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# FINITENESS ASPECTS OF GORENSTEIN HOMOLOGICAL DIMENSIONS

BY

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**Abstract.** We present an alternative way of measuring the Gorenstein projective (resp., injective) dimension of modules via a new type of complete projective (resp., injective) resolutions. As an application, we easily recover well known theorems such as the Auslander–Bridger formula. Our approach allows us to relate the Gorenstein global dimension of a ring $R$ to the cohomological invariants $\mathrm{silp}(R)$ and $\mathrm{spli}(R)$ introduced by Gedrich and Gruenberg by proving that $\mathrm{leftG\text{-}gldim}(R) = \max\{\mathrm{leftsilp}(R), \mathrm{leftspli}(R)\}$, recovering a recent theorem of [I. Emmanouil, J. Algebra 372 (2012), 376–396]. Moreover, this formula permits to recover the main theorem of [D. Bennis and N. Mahdou, Proc. Amer. Math. Soc. 138 (2010), 461–465]. Furthermore, we prove that, in the setting of a left and right Noetherian ring, the Gorenstein global dimension is left-right symmetric, generalizing a theorem of Enochs and Jenda. Finally, using recent work of I. Emmanouil and O. Talelli, we compute the Gorenstein global dimension for various types of rings such as commutative $\aleph_0$-Noetherian rings and group rings.

**1. Introduction.** Throughout this paper, $R$ denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left $R$-modules. Also, for any $R$-module $A$, $Z(A)$ denotes the set of all zerodivisors of $A$.

Recall that Gorenstein projective (resp., injective) modules originate from the classical notion of projective (resp., injective) modules, being images and kernels of the differentials of complete projective (resp., injective) resolutions. Specifically, a module $M$ is said to be *Gorenstein projective* if there exists an exact sequence of projective modules, called a complete projective resolution,

$$\mathbf{P} := \cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots,$$

such that $\mathbf{P}$ remains exact after applying the functor $\mathrm{Hom}_R(-, P)$ for each projective module $P$ and $M := \mathrm{Im}(P_0 \to P_{-1})$. Gorenstein injective modules are defined dually. These new concepts allow Enochs and Jenda [17] to

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introduce new (Gorenstein homological) dimensions in order to extend the G-dimension defined by Auslander and Bridger in [1, 2]. It turns out, in particular, that these Gorenstein homological dimensions are refinements of the classical dimensions of a module $M$, in the sense that $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ and $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ with equality each time the corresponding classical homological dimension is finite. The reader is referred to [4, 6, 8, 10, 12, 17, 18, 20, 23, 30–33] for basics and recent investigations on Gorenstein homological theory as well as some related themes to our subject. Nevertheless, the finiteness of Gorenstein homological dimensions remains one of the key problems of Gorenstein homological algebra (see the survey [11] and the introduction of [12] for a further discussion of this issue). The problem was partly solved by Christensen, Frankild and Holm [12]: *If $R$ has a dualizing complex, that is, if $R$ is a homomorphic image of a Gorenstein local ring, and $M$ is an $R$-module, then* $\mathrm{Gpd}_R(M)$ *(resp.,* $\mathrm{Gid}_R(M)$*) is finite if and only if $M$ belongs to $A(R)$ (resp., $B(R)$), where $A(R)$ and $B(R)$ stand for the Auslander class and the Bass class of $R$, respectively.*

The main purpose of this paper is to unify the study of modules of finite Gorenstein homological dimensions. Specifically, we present an alternative way of measuring the Gorenstein projective (resp., injective) dimension of modules via a new type of complete projective (resp., injective) resolutions. It is worth recalling that complete projective (resp., injective) resolutions are at the heart of Gorenstein homological algebra. The images and kernels of the differentials of such resolutions are called Gorenstein projective (resp., injective) modules. We introduce generalized Gorenstein projective modules and generalized Gorenstein injective modules, and show that they inherit all properties of corresponding Gorenstein homological modules. In accordance with the new concepts, we introduce the generalized Gorenstein projective dimension and generalized Gorenstein injective dimension of a module $M$, denoted, respectively, by $\mathrm{GGpd}_R(M)$ and $\mathrm{GGid}_R(M)$. In Section 2, we prove that

$$\mathrm{GGpd}_R(M) = \mathrm{Gpd}_R(M), \qquad \mathrm{GGid}_R(M) = \mathrm{Gid}_R(M),$$

for each $R$-module $M$. This is mainly due to the fact that the category $\mathcal{GP}(R)$ (resp., $\mathcal{GI}(R)$) of Gorenstein projective (resp., injective) modules is projectively resolving (resp., injectively resolving).

As an application, we easily recover known theorems of Gorenstein homological algebra such as the Auslander–Bridger formula and the main theorem of Bennis and Mahdou [7]. Moreover, we relate the Gorenstein global dimension of a ring $R$ to the cohomological invariants $\mathrm{slip}(R)$ and $\mathrm{silp}(R)$ of $R$ introduced by Gedrich and Gruenberg [21]. Recall that Bennis and Mahdou [7] introduce the following invariants for a ring $R$:

$$\text{l-GPD}(R) := \sup\{\text{Gpd}_R(M) : M \text{ is an } R\text{-module}\},$$
$$\text{l-GID}(R) := \sup\{\text{Gid}_R(M) : M \text{ is an } R\text{-module}\}.$$

The main theorem of [7] states that for an arbitrary ring $R$,

$$\text{l-GPD}(R) = \text{l-GID}(R).$$

This allows one to define the left Gorenstein global dimension of $R$, denoted by l-G-gldim$(R)$, to be this common value. Further, recall that in connection with the existence of complete cohomological functors in the category of left $R$-modules, Gedrich and Gruenberg [21] have defined the following invariants for a given ring $R$:

$$\text{l-spli}(R) := \sup\{\text{pd}_R(I) : I \text{ is an injective left module}\},$$
$$\text{l-silp}(R) := \sup\{\text{id}_R(P) : P \text{ is a projective left module}\}.$$

The relation between l-spli$(R)$ and l-silp$(R)$, as noted by Gedrich and Gruenberg, still remains unclear for a general ring $R$. These two invariants are easily checked to be equal if they are both finite and Gedrich and Gruenberg asked whether the finiteness of one implies that of the other. A positive answer to this latter question would prove a long-standing conjecture in representation theory, the well known Gorenstein symmetry conjecture [3, conjecture 13], which states that $\text{id}_R({}_RR) = \text{id}_R(R_R)$ for any Artin algebra $R$. This conjecture, as asserted by Beligiannis and Reiten [5], is equivalent to l-silp$(R)$ = l-spli$(R)$ for any Artin algebra $R$. In this context, we recover Bennis and Mahdou's theorem as well as Emmanouil's theorem [14, Theorem 4.1] by proving that, for an arbitrary ring $R$,

$$\text{l-GPD}(R) = \text{l-GID}(R) = \max\{\text{l-silp}(R), \text{l-spli}(R)\},$$

showing, in particular, via Theorem 3.10, that for any left and right Noetherian ring $R$,

$$\text{l-G-gldim}(R) = \text{r-G-gldim}(R) =: \text{G-gldim}(R).$$

This last formula is the analog in Gorenstein homological algebra of the classical Auslander theorem stating that l-gldim$(R)$ = r-gldim$(R)$ for a left and right Noetherian ring $R$ [29, Corollary 9.23]. Also, it generalizes a result of Enochs and Jenda [19, Theorem 12.3.1 and Corollary 12.3.2] establishing the equality l-G-gldim$(R)$ = r-G-gldim$(R)$ in the restricted setting of Iwanaga–Gorenstein rings. At the end of Section 3, using recent work of I. Emmanouil and O. Talelli [14, 16], we compute the Gorenstein global dimension for various types of rings such as commutative $\aleph_0$-Noetherian rings and group rings.

**2. Generalized Gorenstein projective and injective dimensions.** The goal of this section is to introduce and study Gorenstein $n$-projective modules and Gorenstein $n$-injective modules over $R$.

Definition 2.1. Let $R$ be a ring.

(1) Let $n \geq 0$ be an integer. An exact sequence of $R$-modules

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

is called a *complete $n$-projective resolution* if $\{\mathrm{pd}_R(E_i) : i \in \mathbb{Z}\}$ is a bounded set and $\mathrm{Ext}_R^{k+1}(M_i, P) = 0$ for each integer $k \geq n$, each $M_i := \mathrm{Im}(d_i)$, each integer $i$ and each projective $R$-module $P$.

(2) Let $n \geq 0$ be an integer. An $R$-module $M$ is called a *Gorenstein $n$-projective $R$-module* if $M$ is a kernel or image of a differential of a complete $n$-projective resolution.

(3) An $R$-module $M$ is called a *generalized Gorenstein projective $R$-module* if $M$ is Gorenstein $n$-projective for some positive integer $n$.

(4) We define the *generalized Gorenstein projective dimension* of an $R$-module $M$ as follows:

$$\mathrm{GGpd}_R(M) = \begin{cases} \sup\{r : \mathrm{Ext}_R^r(M,P) \neq 0 \text{ for some projective } R\text{-module } P\} \\ \qquad \text{if } M \text{ is a generalized Gorenstein projective module,} \\ +\infty \quad \text{otherwise.} \end{cases}$$

Gorenstein $n$-injective $R$-modules, generalized Gorenstein injective $R$-modules and generalized Gorenstein injective dimension are defined dually.

Remark 2.2. (1) Let $R$ be a commutative ring and $(x_1, \ldots, x_n)$ be a regular $R$-sequence. If $G$ is a Gorenstein projective $R$-module, then $G/(x_1, \ldots, x_n)$ is a Gorenstein $n$-projective $R$-module. This has a routine proof using Rees's theorem and its dual [28, Exercises 2 and 3, p. 155].

(2) Let $M$ be an $R$-module. Then $M$ is generalized Gorenstein projective if and only if $\mathrm{GGpd}_R(M) < \infty$.

(3) Each Gorenstein projective (resp., Gorenstein injective) module is Gorenstein 0-projective (resp., Gorenstein 0-injective). In fact, we will see next that these two notions coincide.

(4) Let $n \geq 0$ be an integer. Then any $R$-module $M$ such that $\mathrm{pd}_R(M) \leq n$ (resp., $\mathrm{id}_R(M) \leq n$) is a Gorenstein $n$-projective (resp., Gorenstein $n$-injective) $R$-module. Indeed, given an $R$-module $M$ such that $\mathrm{pd}_R(M) \leq n$, it suffices to note that the exact sequence of $R$-modules $0 \to M \to M \to 0$ is a complete $n$-projective resolution and thus $M$ is Gorenstein $n$-projective.

Proposition 2.3. *Let $M$ be an $R$-module and $n \geq 0$ an integer. Then the following assertions are equivalent:*

(1) *$M$ is Gorenstein $n$-projective over $R$.*

(2) *There exists an exact sequence of $R$-modules*

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

*such that* $\{\mathrm{pd}_R(E_i) : i \in \mathbb{Z}\}$ *is a bounded set and* $\mathrm{Ext}_R^{k+1}(M_i, Q) = 0$ *for each integer* $k \geq n$*, each* $M_i := \mathrm{Im}(d_i)$ *and each* $R$*-module* $Q$ *with finite projective dimension.*

*Proof.* (1)⇒(2). Assume that $M$ is Gorenstein $n$-projective. Then there exists an exact sequence of $R$-modules

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

such that $\{\mathrm{pd}_R(E_i) : i \in \mathbb{Z}\}$ is a bounded set and $\mathrm{Ext}_R^{k+1}(M_i, P) = 0$ for each integer $k \geq n$, each $M_i := \mathrm{Im}(d_i)$ and each projective $R$-module $P$. Now, let $Q$ be an $R$-module of finite projective dimension $r$. We use induction on $r$. By (1), the property is true when $r = 0$. Assume that $r \geq 1$ and let $0 \to H \to P \to Q \to 0$ be an exact sequence of $R$-modules such that $P$ is projective over $R$. Considering the next portion of its associated long exact sequence yields, by inductive assumptions since $\mathrm{pd}_R(H) = r - 1$,

$$\mathrm{Ext}_R^{k+1}(M_i, P) = 0 \to \mathrm{Ext}_R^{k+1}(M_i, Q) \to \mathrm{Ext}_R^{k+2}(M_i, H) = 0$$

for each integer $k \geq n$ and each integer $i$. Then $\mathrm{Ext}_R^{k+1}(M_i, Q) = 0$ for each integer $k \geq n$ and each integer $i$, establishing (2).

(2)⇒(1) is straightforward, completing the proof. ∎

Next, we formulate one of the main theorems of this section. It states that the generalized Gorenstein projective dimension and the Gorenstein projective dimension coincide. Also, it highlights the relation between [12, Lemma 2.13], which approximates a module of finite Gorenstein projective dimension by a module of finite projective dimension, and our Gorenstein $n$-projective modules.

THEOREM 2.4. *Let* $n \geq 0$ *be an integer and* $M$ *be an* $R$*-module. Then the following statements are equivalent:*

(1) $\mathrm{Gpd}_R(M) \leq n$.
(2) *There exists an exact sequence of* $R$*-modules* $0 \to M \to E \to G \to 0$ *such that* $\mathrm{pd}_R(E) \leq n$ *and* $G$ *is a Gorenstein projective* $R$*-module.*
(3) *There exists an exact sequence of* $R$*-modules*

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

*such that* $M = \mathrm{Im}(d_0)$, $\mathrm{pd}_R(E_i) \leq n$ *and* $\mathrm{Ext}_R^{n+1}(M_i, Q) = 0$ *for each* $M_i := \mathrm{Im}(d_i)$*, each integer* $i$*, and each projective* $R$*-module* $Q$*.*
(4) $\mathrm{GGpd}_R(M) \leq n$.
(5) $M$ *is a Gorenstein* $n$*-projective module.*

*Consequently,* $\mathrm{GGpd}_R(M) = \mathrm{Gpd}_R(M)$.

*Proof.* (1)⇔(2). This is straightforward by [12, Lemma 2.13], [23, Theorem 2.20] and [23, Theorem 2.24].

(2)⇒(3). If (2) holds, then $\mathrm{Gpd}_R(M) \leq n$, and thus $\mathrm{Ext}_R^{k+1}(M,P) = 0$ for each integer $k \geq n$ and each projective module $P$. Let $\cdots \to P_2 \to P_1 \to M \to 0$ be a projective resolution of $M$ and let $M_i := \mathrm{Im}(P_{i+1} \to P_i)$ for each integer $i \geq 1$. Then it is readily checked that $\mathrm{Ext}_R^{k+1}(M_i, Q) = 0$ for each integer $k \geq n$, each integer $i \geq 1$, and each projective $R$-module $Q$. Moreover, as $G$ is Gorenstein projective, there exists an exact sequence $0 \to G \to P_{-1} \to P_{-2} \to \cdots$ of $R$-modules such that each $P_i$ is projective over $R$ and $\mathrm{Ext}_R^{k+1}(M_i, P) = 0$ for each $M_i := \mathrm{Im}(P_i \to P_{i-1})$ with $i \leq -1$ an integer, each integer $k \geq 0$ and each projective $R$-module $P$. Pasting the above three resolutions yields the desired exact sequence

$$\cdots \to P_2 \to P_1 \to E \to P_{-1} \to P_{-2} \to \cdots .$$

(3)⇒(4)⇔(5) are straightforward.

(5)⇒(1). Assume that $M$ is a Gorenstein $n$-projective $R$-module. Then there exists a complete $n$-projective resolution

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

such that $M = \mathrm{Im}(d_0)$. Let $M_i := \mathrm{Im}(d_i)$ with $M = M_0$ for each integer $i$. Let $r := \sup\{\mathrm{pd}_R(E_i) : i \in \mathbb{Z}\}$ and $m := \max\{n, r\}$. By hypothesis, $r$ and $m$ are positive integers. Fix an integer $i$ and consider the short exact sequence $0 \to M_{i+1} \to E_i \to M_i \to 0$ and the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'_{i+1} & \to & P_i & \to & M'_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{i+1,m-1} & \to & P_{i+1,m-1} \oplus P_{i,m-1} & \to & P_{i,m-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{i+1,0} & \to & P_{i+1,0} \oplus P_{i,0} & \to & P_{i,0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{i+1} & \to & E_i & \to & M_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the $P_{i,j}, P_{i+1,j}$ are projective modules. As $\mathrm{pd}_R(E_i) \leq m$, $P_i$ is a projective $R$-module. Moreover, as $n \leq m$ and $M_i, M_{i+1}$ are Gorenstein $n$-projective,

$$\mathrm{Ext}^1_R(M'_{i+1}, Q) = \mathrm{Ext}^{m+1}_R(M_{i+1}, Q) = 0,$$
$$\mathrm{Ext}^1_R(M'_i, Q) = \mathrm{Ext}^{m+1}_R(M_i, Q) = 0,$$

for each projective module $Q$. It follows that the derived exact sequence

$$\mathbf{P} = \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \to \cdots$$

is a complete projective resolution and thus each $M'_i$ is Gorenstein projective over $R$. Hence $\mathrm{Gpd}_R(M) \le m$ since $0 \to M'_0 \to P_{0,m-1} \to P_{0,m-2} \to \cdots \to P_{0,0} \to M_0 = M \to 0$ is an exact sequence with $M'_0$ Gorenstein projective. Therefore, by [23, Theorem 2.20],

$$\mathrm{Gpd}_R(M) = \sup\{k \in \mathbb{N} : \mathrm{Ext}^k_R(M, P) \neq 0 \text{ for some projective } R\text{-module } P\} \le n,$$

as $M$ is Gorenstein $n$-projective, completing the proof. ∎

The following result simplifies the definition of a complete $n$-projective resolution.

Corollary 2.5. *Let $n \ge 0$ be an integer and $\mathbf{E} = \cdots \to E_2 \to E_1 \to E_0 \to E_{-1} \to \cdots$ be an exact sequence of $R$-modules with $M_i := \mathrm{Im}(E_i \to E_{i-1})$, $i \in \mathbb{Z}$. Then $\mathbf{E}$ is a complete $n$-projective resolution if and only if $\mathrm{pd}_R(E_i) \le n$ and $\mathrm{Ext}^{n+1}_R(M_i, P) = 0$ for each integer $i \in \mathbb{Z}$ and each projective $R$-module $P$.*

*Proof.* Assume that $\mathbf{E}$ is a complete $n$-projective resolution. Fix an integer $i$ and consider the short exact sequence $0 \to M_i \to E_i \to M_{i-1} \to 0$. Note that $\mathrm{pd}_R(E_i) = \mathrm{Gpd}_R(E_i) \le \max\{\mathrm{Gpd}_R(M_i), \mathrm{Gpd}_R(M_{i-1})\}$. Then, by Theorem 2.4, $\mathrm{pd}_R(E_i) \le n$, as desired. ∎

It is well known that to link the Gorenstein projective dimension of a module $M$ to the vanishing of the functor Ext, one has to assume finiteness of $\mathrm{Gpd}_R(M)$ (see [23, Theorem 2.20]). In light of Theorem 2.4, which now guarantees the equality $\mathrm{GGpd}_R(M) = \mathrm{Gpd}_R(M)$ for any module $M$, a new interpretation of [23, Theorem 2.20] arises. In fact, it turns out that the latter serves to compute the Gorenstein projective dimension of modules emerging as images of complete $n$-projective resolutions, as is noted next.

Corollary 2.6. *Let $M$ be an $R$-module. If $M$ is a generalized Gorenstein projective module, then*

$$\mathrm{Gpd}_R(M) = \sup\{n \in \mathbb{N} : \mathrm{Ext}^n_R(M, P) \neq 0 \textit{ for some projective } R\textit{-module } P\}.$$

Assertion (2) of Theorem 2.4 makes it legitimate to introduce the following concept.

Definition 2.7. Let $M$ be an $R$-module of finite Gorenstein projective dimension. An $R$-module $E$ is said to be a *homological associate* to $M$ if

$E$ is of finite projective dimension and there exists an exact sequence $0 \to M \to E \to G \to 0$ with $G$ Gorenstein projective.

Homological associates for modules of finite Gorenstein injective dimension are defined dually. Next, we seek common properties shared by modules of finite Gorenstein projective dimension and their homological associates. Our investigation on homological associates sheds a new light on the strong correlation between the Auslander–Buchsbaum and Auslander–Bridger formulas. In fact, the latter turns out to be a consequence of the former. That is the object of the theorem.

Remark 2.8. Let $M$ be an $R$-module of finite Gorenstein projective dimension. It is easy to check that if $E$ is a homological associate to $M$, then $\mathrm{pd}_R(E) = \mathrm{Gpd}_R(M)$.

Theorem 2.9. *Let $R$ be a commutative Noetherian ring and $M$ a finitely generated nonzero $R$-module of finite Gorenstein projective dimension. Then:*

(1) *The module $M$ admits a finitely generated homological associate $E$.*
(2) *Moreover, if $R$ is local, then any finitely generated homological associate $E$ to $M$ satisfies* $\mathrm{depth}(E) = \mathrm{depth}(M)$.

*Proof.* First, recall that if $R$ is a commutative Noetherian local ring such that $\mathrm{depth}(R) = 0$ and if $M$ is a finitely generated $R$-module, then $M^+ := \mathrm{Hom}_R(M, R) = 0$ if and only if $M = 0$ [25, Lemma 4.1].

(1) Assume that $\mathrm{Gpd}_R(M) = n \in \mathbb{N}$. By [23, Remark 2.12 and Theorem 2.13], there exists an exact sequence of $R$-modules $0 \to K \to D \to M \to 0$ such that $D$ is a finitely Gorenstein projective module and $K$ is a finitely generated $R$-module with $\mathrm{pd}_R(K) = n - 1$. Thus, as $R$ is Noetherian, there exists a finitely generated projective $R$-module $P$ and a finitely generated Gorenstein projective $R$-module $G$ such that the sequence $0 \to D \to P \to G \to 0$ is exact (see [22, Thereom 4.46]). Now, consider the pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & D & \to & P & \to & G & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \dashrightarrow & E & \to & G & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Then $E$ is finitely generated and $\mathrm{pd}_R(E) \leq 1 + \mathrm{pd}_R(K) \leq n$. Hence $E$ is a finitely generated homological associate to $M$, as desired.

(2) The argument uses induction on $\mathrm{depth}(R)$. Let $E$ be a finitely generated homological associate to $M$ and let $0 \to M \to E \to G \to 0$ be an exact sequence of $R$-modules such that $G$ is a (finitely generated) Gorenstein projective $R$-module. Assume that $\mathrm{depth}(R) = 0$. As $E$ is finitely generated over $R$ and $\mathrm{pd}_R(E) = \mathrm{Gpd}_R(M) < \infty$, we infer, by the Auslander–Buchsbaum formula, that $E$ is projective over $R$ and $\mathrm{depth}(E) = 0$, and thus $M$ is Gorenstein projective over $R$.

Assume by way of contradiction that $\mathrm{depth}(M) \geq 1$ and let $x$ be a nonunit element of $R \setminus Z(M)$. Applying the functor $\mathrm{Hom}_R(-, R)$ to the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ yields the exact sequence

$$0 \to (M/xM)^+ \to M^+ \xrightarrow{x} M^+ \to \mathrm{Ext}^1_R(M/xM, R) \to 0.$$

A second application of $\mathrm{Hom}_R(-, R)$ gives the following sequence of $R$-modules:

$$0 \to \mathrm{Ext}^1_R(M/xM, R)^+ \to M^{++} \xrightarrow{x} M^{++}.$$

Now, since $M$ is Gorenstein projective, we get the exact sequence of $R$-modules

$$0 \to \mathrm{Ext}^1_R(M/xM, R)^+ \to M \xrightarrow{x} M.$$

Hence $\mathrm{Ext}^1_R(M/xM, R)^+ = 0$, as $x \notin Z(M)$, yielding $\mathrm{Ext}^1_R(M/xM, R) = 0$ by (the above-mentioned) [25, Lemma 4.1]. Therefore $0 \to (M/xM)^+ \to M^+ \xrightarrow{x} M^+ \to 0$ is exact, so that $M^+ = xM^+$. As $M$ is finitely generated over $R$, and thus $M^+$ is finitely generated over $R$, by Nakayama's lemma, we get $M^+ = 0$, yielding $M = 0$, by [25, Lemma 4.1], which is absurd.

It follows that $\mathrm{depth}(M) = 0 = \mathrm{depth}(E)$, as desired. Now, suppose that $\mathrm{depth}(R) \geq 1$. If $\mathrm{depth}(M) = 0$, then $\mathrm{depth}(E) = 0$. Assume that $\mathrm{depth}(M) \geq 1$. Hence, there exists a nonunit element $x \in R \setminus Z(M) \cup Z(R)$. As $G$ is Gorenstein projective, thus a submodule of a projective module, we get $x \notin Z(G)$. It follows, by tensoring the above sequence by $R/xR$, that

$$0 \to M/xM \to E/xE \to G/xG \to 0$$

is an exact sequence of $R/xR$-modules with $G/xG$ Gorenstein projective over $R/xR$ and $\mathrm{pd}_{R/xR}(E/xE) = \mathrm{pd}_R(E)$ [28, Theorem E, p. 124] since $x \notin Z(E)$. Hence, as $\mathrm{depth}(R/xR) = \mathrm{depth}(R) - 1$, we get, by induction,

$$\mathrm{depth}_{R/xR}(M/xM) = \mathrm{depth}_{R/xR}(E/xE).$$

It follows that $\mathrm{depth}(M) = \mathrm{depth}(E)$, completing the proof. ■

Theorem 2.9 allows us to give an alternative proof to the Auslander–Bridger formula. Recall that this formula is the Gorenstein version of the well-known Auslander–Buchsbaum equality [10, p. 13].

Corollary 2.10 (Auslander–Bridger's theorem). *Let $R$ be a local Noetherian ring. Let $M$ be a nonzero finitely generated $R$-module such that $\mathrm{Gpd}_R(M) < \infty$. Then*

$$\mathrm{Gpd}_R(M) + \mathrm{depth}(M) = \mathrm{depth}(R).$$

*Proof.* By Theorem 2.9, there exists a finitely generated homological associate $E$ to $M$ which satisfies $\mathrm{depth}(M) = \mathrm{depth}(E)$. Applying the Auslander–Buchsbaum formula, we get $\mathrm{pd}_R(E) + \mathrm{depth}(E) = \mathrm{depth}(R)$. It follows that $\mathrm{Gpd}_R(M) + \mathrm{depth}(M) = \mathrm{depth}(R)$, as contended. ∎

Note that, by definition, the images (or kernels) of the differentials of a complete Gorenstein projective resolution are all Gorenstein projective, that is, their Gorenstein projective dimension is zero. This is no longer the case for a complete Gorenstein $n$-projective resolution (for some positive integer $n$) within which these images might have different Gorenstein projective dimensions. So, it is legitimate to seek relations between the Gorenstein projective dimensions of the different differential images arising from a complete $n$-projective resolution $\mathbf{E} = \cdots \to E_2 \to E_1 \to E_0 \to E_{-1} \to E_{-2} \to \cdots$. This is the goal of the next result.

Theorem 2.11.

(1) *Let $0 \to N \to E \to M \to 0$ be an exact sequence of $R$-modules.*
   (a) *If $\mathrm{Gpd}_R(N) > \mathrm{Gpd}_R(E)$, then $\mathrm{Gpd}_R(M) > \mathrm{Gpd}_R(E)$.*
   (b) *If $\mathrm{Gpd}_R(M) > \mathrm{Gpd}_R(E)$, then $\mathrm{Gpd}_R(M) = 1 + \mathrm{Gpd}_R(N)$.*

(2) *Let $n \geq 0$ be an integer and let*

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

*be a complete $n$-projective resolution. Let $M_i := \mathrm{Im}(d_i)$ for each integer $i$.*

   (a) *For a fixed integer $i$, consider the short exact sequence*

$$0 \to M_{i+1} \to E_i \xrightarrow{d_i} M_i \to 0.$$

   (i) *If $\mathrm{Gpd}_R(M_i) \leq \mathrm{pd}_R(E_i)$, then*

$$\max\{\mathrm{Gpd}_R(M_i), \mathrm{Gpd}_R(M_{i+1})\} = \mathrm{pd}_R(E_i).$$

   (ii) *If $\mathrm{Gpd}_R(M_i) > \mathrm{pd}_R(E_i)$, then $\mathrm{Gpd}_R(M_i) = 1 + \mathrm{Gpd}_R(M_{i+1})$.*

   (b) $\sup\{\mathrm{Gpd}_R(M_i) : i \in \mathbb{Z}\} = \sup\{\mathrm{pd}_R(E_i) : i \in \mathbb{Z}\}$.

*Proof.* (1) (a) Assume that $\mathrm{Gpd}_R(N) > \mathrm{Gpd}_R(E)$. First, note that $\mathrm{Gpd}_R(E)$ is finite and let $r := \mathrm{Gpd}_R(E)$. Then $\mathrm{Gpd}_R(M)$ and $\mathrm{Gpd}_R(N)$ are simultaneously finite. Suppose that $\mathrm{Gpd}_R(N)$ is finite and, by way of contradiction, that $\mathrm{Gpd}_R(M) \leq r$. Considering the next portion of the associated long exact sequence

$$\mathrm{Ext}_R^{r+1}(E,P)=0 \to \mathrm{Ext}_R^{r+1}(N,P) \to \mathrm{Ext}_R^{r+2}(M,P)=0$$

yields $\mathrm{Ext}_R^{r+1}(N,P)=0$ for each projective module $P$. Hence $\mathrm{Gpd}_R(N)\leq r$, which is absurd. Therefore $\mathrm{Gpd}_R(M)>r$ as desired.

(b) Assume that $\mathrm{Gpd}_R(M)>\mathrm{Gpd}_R(E)$. As noted above, $\mathrm{Gpd}_R(E)$ is finite, and so let $r:=\mathrm{Gpd}_R(E)$. Then $\mathrm{Gpd}_R(M)$ and $\mathrm{Gpd}_R(N)$ are simultaneously finite. Suppose that $\mathrm{Gpd}_R(M)=:m$ and $\mathrm{Gpd}_R(N)=:n$ are finite. Hence, considering the long exact sequence associated to the above exact sequence yields easily $n\leq m-1$ and $m\leq 1+\max\{n,r\}$. If $n<r$, then the following portion of the long exact sequence gives, for each projective module $P$,

$$\mathrm{Ext}_R^{r}(N,P)=0 \to \mathrm{Ext}_R^{r+1}(M,P) \to \mathrm{Ext}_R^{r+1}(E,P)=0,$$

so that $\mathrm{Ext}_R^{r+1}(M,P)=0$ for each projective $R$-module $P$. Hence $m\leq r$, which is absurd. It follows that $n\geq r$, yielding $m=1+n$, as desired.

(2) (a) (i) If $\mathrm{Gpd}_R(M_i)\leq \mathrm{pd}_R(E_i)$, then, by (1)(a), $\mathrm{Gpd}_R(M_{i+1})\leq \mathrm{pd}_R(E_i)$, so that $\sup\{\mathrm{Gpd}_R(M_i),\ \mathrm{Gpd}_R(M_{i+1})\}\leq \mathrm{pd}_R(E_i)$. The inverse inequality is easy.

(ii) Follows from (1)(b).

(b) Note that, by Theorem 2.4, the two terms are finite and less than $n$. Also, as $\mathrm{pd}_R(E_i)\leq \max\{\mathrm{Gpd}_R(M_i),\mathrm{Gpd}_R(M_{i+1})\}$ for each integer $i$, we get

$$w:=\sup\{\mathrm{pd}_R(E_i) : i\in\mathbb{Z}\}\leq \sup\{\mathrm{Gpd}_R(M_i) : i\in\mathbb{Z}\}.$$

Now, assume that $\sup\{\mathrm{Gpd}_R(M_i) : i\in\mathbb{Z}\}>w$ and let $j$ be an integer such that $\mathrm{Gpd}_R(M_j)>w$. Then, in particular, $\mathrm{Gpd}_R(M_j)>\mathrm{pd}_R(E_{j-1})$. Hence, by (1)(a), $\mathrm{Gpd}_R(M_{j-1})>\mathrm{pd}_R(E_{j-1})$, so that, by (1)(b),

$$\mathrm{Gpd}_R(M_{j-1})=1+\mathrm{Gpd}_R(M_j)>1+w.$$

Iterating the above process, one easily proves that, for each integer $k\leq j-1$,

$$\mathrm{Gpd}_R(M_k)>j-k+w.$$

It follows that $\sup\{\mathrm{Gpd}_R(M_i) : i\in\mathbb{Z}\}=\infty$, which is absurd. ∎

The following corollary gives mild conditions for a module to be Gorenstein projective.

Corollary 2.12. *Let $n\geq 0$ be an integer and let* $\mathbf{P}=\cdots\to P_2\xrightarrow{d_2} P_1\xrightarrow{d_1} P_0\xrightarrow{d_0} P_{-1}\xrightarrow{d_{-1}} P_{-2}\to\cdots$ *be a complete $n$-projective resolution such that each $P_i$ is projective. Then* $\mathbf{P}$ *is a complete projective resolution.*

*Proof.* Apply Theorem 2.11(2)(b). ∎

Given a positive integer $n$, consider a complete $n$-projective resolution

$$\mathbf{E}=\cdots\to E_2\xrightarrow{d_2} E_1\xrightarrow{d_1} E_0\xrightarrow{d_0} E_{-1}\xrightarrow{d_{-1}} E_{-2}\to\cdots.$$

By shifting and summing we get the periodic exact sequence

$$\cdots \xrightarrow{\oplus d_i} \bigoplus_{i\in\mathbb{Z}} E_i \xrightarrow{\oplus d_i} \bigoplus_{i\in\mathbb{Z}} E_i \xrightarrow{\oplus d_i} \bigoplus_{i\in\mathbb{Z}} E_i \xrightarrow{\oplus d_i} \cdots$$

It is easy to check that this sequence is a complete $n$-projective resolution as well. This shows that any complete $n$-projective resolution is a direct summand of such a periodic complete $n$-projective resolution.

The following result determines exactly the Gorenstein projective dimension of modules that emerge as images of such periodic exact sequences.

Corollary 2.13. *Let $n$ be a positive integer and*

$$\cdots \xrightarrow{d} E \xrightarrow{d} E \xrightarrow{d} E \xrightarrow{d} E \xrightarrow{d} \cdots$$

*be a complete $n$-projective resolution. Let $M := \mathrm{Im}(d)$. Then*

$$\mathrm{Gpd}_R(M) = \mathrm{pd}_R(E).$$

*Moreover, if $\mathrm{fd}_R(M) < \infty$, then*

$$\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M) = \mathrm{pd}_R(E).$$

*Proof.* The first statement is a direct consequence of Theorem 2.11(2)(b). The second statement holds via [9, Theorem 2.3]. ∎

Next, we denote by

$$\mathrm{FFD}(R) := \sup\{\mathrm{fd}_R(M) : M \text{ is an } R\text{-module of finite flat dimension}\}$$

the *finitistic flat dimension* of $R$.

Corollary 2.14. *Let $R$ be a ring. If $FFD(R) < \infty$ and if $M$ is an $R$-module such that $fd_R(M) < \infty$, then $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$.*

*Consequently, if $\mathrm{wgldim}(R) < \infty$, then $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ for each $R$-module $M$.*

*Proof.* Let $M$ be an $R$-module such that $\mathrm{fd}_R(M) < \infty$. We are done if $\mathrm{Gpd}_R(M) = \infty$. Now assume that $\mathrm{Gpd}_R(M) =: n < \infty$, that is, $M$ is Gorenstein $n$-projective. Let $\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$ be a complete $n$-projective resolution such that $M = \mathrm{Im}(d_0)$. Let $M_i := \mathrm{Im}(d_i)$ for each integer $i$ with $M = M_0$. It is easily seen that, as $\mathrm{fd}_R(M) < \infty$, $\mathrm{fd}_R(M_i) < \infty$ for each integer $i$. Since $\mathrm{FFD}(R) < \infty$, it follows that $\mathrm{fd}_R(\bigoplus_{i\in\mathbb{Z}} M_i) < \infty$. Considering the derived exact sequence

$$\cdots \to \bigoplus_{i\in\mathbb{Z}} E_i \to \bigoplus_{i\in\mathbb{Z}} E_i \to \bigoplus_{i\in\mathbb{Z}} E_i \to \cdots$$

it follows, by Corollary 2.13, that

$$\mathrm{Gpd}_R\Big(\bigoplus_{i\in\mathbb{Z}} M_i\Big) = \mathrm{pd}_R\Big(\bigoplus_{i\in\mathbb{Z}} M_i\Big) = \mathrm{pd}_R\Big(\bigoplus_{i\in\mathbb{Z}} E_i\Big) < \infty,$$

so that $\mathrm{pd}_R(M_i) < \infty$ for each integer $i$, and in particular $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$, as desired. ∎

Next, we record the dual results to Theorem 2.4 and Corollary 2.6 for the Gorenstein injective dimension.

THEOREM 2.15. *Let $n \geq 0$ be an integer and $M$ be an $R$-module. Then the following statements are equivalent:*

(1) $\mathrm{Gid}_R(M) \leq n$.
(2) *There exists an exact sequence of $R$-modules $0 \to G \to E \to M \to 0$ such that $\mathrm{id}_R(E) \leq n$ and $G$ is a Gorenstein injective $R$-module.*
(3) *There exists an exact sequence of $R$-modules*

$$\mathbf{E} = \cdots \to E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \to \cdots$$

*such that $M = \mathrm{Im}(d_0)$, $\mathrm{id}_R(E_i) \leq n$ and $\mathrm{Ext}_R^{n+1}(I, M_i) = 0$ for each $M_i := \mathrm{Im}(d_i)$, each integer $i$, and each injective $R$-module $I$.*
(4) $\mathrm{GGid}_R(M) \leq n$.
(5) *$M$ is a Gorenstein $n$-injective module.*

*Consequently,* $\mathrm{GGid}_R(M) = \mathrm{Gid}_R(M)$.

COROLLARY 2.16. *Let $n \geq 0$ be an integer and $M$ an $R$-module. If $M$ is a generalized Gorenstein injective module, then*

$$\mathrm{Gid}_R(M) = \sup\{n \in \mathbb{N} : \mathrm{Ext}_R^n(I, M) \neq 0 \textit{ for some injective } R\textit{-module } I\}.$$

The following result highlights the close relation between the generalized Gorenstein projective modules and Gorenstein $n$-projective modules (for some positive integer $n$), on the one hand, and the known Gorenstein projective modules, on the other. More precisely, we give the corresponding versions of Theorem 2.5, Corollary 2.11 and Proposition 2.18 of [23] for the new invariants.

THEOREM 2.17. *Let $n \geq 0$ be an integer. Then the class $\mathcal{GGP}(R)$ (resp., $\mathcal{G}_n\mathcal{P}(R)$) of all generalized Gorenstein projective $R$-modules (resp., Gorenstein $n$-projective $R$-modules) is projectively resolving. Furthermore $\mathcal{GGP}(R)$ (resp., $\mathcal{G}_n\mathcal{P}(R)$) is closed under arbitrary direct sums and under direct summands.*

*Proof.* [23, Theorem 2.24] along with Theorem 2.4 prove that $\mathcal{GGP}(R)$ is projectively resolving. Also, a direct proof shows that $\mathcal{GGP}(R)$ and $\mathcal{G}_n\mathcal{P}(R)$ are stable under direct sums. Now, let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R$-modules with $M''$ Gorenstein $n$-projective. By [23, Theorem 2.24], $\mathrm{Gpd}_R(M) < \infty$ if and only if $\mathrm{Gpd}_R(M') < \infty$. Moreover, consider the next portion of the associated long exact sequence

$$\operatorname{Ext}_R^{n+i}(M'',Q)=0\to \operatorname{Ext}_R^{n+i}(M,Q)\to \operatorname{Ext}_R^{n+i}(M',Q)\\ \to \operatorname{Ext}_R^{n+i+1}(M'',Q)=0$$

for each projective $R$-module $Q$ and each integer $i\geq 1$. Then $\operatorname{Ext}_R^{n+i}(M,Q)\cong \operatorname{Ext}_R^{n+i}(M',Q)$ for each projective $R$-module $Q$ and each integer $i\geq 1$. It follows, by Theorem 2.4, that $M$ is Gorenstein $n$-projective if and only if so is $M'$. Hence $\mathcal{G}_n\mathcal{P}(R)$ is projectively resolving. Finally, the Eilenberg swindle [23, Proposition 1.4] shows that $\mathcal{GGP}(R)$ and $\mathcal{G}_n\mathcal{P}(R)$ are both closed under direct summands, as desired. ∎

PROPOSITION 2.18. *Let $n\geq 0$ be an integer and let $0\to K\to G\to M\to 0$ be an exact sequence of $R$-modules, where $G$ is a Gorenstein $n$-projective module. Then either the three modules are Gorenstein $n$-projective, or* $\operatorname{Gpd}_R(M)=1+\operatorname{Gpd}_R(K)$.

*Proof.* If $M$ is Gorenstein $n$-projective, then, by Theorem 2.17, $K$ is Gorenstein $n$-projective. Now, assume that $M$ is not Gorenstein $n$-projective. Then, by Theorem 2.4, $\operatorname{Gpd}_R(M)\geq n+1$, so that, by Theorem 2.11(2)(b), $\operatorname{Gpd}_R(M)=1+\operatorname{Gpd}_R(K)$, as desired. ∎

COROLLARY 2.19. *Let $n\geq 0$ be an integer and let $0\to G'\to G\to M\to 0$ be an exact sequence of $R$-modules, where $G'$ and $G$ are Gorenstein $n$-projective modules and $\operatorname{Ext}_R^{r+1}(M,P)=0$ for each projective module $P$ with $r:=\max\{\operatorname{Gpd}_R(G'),\operatorname{Gpd}_R(G)\}$. Then $M$ is Gorenstein $n$-projective.*

*Proof.* First, it is easy to check that $\operatorname{Ext}_R^{k+1}(M,P)=0$ for each projective module $P$ and each integer $k\geq r$. Also, note that $\operatorname{Gpd}_R(M)<\infty$. Then, by [23, Theorem 2.20],

$$\operatorname{Gpd}_R(M)=\sup\{k\in\mathbb{N}:\operatorname{Ext}_R^k(M,P)\neq 0 \text{ for some projective } R\text{-module } P\}.$$

Hence $\operatorname{Gpd}_R(M)\leq r\leq n$, that is, $M$ is Gorenstein $n$-projective, completing the proof. ∎

**3. Gorenstein global dimension.** This section is devoted to some applications of the results of Section 2. We give alternative and short proofs of many known theorems such as the main theorems of [23] and the main theorem of [7] concerning the Gorenstein global dimension. Also, we recover a recent theorem of Emmanouil [15, Theorem 4.1] and compute the Gorenstein global dimension for various types of rings such as $\aleph_0$-Noetherian rings and group rings.

Recall that a *Gorenstein ring* is a commutative Noetherian ring of finite self-injective dimension. In the noncommutative setting, a ring $R$ is defined to be *Iwanaga–Gorenstein* if it is left and right Noetherian and has finite left and right injective dimensions [19, Definition 9.1.1].

First, we give an easy proof of a theorem of Holm [23, Theorem 2.28], which equates the finitistic Gorenstein projective dimension $\mathrm{FGPD}(R) := \sup\{\mathrm{Gpd}_R(M) : M \text{ is an } R\text{-module of finite Gorenstein projective dimension}\}$ (resp., $\mathrm{FGID}(R) := \sup\{\mathrm{Gid}_R(M) : M$ is an $R$-module of finite Gorenstein injective dimension$\}$) with the well known finitistic projective dimension $\mathrm{FPD}(R)$ (resp., finitistic injective dimension $\mathrm{FID}(R)$).

THEOREM 3.1. *Let $R$ be a ring. Then*

$$\mathrm{FGPD}(R) = \mathrm{FPD}(R), \qquad \mathrm{FGID}(R) = \mathrm{FID}(R).$$

*Proof.* Clearly, $\mathrm{FPD}(R) \leq \mathrm{GFPD}(R)$. Let $M$ be an $R$-module such that $\mathrm{Gpd}_R(M) < \infty$. Then, by Theorem 2.4, $M$ admits a homological associate $E$, and thus $\mathrm{pd}_R(E) = \mathrm{Gpd}_R(M) < \infty$. Hence $\mathrm{Gpd}(M) \leq \mathrm{FPD}(R)$, so that $\mathrm{GFPD}(R) \leq \mathrm{FPD}(R)$ and equality holds. ∎

In [10, Theorem 6.3.2], Christensen proved that if $(R, m, k)$ is a commutative local (Noetherian) Cohen–Macaulay ring with a dualizing module such that $\mathrm{Gid}_R(R) < \infty$, then $\mathrm{id}_R(R) < \infty$, that is, $R$ is a Gorenstein ring. The main theorem of [24], namely [24, Theorem 2.1], generalized this result by proving, in fact, that for an arbitrary ring $R$, if $M$ is an $R$-module such that $\mathrm{pd}_R(M) < \infty$, then $\mathrm{Gid}_R(M) = \mathrm{id}_R(M)$. Next, we recover concisely this theorem as well as its dual [24, Theorem 2.2] for the Gorenstein projective dimension.

First, it is worth noticing, as is pointed out in [12, Lemma 3.1], that if $G$ is a Gorenstein projective $R$-module and $H$ is an $R$-module with $n := \mathrm{id}_R(H) < \infty$, then $\mathrm{Ext}^1_R(G, H) = 0$ since $G$ is the $n$th syzygy of a projective resolution of some module $K$.

THEOREM 3.2. *Let $M$ be an $R$-module such that* $\min\{\mathrm{pd}_R(M), \mathrm{id}_R(M)\} < \infty$. *Then*

$$\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M), \qquad \mathrm{Gid}_R(M) = \mathrm{id}_R(M).$$

*Proof.* It suffices to prove that $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ since a similar argument yields the second equality. If $\mathrm{Gpd}_R(M) = \infty$, the desired equality easily holds. If $\mathrm{id}_R(M) = \infty$, then $\mathrm{pd}_R(M) = \min\{\mathrm{pd}_R(M), \mathrm{id}_R(M)\} < \infty$, yielding easily $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$. Now, assume that $\mathrm{Gpd}_R(M) = n < \infty$ and $\mathrm{id}_R(M) < \infty$. Then, by Theorem 2.4, $M$ admits a homological associate $E$ and thus there exists an exact sequence $0 \to M \to E \to G \to 0$ such that $\mathrm{pd}_R(E) = n$ and $G$ is Gorenstein projective. Hence, as $\mathrm{id}_R(M) < \infty$, we have $\mathrm{Ext}^1_R(G, M) = 0$. It follows that the above sequence splits and thus $\mathrm{pd}_R(M) \leq n$, yielding $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$, as desired. ∎

In [5], A. Beligiannis and I. Reiten define the following concepts for an abelian category $\mathcal{C}$ with enough injective and projective objects:

$$\operatorname{spli}(\mathcal{C}) = \sup\{\operatorname{pd}(I) : I \text{ is an injective object of } \mathcal{C}\},$$
$$\operatorname{silp}(\mathcal{C}) = \sup\{\operatorname{id}(P) : P \text{ is a projective object of } \mathcal{C}\}.$$

The abelian category $\mathcal{C}$ is said to be *Gorenstein* if $\operatorname{spli}(\mathcal{C}) < \infty$ and $\operatorname{silp}(\mathcal{C}) < \infty$ [5, Definition 2.1, p. 121].

Moreover, Beligiannis and Reiten [5] called a ring $R$ a *Gorenstein ring* if the category of left $R$-modules $\operatorname{Mod}(R)$ is a Gorenstein category, and checked that this definition agrees with the usual definition of a Gorenstein ring once restricted to left and right Noetherian rings.

On the other hand, recall that in [7], Bennis and Mahdou introduced the following invariants for a ring $R$:

$$\text{l-GPD}(R) := \sup\{\operatorname{Gpd}_R(M) : M \text{ is an } R\text{-module}\},$$
$$\text{l-GID}(R) := \sup\{\operatorname{Gid}_R(M) : M \text{ is an } R\text{-module}\}.$$

In this context, they prove [7, Theorem 2.1] that for an arbitrary ring $R$,

$$\text{l-GPD}(R) = \text{l-GID}(R),$$

and define the left Gorenstein global dimension, denoted by l-G-gldim$(R)$, to be this common value. Our next theorem, Theorem 3.3, recovers this result of [7] as well as a recent theorem of Emmanouil [15, Theorem 4.1].

Furthermore, given a ring $R$, one might consider the counterparts of the cohomological invariants l-spli$(R)$ and l-silp$(R)$ in Gorenstein homological algebra. The natural way to define these new invariants is the following:

$$\text{l-G-spli}(R) = \sup\{\operatorname{Gpd}_R(M) : M \text{ is a Gorenstein injective } R\text{-module}\},$$
$$\text{l-G-silp}(R) = \sup\{\operatorname{Gid}_R(M) : M \text{ is a Gorenstein projective } R\text{-module}\}.$$

It is obvious that $\text{l-spli}(R) \leq \text{l-G-spli}(R)$ and $\text{l-silp}(R) \leq \text{l-G-silp}(R)$ since the projective dimension and Gorenstein projective dimension (resp., injective dimension and Gorenstein injective dimension) coincide for an injective module $I$ (resp., for a projective module $P$). So, it is legitimate to wonder if $\text{l-silp}(R) = \text{l-G-silp}(R)$ and $\text{l-spli}(R) = \text{l-G-spli}(R)$. The next theorem and corollary shed more light on this issue as well as on the relations between these cohomological and Gorenstein cohomological invariants and the Gorenstein global dimensions l-GPD$(R)$ and l-GID$(R)$.

THEOREM 3.3. *Given a ring $R$, the following are identical:*

- l-GPD$(R)$,
- l-GID$(R)$,
- $\max\{\text{l-silp}(R), \text{l-spli}(R)\}$,
- $\max\{\text{l-G-silp}(R), \text{l-G-spli}(R)\}$.

*Proof.* We prove that l-GPD$(R) = \max\{$l-silp$(R)$, l-spli$(R)\}$. Let $n \geq 0$ be an integer. Assume that l-GPD$(R) \leq n$. Then $\mathrm{Gpd}_R(M) \leq n$ for each $R$-module $M$. Therefore, by [23, Theorem 2.20], $\mathrm{Ext}_R^{n+1}(M, P) = 0$ for each $R$-module $M$ and each projective module $P$. Hence $\mathrm{id}_R(P) \leq n$ for each projective $R$-module $P$, so that l-silp$(R) \leq n$. Further, by Theorem 3.2, $\mathrm{pd}_R(I) = \mathrm{Gpd}_R(I) \leq$ l-GPD$(R) \leq n$ for each injective $R$-module $I$, yielding l-spli$(R) \leq n$. It follows that $\max\{$l-silp$(R)$, l-spli$(R)\} \leq n$.

Now, assume that $\max\{$l-silp$(R)$, l-spli$(R)\} \leq n$. Let $M$ be an $R$-module. Let $\cdots \to P_1 \to P_0 \to M \to 0$ and $0 \to M \to I_0 \to I_1 \to \cdots$ be, respectively, a projective resolution and injective resolution of $M$. Pasting these two resolutions yields the exact sequence

$$\mathbf{E} = \cdots \to P_1 \to P_0 \to I_0 \to I_1 \to \cdots$$

with $\mathrm{Im}(P_0 \to I_0) = M$. Note that $\mathrm{pd}_R(I_j) \leq$ l-spli$(R) \leq n$ for each integer $j \geq 0$. Let $P$ be a projective $R$-module. Then $\mathrm{id}_R(P) \leq$ l-silp$(R) \leq n$, so that $\mathrm{Ext}_R^{n+1}(N, P) = 0$ for each $R$-module $N$. It follows that $\mathbf{E}$ is a complete $n$-projective resolution, and thus $M$ is Gorenstein $n$-projective. Hence, via Theorem 2.4, $\mathrm{Gpd}_R(M) \leq n$. Therefore l-GPD$(R) \leq n$. This establishes the claim. Now, as l-silp$(R) \leq$ l-G-silp$(R) \leq$ l-GPD$(R)$ and l-spli$(R) \leq$ l-G-spli$(R) \leq$ l-GPD$(R)$, we get

$$\text{l-GPD}(R) = \max\{\text{l-silp}(R), \text{l-spli}(R)\} = \max\{\text{l-G-silp}(R), \text{l-G-spli}(R)\}.$$

Using a similar argument one easily proves that

$$\text{l-GID}(R) = \max\{\text{l-silp}(R), \text{l-spli}(R)\},$$

as contended. ∎

Corollary 3.4. *Let $R$ be an arbitrary ring.*

(1) *Either* l-spli$(R)$ = l-G-spli$(R)$ *or* l-silp$(R)$ = l-G-silp$(R)$.
(2) *If* l-silp$(R)$ = l-spli$(R)$, *then* l-G-silp$(R)$ = l-G-spli$(R)$.
(3) *If both* l-G-silp$(R)$ *and* l-G-spli$(R)$ *are finite, then* l-G-silp$(R)$ = l-G-spli$(R)$.

*Proof.* (1) and (2) are obvious since

$$\text{l-silp}(R) \leq \text{l-G-silp}(R) \leq \text{l-G-gldim}(R),$$
$$\text{l-spli}(R) \leq \text{l-G-spli}(R) \leq \text{l-G-gldim}(R),$$
$$\text{l-G-gldim}(R) = \max\{\text{l-spli}(R), \text{l-silp}(R)\}.$$

(3) If both l-G-silp$(R)$ and l-G-spli$(R)$ are finite, then both l-silp$(R)$ and l-spli$(R)$ are finite, so that l-silp$(R)$ = l-spli$(R)$ and the desired equality follows from (2). ∎

In view of the Beligiannis–Reiten definition of a (not necessarily Noetherian) Gorenstein ring (see [5]), we get the following result.

Corollary 3.5. *A ring $R$ is Gorenstein if and only if* l-G-gldim$(R) < \infty$ *and* r-G-gldim$(R) < \infty$.

Next, we compute the Gorenstein global dimension of various types of rings. Recall that a *left* $\aleph_0$*-Noetherian ring* is a ring all of whose left ideals are countably generated.

Corollary 3.6. *If $R$ is an $\aleph_0$-Noetherian ring which is isomorphic to its opposite ring $R^{\mathrm{op}}$, then*

$$\text{G-gldim}(R) = \mathrm{silp}(R) = \mathrm{id}_R(R^{(\mathbb{N})})$$

*where $R^{(\mathbb{N})}$ is the free $R$-module with an infinite countable basis.*

*In particular, if $R$ is a commutative $\aleph_0$-Noetherian ring, then*

$$\text{G-gldim}(R) = \mathrm{silp}(R) = \mathrm{id}_R(R^{(\mathbb{N})}).$$

*Proof.* By [14, Corollary 3.7], $\mathrm{spli}(R) \le \mathrm{silp}(R)$. Then, in view of Theorem 3.3, G-gldim$(R) = \mathrm{silp}(R)$. Now, by [16, Corollary 3.2], $\mathrm{silp}(R) = \mathrm{id}_R(R^{(\mathbb{N})})$, as desired. ∎

Corollary 3.7. *Let $R$ be a commutative $\aleph_0$-Noetherian ring and $G$ be a group. Then*

$$\text{G-gldim}(RG) = \mathrm{silp}(RG).$$

*In particular,*

$$\text{G-gldim}(\mathbb{Z}G) = \mathrm{silp}(\mathbb{Z}G) = \mathrm{spli}(\mathbb{Z}G).$$

*Proof.* This follows from [14, Proposition 4.3 and Corollary 4.5] and Theorem 3.3. ∎

Corollary 3.8. *If $G$ is a finite group, then* G-gldim$(\mathbb{Z}G) = 1$.

*Proof.* Apply [14, Theorem 4.6] and Theorem 3.3. ∎

Ikenaga [26] introduced the generalized cohomological dimension $\underline{\mathrm{cd}}(G)$ of a group $G$ by defining $\underline{\mathrm{cd}}(G)$ to be the supremum of all integers $n$ for which there exist a $\mathbb{Z}$-free $\mathbb{Z}G$-module $M$ and a projective $\mathbb{Z}G$-module $P$ such that $\mathrm{Ext}^n_{\mathbb{Z}G}(M, P) \neq 0$. An interesting recent theorem of Emmanouil (see [14]) characterizes the finiteness of a group $G$ in terms of $\underline{\mathrm{cd}}(G)$ by proving that a group $G$ is finite if and only if $\underline{\mathrm{cd}}(G) = 0$ [14, Theorem 4.6]. The object of the next result is to link the cohomological invariant $\underline{\mathrm{cd}}(G)$ to the Gorenstein global dimension of $\mathbb{Z}G$.

Corollary 3.9. *Let $G$ be a group. Then:*

(1) $\underline{\mathrm{cd}}(G) \le$ G-gldim$(\mathbb{Z}G) \le \underline{\mathrm{cd}}(G) + 1$.
(2) *If* $\underline{\mathrm{cd}}(G) \le 1$, *then* G-gldim$(\mathbb{Z}G) = 1$.

*Proof.* Apply [14, Corollary 4.7] and Corollary 3.7. ∎

Next, we envisage to transfer to Gorenstein homological theory a classical theorem due to Auslander stating that the left and right global dimensions coincide for a left and right Noetherian ring $R$ [29, Corollary 9.23]. In effect, we prove that, when $R$ is left and right Noetherian, the left Gorenstein global dimension and right Gorenstein global dimension coincide, generalizing a theorem of Enochs and Jenda [19, Theorem 12.3.1 and Corollary 12.3.2], who established the equality in the restricted setting of Iwanaga–Gorenstein rings.

Recall that $\text{l-sfli}(R) := \sup\{\text{fd}_R(I) : I \text{ is an injective left } R\text{-module}\}$ and the (*left*) *Gorenstein weak global dimension* of $R$ is the invariant $\text{l-G-wgldim}(R) := \sup\{\text{Gfd}_R(M) : M \text{ is a left } R\text{-module}\}$.

Theorem 3.10. *The following are identical for a left and right Noetherian ring $R$:*

- $\text{l-G-gldim}(R)$,
- $\text{r-G-gldim}(R)$,
- $\text{l-G-wgldim}(R)$,
- $\text{r-G-wgldim}(R)$,
- $\max\{\text{l-silp}(R), \text{l-spli}(R)\}$,
- $\max\{\text{r-silp}(R), \text{r-spli}(R)\}$,
- $\max\{\text{l-silp}(R), \text{r-silp}(R)\}$,
- $\max\{\text{l-spli}(R), \text{r-spli}(R)\}$,
- $\max\{\text{l-sfli}(R), \text{r-sfli}(R)\}$,
- $\max\{\text{id}_R({}_RR), \text{id}_R(R_R)\}$.

First, we prove the following lemma.

Lemma 3.11. *Let $R$ be a left Noetherian ring. Then*

$$\text{l-silp}(R) \le \text{l-G-wgldim}(R).$$

*Proof.* Given an $R$-module $K$, we denote by $K^*$ the Pontryagin dual $\text{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$ of $K$. Let $n \ge 0$ be an integer and assume that $\text{l-G-wgldim}(R) \le n$. Let $P$ be a projective left $R$-module and $M$ be a finitely generated left $R$-module. Then, by [29, Theorem 9.51],

$$\text{Tor}_{n+1}^R(P^*, M) \cong \text{Ext}_R^{n+1}(M, P)^*.$$

As $\text{Gfd}_R(M) \le \text{l-G-wgldim}(R) \le n$, we get $\text{Tor}_{n+1}^R(P^*, M) = 0$ since $P^*$ is an injective right $R$-module. Hence $\text{Ext}_R^{n+1}(M, P)^* = 0$, yielding $\text{Ext}_R^{n+1}(M, P) = 0$ for each finitely generated left $R$-module $M$ and each projective left $R$-module $P$. It follows that $\text{id}_R(P) \le n$ for each projective left $R$-module $P$, and thus $\text{l-silp}(R) \le n$, establishing the desired result. ∎

*Proof of Theorem 3.10.* By [19, Proposition 9.1.2], $\text{id}_R(P) \le \text{id}_R({}_RR)$ for each projective left $R$-module $P$. Then $\text{l-silp}(R) \le \text{id}_R({}_RR) \le \text{l-silp}(R)$, yielding the equality l-silp$(R)$ = $\mathrm{id}_R({}_RR)$. Similarly, we get r-silp$(R)$ = $\mathrm{id}_R(R_R)$. Also, by [27, Proposition 1] or [13, Corollary 3.9], $\mathrm{id}({}_RR)$ = r-sfli$(R)$ and $\mathrm{id}(R_R)$ = l-sfli$(R)$. Therefore

$$\max\{\text{l-sfli}(R), \text{r-sfli}(R)\} = \max\{\mathrm{id}_R({}_RR), \mathrm{id}_R(R_R)\}.$$

Moreover, l-sfli$(R) = \mathrm{id}_R(R_R) \leq$ l-spli$(R)$. It follows that

$$\max\{\mathrm{id}_R({}_RR), \mathrm{id}_R(R_R)\} \leq \max\{\text{l-silp}(R), \text{l-spli}(R)\}.$$

Now, suppose that $\max\{\mathrm{id}_R({}_RR), \mathrm{id}_R(R_R)\} =: n < \infty$. Then l-sfli$(R) = \mathrm{id}_R(R_R) < \infty$, so that, by [14, Proposition 3.3 and Lemma 3.5],

$$\text{l-spli}(R) \leq \text{l-silp}(R) = \mathrm{id}_R({}_RR) \leq n.$$

Hence $\max\{\text{l-silp}(R), \text{l-spli}(R)\} \leq n$. Consequently, by Theorem 3.3,

$$\max\{\mathrm{id}_R({}_RR), \mathrm{id}_R(R_R)\} = \max\{\text{l-silp}(R), \text{l-spli}(R)\} = \text{l-G-gldim}(R).$$

A similar argument yields

$$\text{r-G-gldim}(R) = \max\{\mathrm{id}_R({}_RR), \mathrm{id}_R(R_R)\} = \max\{\text{r-silp}(R), \text{r-spli}(R)\},$$

proving that

$$\text{l-G-gldim}(R) = \text{r-G-gldim}(R) = \max\{\mathrm{id}_R({}_RR), \mathrm{id}_R(R_R)\}.$$

Let $I$ be an injective left $R$-module. Assume that $\mathrm{Gfd}_R(I) = n < \infty$. Then, by [12, Lemma 2.15], there exists an exact sequence of left $R$-modules $0 \to I \to E \to G \to 0$ such that $\mathrm{fd}_R(E) = n$ and $G$ is Gorenstein flat over $R$. Since $I$ is injective, this sequence splits, so that $I$ is a direct summand of $E$. Hence $\mathrm{fd}_R(I) \leq n$. It follows that $\mathrm{Gfd}_R(I) = \mathrm{fd}_R(I)$ for each injective left $R$-module. Therefore,

$$\begin{aligned}\text{l-sfli}(R) &= \max\{\mathrm{fd}_R(I) : I \text{ is an injective left } R\text{-module}\}\\ &= \max\{\mathrm{Gfd}_R(I) : I \text{ is an injective left } R\text{-module}\}\\ &\leq \text{l-G-wgldim}(R),\end{aligned}$$

and similarly r-sfli$(R) \leq$ r-G-wgldim$(R)$. Now, if l-sfli$(R) < \infty$, then, by [14, Proposition 3.3 and Lemma 3.5] and Lemma 3.11,

$$\text{l-spli}(R) \leq \text{l-silp}(R) \leq \text{l-G-wgldim}(R),$$

and thus, by Theorem 3.3, l-G-gldim$(R) \leq$ l-G-wgldim$(R)$. Also, if l-sfli$(R)$ $= \infty$, then, as l-sfli$(R) \leq$ l-G-wgldim$(R)$, we get l-G-wgldim$(R) = \infty$. It follows that

$$\text{l-G-gldim}(R) \leq \text{l-G-wgldim}(R).$$

Further, since, by [23, Proposition 3.11],

$$\mathrm{Gfd}_R(M) = \mathrm{Gid}_R(M^*) \leq \text{r-G-gldim}(R) = \text{l-G-gldim}(R)$$

for each left $R$-module $M$, we have $\text{l-G-wgldim}(R) \leq \text{l-G-gldim}(R)$. Consequently,

$$\text{l-G-wgldim}(R) = \text{l-G-gldim}(R)$$

and, via a similar argument, $\text{r-G-wgldim}(R) = \text{r-G-gldim}(R)$. This completes the proof. ∎

DEFINITION 3.12. If $R$ is a left and right Noetherian ring, then, we denote by $\text{G-gldim}(R)$ (resp., $\text{G-wgldim}(R)$) the common value $\text{l-G-gldim}(R) = \text{r-G-gldim}(R)$ (resp., $\text{l-G-wgldim}(R) = \text{r-G-wgldim}(R)$).

Our final result is a regularity-like theorem for Iwanaga–Gorenstein rings. In effect, a well known theorem of Serre states that a Noetherian local ring $R$ is regular if and only if $\text{gldim}(R) < \infty$ [29, Theorem 9.58]. The analog of this theorem for Iwanaga–Gorenstein rings is established in [19, Theorem 12.3.1]. Next, we recover this result via Theorem 3.10 and express it in terms of the finiteness of the Gorenstein global dimension.

COROLLARY 3.13. *Let $R$ be a left and right Noetherian ring. The following assertions are equivalent.*

(1) *$R$ is Iwanaga–Gorenstein;*
(2) $\text{G-gldim}(R) < \infty$*;*
(3) $\text{G-wgldim}(R) < \infty$.

*Moreover, if $R$ is Gorenstein, then* $\text{id}_R(R) = \text{G-gldim}(R) = \text{G-wgldim}(R)$.

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