T_xM.\end{aligned} \tag{2.12}$$

Moreover, in both cases the following condition holds at x:

$$T \cdot T = Q(\mathrm{Ric}(T), T). \tag{2.13}$$

3. Identities for generalized curvature tensors. In this section we present results on generalized curvature tensors satisfying some conditions.

Proposition 3.1. *If B is a generalized curvature tensor on a semi-Riemannian manifold (M, g), $n \geq 4$, then the following identities are satisfied on M:*

$$\begin{aligned}(n-2)(B\cdot \mathrm{Weyl}(B) - \mathrm{Weyl}(B)\cdot B) &+ \frac{\kappa(B)}{n-1} Q(g, \mathrm{Weyl}(B))\\ &= (g \wedge \mathrm{Ric}(B))\cdot \mathrm{Weyl}(B) - g \wedge (\mathrm{Weyl}(B)\cdot \mathrm{Ric}(B)),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\mathrm{Weyl}(B)\cdot \mathrm{Weyl}(B) = B\cdot B &\\ - \frac{1}{n-2}&((g \wedge \mathrm{Ric}(B))\cdot B + g \wedge (B\cdot \mathrm{Ric}(B)))\\ + \frac{\kappa(B)}{(n-2)(n-1)}&Q(g, \mathrm{Weyl}(B)) - \frac{1}{(n-2)^2} Q((\mathrm{Ric}(B))^2, G).\end{aligned} \tag{3.2}$$

Proof. Using (2.3), (2.9) and (2.11), we obtain the identities

$$Q(g, \mathrm{Weyl}(B)) = Q(g, B) + \frac{1}{n-2} Q(\mathrm{Ric}(B), G),$$

$$\begin{aligned}\mathrm{Weyl}(B)\cdot \mathrm{Weyl}(B) &= \mathrm{Weyl}(B)\cdot \bigg(B - \frac{1}{n-2} g \wedge \mathrm{Ric}(B)\\ &\qquad + \frac{\kappa(B)}{(n-2)(n-1)} G\bigg)\\ &= \mathrm{Weyl}(B)\cdot B - \frac{1}{n-2} g \wedge (\mathrm{Weyl}(B)\cdot \mathrm{Ric}(B)),\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
(3.4)\quad \mathrm{Weyl}(B)\cdot\mathrm{Weyl}(B) &= \Big(B - \frac{1}{n-2} g\wedge \mathrm{Ric}(B) \\
&\quad + \frac{\kappa(B)}{(n-2)(n-1)}G\Big)\cdot \mathrm{Weyl}(B) \\
&= B\cdot \mathrm{Weyl}(B) - \frac{1}{n-2}(g\wedge \mathrm{Ric}(B))\cdot \mathrm{Weyl}(B) \\
&\quad + \frac{\kappa(B)}{(n-2)(n-1)}Q(g,\mathrm{Weyl}(B)), \\
B\cdot \mathrm{Weyl}(B) &= B\cdot B - \frac{1}{n-2} g\wedge (B\cdot \mathrm{Ric}(B)).
\end{aligned}
$$

From (3.3) and (3.4) we easily get (3.1). Further, by making use of (2.3), (2.10) and (2.11), we find

$$
\begin{aligned}
\mathrm{Weyl}(B)\cdot\mathrm{Weyl}(B) &= \mathrm{Weyl}(B)\cdot B - \frac{1}{n-2}\mathrm{Weyl}(B)\cdot (g\wedge \mathrm{Ric}(B)) \\
&= B\cdot B - \frac{1}{n-2}(g\wedge \mathrm{Ric}(B))\cdot B + \frac{\kappa(B)}{(n-2)(n-1)}Q(g,B) \\
&\quad - \frac{1}{n-2} g\wedge (B\cdot \mathrm{Ric}(B)) + \frac{1}{(n-2)^2}\,(g\wedge \mathrm{Ric}(B))\cdot (g\wedge \mathrm{Ric}(B)) \\
&\quad - \frac{\kappa(B)}{(n-2)^2(n-1)}G\cdot (g\wedge \mathrm{Ric}(B)) \\
&= B\cdot B + \frac{\kappa(B)}{(n-2)(n-1)}Q(g,B) - \frac{1}{(n-2)^2}Q((\mathrm{Ric}(B))^2, G) \\
&\quad - \frac{\kappa(B)}{(n-2)^2(n-1)}Q(g, g\wedge \mathrm{Ric}(B)) \\
&\quad - \frac{1}{n-2}((g\wedge \mathrm{Ric}(B))\cdot B + g\wedge (B\cdot \mathrm{Ric}(B))).
\end{aligned}
$$

By making use of (2.3), we rewrite this as (3.2), completing the proof. ∎

PROPOSITION 3.2. *Let B be a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n \geq 4$, satisfying on $\mathcal{U}_{\mathrm{Ric}(B)} \cap \mathcal{U}_{\mathrm{Weyl}(B)} \subset M$,*

$$B\cdot B - Q(\mathrm{Ric}(B), B) = L_1 Q(g, \mathrm{Weyl}(B)), \tag{3.5}$$

$$\mathrm{Weyl}(B)\cdot \mathrm{Weyl}(B) = L_{\mathrm{Weyl}(B)} Q(g, \mathrm{Weyl}(B)), \tag{3.6}$$

$$B\cdot \mathrm{Ric}(B) = Q(g, D), \tag{3.7}$$

where D is a symmetric $(0,2)$-tensor and L_1 and $L_2 = L_{\mathrm{Weyl}(B)}$ some functions on this set. Then on $\mathcal{U}_{\mathrm{Ric}(B)} \cap \mathcal{U}_{\mathrm{Weyl}(B)}$ we have

$$\mathrm{Weyl}(B)\cdot \mathrm{Ric}(B) = L_2 Q(g, \mathrm{Ric}(B))), \tag{3.8}$$

$$\mathrm{Weyl}(B)\cdot B = L_2 Q(g, B), \tag{3.9}$$

$$(\mathrm{Ric}(B))^2 = \lambda_0\,\mathrm{Ric}(B) + \lambda_3 g, \tag{3.10}$$

$$D = \Big(L_2 - \frac{\kappa(B)}{(n-2)(n-1)} + \frac{\lambda_0}{n-2}\Big) \operatorname{Ric}(B) + \lambda_4 g, \tag{3.11}$$

$$B \cdot \operatorname{Ric}(B) = \left(L_2 - \frac{\kappa(B)}{(n-2)(n-1)} + \frac{\lambda_0}{n-2}\right) Q(g, \operatorname{Ric}(B)), \tag{3.12}$$

$$\begin{aligned} Q(\operatorname{Ric}(B) - \alpha_1 g, \operatorname{Weyl}(B) + \alpha_2 G) \\ = \frac{1}{4(n-2)} Q(g, \operatorname{Ric}(B) \wedge \operatorname{Ric}(B)), \end{aligned} \tag{3.13}$$

where λ_0, λ_3 and λ_4 are some functions and

$$2\alpha_1 = \frac{\kappa(B)}{n-1} - L_1 + L_2, \qquad (n-2)\alpha_2 = \frac{\lambda_0 - \kappa(B)}{2(n-2)} + L_2.$$

Proof. First of all we note that the following identity is satisfied on a coordinate domain of a point of $\mathcal{U}_{\operatorname{Ric}(B)} \cap \mathcal{U}_{\operatorname{Weyl}(B)}$:

$$\begin{aligned} ((g \wedge \operatorname{Ric}(B)) \cdot B)_{hijklm} = Q(\operatorname{Ric}(B), B)_{hijklm} \\ + g_{hl} A_{mijk} - g_{hm} A_{lijk} - g_{il} A_{mhjk} + g_{im} A_{lhjk} \\ + g_{jl} A_{mkhi} - g_{jm} A_{lkhi} - g_{kl} A_{mjhi} + g_{km} A_{ljhi}, \end{aligned} \tag{3.14}$$

where $A_{mijk} = g^{rs} \operatorname{Ric}(B)_{rm} B_{sijk}$. Further, applying (2.8), (2.9), (3.5), (3.6), (3.7) and (3.14) to (3.2) we get

$$\begin{aligned} & g_{hl} A_{mijk} - g_{hm} A_{lijk} - g_{il} A_{mhjk} + g_{im} A_{lhjk} \\ & \quad + g_{jl} A_{mkhi} - g_{jm} A_{lkhi} - g_{kl} A_{mjhi} + g_{km} A_{ljhi} \\ & = \left(\frac{\kappa(B)}{n-1} + (n-2)(L_1 - L_{\operatorname{Weyl}(B)})\right) Q(g, \operatorname{Weyl}(B))_{hijklm} \\ & \quad + (n-3) Q(\operatorname{Ric}(B), B)_{hijklm} \\ & \quad - Q\left(g, g \wedge \left(D - \frac{1}{n-2}(\operatorname{Ric}(B))^2\right)\right)_{hijklm}. \end{aligned} \tag{3.15}$$

Furthermore (2.6) and (3.7) yield

$$\begin{aligned} &\text{(a)} \quad A_{iljk} = -A_{lijk} + Q(g, D)_{lijk}, \\ &\text{(b)} \quad E_{ij} = \operatorname{Ric}(B)^{rs} B_{sijr} = ((\operatorname{Ric}(B))^2)_{ij} - n D_{ij} + \operatorname{tr}(D) g_{ij}. \end{aligned} \tag{3.16}$$

Contracting (3.15) with g^{hm} and using (2.4), (2.5), (2.7) and (3.16)(a) we find

$$\begin{aligned} & -2(n-2) A_{lijk} + g_{kl} E_{ij} - g_{jl} E_{ik} \\ & = -(\kappa(B) + (n-2)(n-1)(L_1 - L_2)) \operatorname{Weyl}(B)_{lijk} \\ & \quad + (n-3)(\operatorname{Ric}(B)_{ij} \operatorname{Ric}(B)_{lk} - \operatorname{Ric}(B)_{ik} \operatorname{Ric}(B)_{jl}) \\ & \quad - (n-3)\kappa(B) B_{lijk} \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&- (n-3)Q(g,D)_{lijk} + (n-1)\Big(g \wedge \Big(D - \frac{1}{n-2}(\mathrm{Ric}(B))^2\Big)\Big)_{lijk}\\
&- (n-2)\Big(g_{lk}\Big(D - \frac{1}{n-2}(\mathrm{Ric}(B))^2\Big)_{ij}\\
&- g_{jl}\Big(D - \frac{1}{n-2}(\mathrm{Ric}(B))^2\Big)_{ik}\Big)\\
&- \Big(\mathrm{tr}(D) - \frac{1}{n-2}\,\mathrm{tr}((\mathrm{Ric}(B))^2)\Big)(g_{lk}g_{ij} - g_{jl}g_{ik}).
\end{aligned}
$$

Now (3.17), combined with (2.3) and (3.16)(b), yields

$$
\begin{aligned}
2(n-2)A_{lijk} &= (n-2)(\kappa(B) + (n-1)(L_1 - L_2))\mathrm{Weyl}(B)_{lijk}\\
&- (n-3)(\mathrm{Ric}(B)_{lk}\,\mathrm{Ric}(B)_{ij} - \mathrm{Ric}(B)_{ik}\,\mathrm{Ric}(B)_{jl})\\
&+ (n-3)Q(g,D)_{lijk} - 2(g_{kl}D_{ij} - g_{jl}D_{ik})\\
&- \Big(g \wedge \Big((n-1)D - \frac{n-1}{n-2}(\mathrm{Ric}(B))^2 - \frac{(n-3)\kappa(B)}{n-2}\,\mathrm{Ric}(B)\Big)\Big)_{lijk}\\
&- \Big(\frac{(n-3)(\kappa(B))^2}{(n-2)(n-1)} - 2\,\mathrm{tr}(D) + \frac{1}{n-2}\,\mathrm{tr}((\mathrm{Ric}(B))^2)\Big)G_{lijk}.
\end{aligned} \tag{3.18}
$$

Multiplying (3.15) by $2(n-2)$ and using (3.18) we obtain

$$
\begin{aligned}
&(n-2)(\kappa(B) + (n-1)(L_1 - L_2))Q(g,\mathrm{Weyl}(B))\\
&\qquad - \frac{n-3}{2}Q(g,\mathrm{Ric}(B)\wedge\mathrm{Ric}(B)) - (n-1)Q(g, g\wedge D)\\
&\qquad + \frac{n-1}{n-2}Q(g, g\wedge(\mathrm{Ric}(B))^2) + \frac{(n-3)\kappa(B)}{n-2}Q(g, g\wedge\mathrm{Ric}(B))\\
&\qquad + (n-3)Q(g, g\wedge D)\\
&\quad = 2(n-2)\Big(\frac{\kappa(B)}{n-1} + (n-2)(L_1 - L_2)\Big)Q(g,\mathrm{Weyl}(B))\\
&\qquad + 2(n-2)(n-3)Q(\mathrm{Ric}(B),B) - 2(n-2)Q(g, g\wedge D)\\
&\qquad + 2Q(g, g\wedge(\mathrm{Ric}(B))^2),
\end{aligned}
$$

which leads to

$$
Q(\mathrm{Ric}(B),B) = Q(g,T_1), \tag{3.19}
$$

where

$$
\begin{aligned}
T_1 = \frac{1}{2}\Big(\frac{\kappa(B)}{n-1} - L_1 + L_2\Big)\mathrm{Weyl}(B) - \frac{1}{4(n-2)}\,\mathrm{Ric}(B)\wedge\mathrm{Ric}(B)\\
+ \frac{1}{2(n-2)^2} g\wedge(\kappa(B)\,\mathrm{Ric}(B) - (\mathrm{Ric}(B))^2 + 2(n-2)D),
\end{aligned}
$$

$$D = \frac{1}{n-2}(\mathrm{Ric}(B))^2 + \left(L_2 - \frac{\kappa(B)}{(n-2)(n-1)}\right)\mathrm{Ric}(B) + \lambda g, \tag{3.20}$$

and λ is some function. Now (3.5) and (3.19) yield

$$B \cdot B = Q(g, T) \quad \text{where} \quad T = T_1 + L_1 \,\mathrm{Weyl}(B).$$

We can easily check that

$$\mathrm{Weyl}(B) \cdot \mathrm{Ric}(B) = Q\left(g, D - \frac{1}{n-2}\left((\mathrm{Ric}(B))^2 - \frac{\kappa(B)}{n-1}\,\mathrm{Ric}(B)\right)\right), \tag{3.21}$$

is an immediate consequence of Lemma 2.2(ii), (2.3) and (3.7). Next (3.6) and (3.21), in view of Remark 2.3(iv), lead to (3.20). Further, (3.20) and (3.21) give (3.8). Now (3.9) is a consequence of (2.3), (2.8), (2.9), (3.3), (3.6) and (3.8).

Transvecting (3.18) with $\mathrm{Ric}(B)^k_m = \mathrm{Ric}(B)_{mr} g^{rk}$ we get

$$\begin{aligned} 2(n-2)\,\mathrm{Ric}(B)^r_m\,\mathrm{Ric}(B)^s_l B_{rjis} &= \alpha_1\,\mathrm{Ric}(B)^r_m \mathrm{Weyl}(B)_{rjil} \\ &\quad - (n-3)(\mathrm{Ric}(B)_{ij}(\mathrm{Ric}(B))^2_{lm} - \mathrm{Ric}(B)_{jl}(\mathrm{Ric}(B))^2_{im}) \\ &\quad + (n-3)(g_{lj}F_{im} + g_{ij}F_{lm} - \mathrm{Ric}(B)_{lm}D_{ij} - \mathrm{Ric}(B)_{im}D_{lj}) \\ &\quad - 2(\mathrm{Ric}(B)_{lm}D_{ij} - g_{jl}F_{im}) + \lambda_1(g_{ij}\,\mathrm{Ric}(B)_{lm} - g_{lj}\,\mathrm{Ric}(B)_{im}) \\ &\quad + (\kappa(B) - (n-1)L_2)(g_{ij}(\mathrm{Ric}(B))^2_{lm} - g_{lj}(\mathrm{Ric}(B))^2_{im} \\ &\quad + \mathrm{Ric}(B)_{ij}\,\mathrm{Ric}(B)_{lm} - \mathrm{Ric}(B)_{im}\,\mathrm{Ric}(B)_{jl}), \end{aligned} \tag{3.22}$$

where $F_{im} = \mathrm{Ric}(B)^r_i D_{rm}$, $\alpha_1 = (n-2)(\kappa(B) + (n-1)(L_1 - L_2))$ and λ_1 is some function. Symmetrizing (3.22) in j, m and using (3.7) and (3.8), we obtain

$$\begin{aligned} 2(n-2)(g_{im}F_{jl} &+ g_{ij}F_{lm} - \mathrm{Ric}(B)_{lm}D_{ij} - \mathrm{Ric}(B)_{jl}D_{im}) \\ &= \alpha_2 Q(g, \mathrm{Ric}(B))_{jmil} - (n-3)\, Q(\mathrm{Ric}(B), (\mathrm{Ric}(B))^2)_{jiml} \\ &\quad + (\kappa(B) - (n-1)L_2) Q(g, (\mathrm{Ric}(B))^2)_{jmil} \\ &\quad - (n-1)(\mathrm{Ric}(B)_{lm}D_{ij} + \mathrm{Ric}(B)_{lj}D_{im}) \\ &\quad - (n-3)(\mathrm{Ric}(B)_{im}D_{lj} + \mathrm{Ric}(B)_{ij}D_{lm}) \\ &\quad + (n-1)(g_{jl}F_{im} + g_{lm}F_{ij}) + (n-3)(g_{il}F_{lm} + g_{im}F_{jl}), \end{aligned}$$

$\alpha_2 = \alpha_1 L_2 + \lambda_1$, and

$$\begin{aligned} \alpha_2 Q(g, \mathrm{Ric}(B)) &- (n-3)Q(\mathrm{Ric}(B), (\mathrm{Ric}(B))^2) \\ &+ (\kappa(B) - (n-1)L_2)Q(g, (\mathrm{Ric}(B))^2) \\ &- (n-3)Q(\mathrm{Ric}(B), D) - (n-1)Q(g, F) = 0, \end{aligned}$$

which, by (3.20), turns into

$$Q(g,F) = Q\left(g, \frac{\alpha_2}{n-1}\operatorname{Ric}(B) + \left(\frac{\kappa(B)}{n-1} - L_2\right)(\operatorname{Ric}(B))^2\right) - \frac{n-3}{n-2}Q(\operatorname{Ric}(B),(\operatorname{Ric}(B))^2). \tag{3.23}$$

Applying a suitable contraction to (3.23) and making use of (2.6), we get

$$F = \left(\frac{\alpha_2}{n-1} + \frac{(n-3)\operatorname{tr}((\operatorname{Ric}(B))^2)}{(n-2)n}\right)\operatorname{Ric}(B) + \left(\frac{(2n-3)\kappa(B)}{(n-2)(n-1)n} - L_2\right)(\operatorname{Ric}(B))^2 + \lambda_2 g,$$

where λ_2 is some function. Substituting this into (3.23) and using [DV, Lemma 2.4(ii)] we obtain (3.10). Now (3.10) and (3.20) lead to (3.11). Next, using (3.7), (3.10) and (3.20) we find (3.12). From (3.11) we have

$$2(n-2)D = \left(2\lambda_0 + 2(n-2)L_2 - \frac{2\kappa(B)}{n-1}\right)\operatorname{Ric}(B) + 2(n-2)\lambda_4 g,$$

which can be rewritten in the form

$$\kappa(B)\operatorname{Ric}(B) - (\operatorname{Ric}(B))^2 + 2(n-2)D = \left(\lambda_0 + 2(n-2)L_2 + \frac{(n-3)\kappa(B)}{n-1}\right)\operatorname{Ric}(B) + \lambda_5 g. \tag{3.24}$$

Now (3.19), by (3.24), turns into

$$\begin{aligned} &Q(\operatorname{Ric}(B),B) \\ &= Q\left(g, \frac{1}{2}\left(\frac{\kappa(B)}{n-1} - L_1 + L_2\right)\operatorname{Weyl}(B) - \frac{1}{4(n-2)}\operatorname{Ric}(B)\wedge\operatorname{Ric}(B)\right. \\ &\quad \left. + \frac{1}{2(n-2)^2}g\wedge\left(\lambda_0 + 2(n-2)L_2 + \frac{(n-3)\kappa(B)}{n-1}\right)\operatorname{Ric}(B)\right). \end{aligned}$$

This together with the identity

$$\begin{aligned} &Q\left(\operatorname{Ric}(B), -\frac{1}{n-2}g\wedge\operatorname{Ric}(B) + \frac{\kappa(B)}{(n-2)(n-1)}G\right) \\ &\quad = Q\left(g, \frac{1}{2(n-2)}\operatorname{Ric}(B)\wedge\operatorname{Ric}(B) - \frac{\kappa(B)}{(n-2)(n-1)}g\wedge\operatorname{Ric}(B)\right) \end{aligned}$$

gives

$$Q\Big(\mathrm{Ric}(B), B - \frac{1}{n-2} g \wedge \mathrm{Ric}(B) + \frac{\kappa(B)}{(n-2)(n-1)} G\Big)$$
$$= Q\Big(g, \frac{1}{2}\Big(\frac{\kappa(B)}{n-1} - L_1 + L_2\Big)\mathrm{Weyl}(B) + \frac{1}{4(n-2)} \mathrm{Ric}(B) \wedge \mathrm{Ric}(B)$$
$$+ g \wedge \Big(\frac{\lambda_0}{2(n-2)^2} + \frac{L_2}{n-2} - \frac{\kappa(B)}{2(n-2)^2}\Big) \mathrm{Ric}(B)\Big),$$

and

$$Q\Big(\mathrm{Ric}(B) - \frac{1}{2}\Big(\frac{\kappa(B)}{n-1} - L_1 + L_2\Big) g, \mathrm{Weyl}(B)\Big)$$
$$= \frac{1}{4(n-2)} Q(g, \mathrm{Ric}(B) \wedge \mathrm{Ric}(B)) + \Big(\frac{\lambda_0 - \kappa(B)}{2(n-2)^2} + \frac{L_2}{n-2}\Big) Q(g, g \wedge \mathrm{Ric}(B)),$$

i.e. (3.13). Our proposition is thus proved. ∎

Theorem 3.3. *Let B be a generalized curvature tensor on a semi-Riemannian manifold (M, g), $n \geq 4$, such that the conditions* (3.5)–(3.7) *hold on $\mathcal{U}_{\mathrm{Ric}(B)} \cap \mathcal{U}_{\mathrm{Weyl}(B)} \subset M$. Then on this set we have*

$$B \cdot B = \Big(L_2 - \frac{\kappa(B)}{(n-2)(n-1)} + \frac{\lambda_0}{n-2}\Big) Q(g, B). \tag{3.25}$$

Moreover, if $\mathrm{rank}(\mathrm{Ric}(B) - \alpha_1 g) \geq 2$ *at a point $x \in \mathcal{U}_{\mathrm{Ric}(B)} \cap \mathcal{U}_{\mathrm{Weyl}(B)}$ then on some open neighbourhood $\mathcal{U}_1 \subset \mathcal{U}_{\mathrm{Ric}(B)} \cap \mathcal{U}_{\mathrm{Weyl}(B)}$ of x we have*

$$B = \frac{\phi}{2} \mathrm{Ric}(B) \wedge \mathrm{Ric}(B) + \mu g \wedge \mathrm{Ric}(B) + \eta G, \tag{3.26}$$

where ϕ, η and μ are some functions on $\mathcal{U}_1$.

Proof. (i) Let $\mathrm{rank}(\mathrm{Ric}(B) - \alpha_1 g) = 1$ at a point $x \in \mathcal{U}_{\mathrm{Ric}(B)} \cap \mathcal{U}_{\mathrm{Weyl}(B)}$, i.e. at this point we have

$$\mathrm{Ric}(B) - \alpha_1 g = \epsilon w \otimes w, \quad \epsilon = \pm 1, \quad w \in T_x^* M. \tag{3.27}$$

Applying (3.27) in (3.13) we find

$$Q(w \otimes w, T) = 0, \quad T = B + \alpha_3 G, \quad \alpha_3 \in \mathbb{R}. \tag{3.28}$$

From (3.28), in view of Proposition 2.4(ii), it follows that (2.12) and (2.13) hold at x. Since $\mathrm{Ric}(T) = \mathrm{Ric}(B) + (n-1)\alpha_3 g$, (2.13) turns into

$$(B + \alpha_3 G) \cdot (B + \alpha_3 G) = Q(\mathrm{Ric}(B), T) + (n-1)\alpha_3 Q(g, T),$$

which, as a consequence of (2.11), turns into

$$B \cdot B = Q(\mathrm{Ric}(B) - \alpha_1 g, T) + (\alpha_1 + (n-2)\alpha_3) Q(g, T).$$

This, by (3.27), takes the form

$$B \cdot B = \epsilon Q(w \otimes w, T) + (\alpha_1 + (n-2)\alpha_3) Q(g, B),$$

which by (3.28) reduces to

$$B \cdot B = (\alpha_1 + (n-2)\alpha_3) Q(g, B). \tag{3.29}$$

This, by a suitable contraction, yields

$$B \cdot \operatorname{Ric}(B) = (\alpha_1 + (n-2)\alpha_3) Q(g, \operatorname{Ric}(B)). \tag{3.30}$$

Comparing the right-hand sides of (3.12) and (3.30) we get

$$\alpha_1 + (n-2)\alpha_3 = L_2 - \frac{\kappa(B)}{(n-2)(n-1)} + \frac{\lambda_0}{n-2}.$$

Therefore (3.29) turns into (3.25).

(ii) Let $\operatorname{rank}(\operatorname{Ric}(B) - \alpha_1 g) \geq 2$ at a point $x \in \mathcal{U}_{\operatorname{Ric}(B)} \cap \mathcal{U}_{\operatorname{Weyl}(B)}$.

If $\alpha_1 \neq 0$ at x then from (3.13) it follows that

$$\begin{aligned} &Q(\operatorname{Ric}(B) - \alpha_1 g, \operatorname{Weyl}(B) + \alpha_2 G) \\ &\qquad = Q\left(\operatorname{Ric}(B) - \alpha_1 g, -\frac{1}{4(n-2)\alpha_1} \operatorname{Ric}(B) \wedge \operatorname{Ric}(B)\right), \end{aligned}$$

i.e.

$$\begin{aligned} Q\Bigg(&\operatorname{Ric}(B) - \alpha_1 g, \\ &\operatorname{Weyl}(B) + \frac{1}{4(n-2)\alpha_1} \operatorname{Ric}(B) \wedge \operatorname{Ric}(B) + \alpha_2 G\Bigg) = 0 \end{aligned} \tag{3.31}$$

holds at x. From (3.31), in view of Proposition 2.4(i), at x we have:

$$\begin{aligned} &\operatorname{Weyl}(B) + \frac{1}{4(n-2)\alpha_1} \operatorname{Ric}(B) \wedge \operatorname{Ric}(B) + \alpha_2 G \\ &\qquad = \tau_1 (\operatorname{Ric}(B) - \alpha_1 g) \wedge (\operatorname{Ric}(B) - \alpha_1 g), \end{aligned}$$

which leads to (3.26).

If $\alpha_1 = 0$ at x then from (3.13) it follows that

$$Q(\operatorname{Ric}(B), \operatorname{Weyl}(B) + \alpha_2 G) = -\frac{1}{2(n-2)} Q(\operatorname{Ric}(B), g \wedge \operatorname{Ric}(B)),$$

i.e.

$$Q\left(\operatorname{Ric}(B), \operatorname{Weyl}(B) + \frac{1}{2(n-2)} g \wedge \operatorname{Ric}(B) + \alpha_2 G\right) = 0 \tag{3.32}$$

at x. From (3.32), in view of Proposition 2.4(i), at x we have

$$\operatorname{Weyl}(B) + \frac{1}{2(n-2)} g \wedge \operatorname{Ric}(B) + \alpha_2 G = \tau_2 \operatorname{Ric}(B) \wedge \operatorname{Ric}(B),$$

which leads to (3.26).

From (3.26), in view of [K2, Proposition 4.1], it follows that

$$B \cdot B = L_R Q(g, B) \tag{3.33}$$

on $\mathcal{U}_1$, where L_R is some function on this set. But (3.33), by a suitable contraction, gives

$$B \cdot \mathrm{Ric}(B) = L_R Q(g, \mathrm{Ric}(B)). \tag{3.34}$$

Comparing the right-hand sides of (3.12) and (3.34) we get

$$L_R = L_2 - \frac{\kappa(B)}{(n-2)(n-1)} + \frac{\lambda_0}{n-2}.$$

Therefore (3.33) turns into (3.25). Our proposition is thus proved. ∎

4. Conditions of pseudosymmetry type. Let (M, g), $n = \dim M$, be a semi-Riemannian manifold. The manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* ([BDG], [D3, Section 3.1], [DH1]) if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$R \cdot R = L_R Q(g, R) \tag{4.1}$$

on $\mathcal{U}_R \subset M$, where L_R is some function on $\mathcal{U}_R$. The class of pseudosymmetric manifolds is an extension of the class of semisymmetric manifolds ($R \cdot R = 0$). We refer to Sections 3 and 4 of [BDG] for a survey of related results. A geometric interpretation of the notion of pseudosymmetry is given in [HV1]. We also refer to [DH1, Chapter 6] and [HV2] for a recent exposition of the notion of pseudosymmetry.

We note that the tensor $Q(g, R)$ of a manifold (M, g), $n \geq 3$, vanishes at $x \in M$ if and only if $R = \frac{\kappa}{(n-1)n} G$ at x (see, e.g., [D2, Remark 1]). Moreover, the last condition also implies $R \cdot R = 0$ at x. Therefore, if $R \cdot R \neq 0$ at a point of M then $Q(g, R)$ is non-zero at this point.

The manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* ([BDG], [D3, Section 4.1], [DH1]) if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$R \cdot S = L_S Q(g, S) \tag{4.2}$$

on $\mathcal{U}_S \subset M$, where L_S is some function on $\mathcal{U}_S$. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric manifolds ($R \cdot S = 0$), as well as of the class of pseudosymmetric manifolds (see [BDG], [D3]). A geometric interpretation of the notion of Ricci-pseudosymmetry is given in [JH2].

We note that the tensor $Q(g, S)$ of a manifold (M, g), $n \geq 3$, vanishes at $x \in M$ if and only if $S = (\kappa/n) g$ at x (see, e.g., [D2, Lemma 3]). The last condition also implies $R \cdot S = 0$ at x. Clearly, if $R \cdot S \neq 0$ at a point of M then $Q(g, S)$ is non-zero at this point.

The manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudosymmetric* ([BDG], [DH2, Section 4.1], [DH1]) if at every point of M the tensors $R \cdot C$ and $Q(g, C)$

are linearly dependent. This is equivalent to

$$R \cdot C = L_1 Q(g, C) \tag{4.3}$$

on $\mathcal{U}_C \subset M$, where L_1 is some function on $\mathcal{U}_C$. It is obvious that on $\mathcal{U}_R$ (4.1) implies (4.3), with $L_1 = L_R$. Conversely, for manifolds (M, g), $n \geq 5$, on $\mathcal{U}_C \subset M$ (4.3) implies (4.1), with $L_R = L_1$. For $n = 4$ the last statement is not true (see, e.g., [D3, Section 4.2] and references therein). A geometric interpretation of the notion of Weyl-pseudosymmetry is given in [JH1].

The manifold (M, g), $n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* ([BDG], [D3, Section 12.6]) if at every point of M the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent. This is equivalent to (1.4) on $\mathcal{U}_C \subset M$, where L_C is some function on $\mathcal{U}_C$.

We note that the tensor $Q(g, C)$ of a manifold (M, g), $n \geq 4$, vanishes at $x \in M$ if and only if $C = 0$ at x (see, e.g., [D2, Remark 1]). The last condition also implies $R \cdot C = C \cdot C = 0$ at x. Clearly, if $R \cdot C \neq 0$ or $C \cdot C \neq 0$ at a point of M then $Q(g, C)$ is non-zero at this point.

We can also investigate manifolds (M, g), $n \geq 4$, satisfying the following condition on $\mathcal{U}_C \subset M$ (see, e.g., [DDP], [DH3]):

$$R \cdot R - Q(S, R) = L Q(g, C). \tag{4.4}$$

We mention that (4.4) holds on every hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$. Precisely, we have ([DV])

$$R \cdot R - Q(S, R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, C), \tag{4.5}$$

where $\widetilde{\kappa}$ is the scalar curvature of the ambient space. We also mention that conformally flat manifolds, of dimension ≥ 4, satisfying $R \cdot R = Q(S, R)$ were investigated in [D1].

The conditions (4.1)–(4.4), as well as other relations of this kind, are called *curvature conditions of pseudosymmetry type*. For a survey of results on such conditions we refer to [BDG], [DG3] and [DG5] (see also [DH1], [G2] and [S2]).

Remark 4.1. (i) Let (M, g), $n \geq 4$, be a semi-Riemannian manifold and let $\mathcal{U}$ be the set of all points of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ at which $\operatorname{rank}(S - \alpha g) \geq 2$ for any $\alpha \in \mathbb{R}$. In [DY4, Theorem 3.1] it is proved that if a pseudosymmetric manifold (M, g), $n \geq 4$, satisfies (1.4) on $\mathcal{U}_C \subset M$ then

$$Q(S - \beta_1 g, C - \beta_2 G) = 0 \tag{4.6}$$

on this set, where β_1 and β_2 are some functions on $\mathcal{U}_C$. From (4.6), in view of Proposition 2.4(i) it follows that

$$C - \beta_2 G = \lambda (S - \beta_1 g) \wedge (S - \beta_1 g)$$

on $\mathcal{U} \subset M$, where λ is some function on this set. The last equation, together with (2.2), leads to (1.3).

(ii) In [DH3, p. 40] it is proved that if a pseudosymmetric manifold (M, g), $n \geq 4$, satisfies (4.4) on $\mathcal{U}_C \subset M$ then

$$Q(S - \beta_3 g, R - \beta_4 G) = 0 \tag{4.7}$$

on this set, where β_3 and β_4 are some functions on $\mathcal{U}_C$. Now from (4.7), in view of Proposition 2.4(i) it follows that (1.3) holds on $\mathcal{U} \subset M$.

But on the other hand we have

Theorem 4.2 (see, e.g., [G2]). *If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying* (1.3) *on $\mathcal{U} \subset M$ then the following conditions hold on $\mathcal{U}$:*

$$S^2 = \alpha S + \beta g, \qquad \alpha = \kappa + \frac{(n-2)\mu - 1}{\phi}, \qquad \beta = \frac{\mu\kappa + (n-1)\eta}{\phi},$$
$$R \cdot C = L_R Q(g, C), \qquad C \cdot R = L_C Q(g, R),$$

and (4.1), (1.4) *and* (4.4), *with the functions $L_R = \frac{1}{\phi}((n-2)(\mu^2 - \phi\eta) - \eta)$, $L_C = L_R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \alpha\right)$ and $L = L_R + \frac{\mu}{\phi}$, respectively.*

Remark 4.3. Let (M, g), $n \geq 4$, be a semi-Riemannian manifold with parallel Weyl conformal curvature tensor ($\nabla C = 0$) which is neither conformally flat nor locally symmetric. Such manifolds are named *essentially conformally symmetric manifolds* (e.c.s. manifolds, for short; see e.g. [DR1]). The local structure of e.c.s. manifolds has been determined ([DR2]). Certain e.c.s. metrics are realized on compact manifolds (see [DR3] and references therein). It is known that on every e.c.s. manifold the following conditions are satisfied: $\kappa = 0$, $R \cdot R = 0$, $C \cdot C = 0$ and $Q(S, C) = 0$. Moreover, (4.4) holds ([DH3, Theorem 4.3]). In addition,

$$FC = \frac{1}{2} S \wedge S \tag{4.8}$$

on M, where F is some function on M, called the *fundamental function* ([DR1]). At every point of M we also have rank $S \leq 2$ ([DR1, Theorem 5]). We note that from (4.8) it follows immediately that (1.3) holds at all points of an e.c.s. manifold at which $\operatorname{rank} S = 2$. It is clear that such points form the set $\mathcal{U}$.

An immediate consequence of Theorem 3.3 is the following result related to Roter type manifolds.

Theorem 4.4. *If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ conditions* (1.4), (4.4) *and*

$$R \cdot S = Q(g, D), \tag{4.9}$$

where D is a symmetric $(0,2)$-tensor, then (4.1) *holds on this set. Moreover, the Roter type equation* (1.3) *is satisfied on $\mathcal{U} \subset \mathcal{U}_S \cap \mathcal{U}_C$.*

From this we have

COROLLARY 4.5. *If (M,g), $n \geq 4$, is a semi-Riemannian manifold satisfying on $\mathcal{U}_C \cap \mathcal{U}_S \subset M$ conditions* (1.4), (4.2) *and* (4.4), *then* (4.1) *holds on this set. Moreover, the Roter type equation* (1.3) *is satisfied on $\mathcal{U} \subset \mathcal{U}_S \cap \mathcal{U}_C$.*

Let $\overline{M} \times_F \widetilde{M}$ be the warped product of the semi-Riemannian manifolds $(\overline{M}, \overline{g})$, $p = \dim \overline{M}$, and $(\widetilde{M}, \widetilde{g})$, $n - p = \dim \widetilde{M}$, $1 \leq p \leq n-1$, $n \geq 4$, with the warping function F. In [DDP] it is proved that (4.4) holds on every 4-dimensional warped product $\overline{M} \times_F \widetilde{M}$ with $p = 1$. Thus we have

COROLLARY 4.6. *If the 4-dimensional warped product $\overline{M} \times_F \widetilde{M}$, with $p = 1$, satisfies on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{M}$ conditions* (1.4) *and* (4.9), *then* (4.1) *holds on this set. Moreover, the Roter type equation* (1.3) *is satisfied on $\mathcal{U} \subset \mathcal{U}_S \cap \mathcal{U}_C$.*

Roter type warped products $\overline{M} \times_F \widetilde{M}$, with $p = 1$, were investigated in [DS]. In [D2] it is proved that (1.4) holds on every 4-dimensional warped product $\overline{M} \times_F \widetilde{M}$, with $p = 2$. Thus we have

COROLLARY 4.7. *If the 4-dimensional warped product $\overline{M} \times_F \widetilde{M}$, with $p = 2$, satisfies on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{M}$ conditions* (4.4) *and* (4.9), *then* (4.1) *holds on this set. Moreover, the Roter type equation* (1.3) *is satisfied on $\mathcal{U} \subset \mathcal{U}_S \cap \mathcal{U}_C$.*

Roter type warped products $\overline{M} \times_F \widetilde{M}$, with $2 \leq p \leq n-2$, were investigated in [DK] and [K2]. We also refer to [DP] for results on Roter type warped products $\overline{M} \times_F \widetilde{M}$, with $p = n-1$.

Finally, we present an application of our main results to hypersurfaces M immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$. Since (4.5) holds on M, Theorem 4.4 and Corollary 4.5 imply

THEOREM 4.8. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ conditions* (1.4) *and* (4.9), *for some symmetric* $(0,2)$*-tensor D, then* (4.1) *holds on this set. Moreover, the Roter type equation* (1.3) *is satisfied on $\mathcal{U} \subset \mathcal{U}_S \cap \mathcal{U}_C$.*

COROLLARY 4.9. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ conditions* (1.4) *and* (4.2), *then* (4.1) *holds on this set. Moreover, the Roter type equation* (1.3) *is satisfied on $\mathcal{U} \subset \mathcal{U}_S \cap \mathcal{U}_C$.*

We refer to [G1, Section 3] for results on hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying the Roter type equation (1.3). We mention that every Clifford torus $S^p(\sqrt{p/n}) \times S^{n-p}(\sqrt{(n-p)/n})$, $2 \leq p \leq n-2$, $n \neq 2p$, is a non-conformally flat and non-Einstein hypersurface satisfying (1.3) ([G1, Corollary 3.1]).

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