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ON THE LYAPUNOV NUMBERS

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Abstract. We introduce and study the Lyapunov numbers—quantitative measures of the sensitivity of a dynamical system (X, f) given by a compact metric space X and a continuous map $f : X \to X$. In particular, we prove that for a minimal topologically weakly mixing system all Lyapunov numbers are the same.

1. Introduction. Throughout this paper, (X, f) denotes a *topological dynamical system*, where X is a compact metric space with metric d and $f : X \to X$ is a continuous map.

The notion of sensitivity (sensitive dependence on initial conditions) was first used by Ruelle [14]. Following Guckenheimer [10], Auslander and Yorke [5] a dynamical system (X, f) is called *sensitive* if there exists a positive ε such that for every $x \in X$ and every neighborhood U_x of x, there exist $y \in U_x$ and a nonnegative integer n with $d(f^n(x), f^n(y)) > \varepsilon$.

Recently several authors studied various properties related to sensitivity (cf. Abraham et al. [1], Akin and Kolyada [3], Moothathu [13], Huang et al. [11]). The following was proved in [3].

Proposition 1.1. *Let (X, f) be a topological dynamical system. The following conditions are equivalent:*

(1) *(X, f) is sensitive.*
(2) *There exists a positive ε such that for every $x \in X$ and every neighborhood U_x of x, there exists $y \in U_x$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$.*
(3) *There exists a positive ε such that in any opene* (1) *U in X there are $x, y \in U$ and a nonnegative integer n with $d(f^n(x), f^n(y)) > \varepsilon$.*
(4) *There exists a positive ε such that in any opene $U \subset X$ there are $x, y \in U$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$.*

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(1) Because we so often have to refer to open, nonempty subsets, we will call such subsets *opene*.

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For a dynamical system (X, f) a point $x \in X$ is Lyapunov stable if the dependence of the orbit upon the initial position is continuous at x (see [3]). This is most easily defined using the f-extension of the metric d:

$$d_f(x, y) = \sup\{d(f^n(x), f^n(y)) : n \geq 0\}$$

for $x, y \in X$. Clearly, d_f is a metric on X and

$$d_f(x, y) = \max[d(x, y), d_f(f(x), f(y))].$$

Using these metrics we define the *diameter* and *f-diameter* for $A \subset X$, and the *radius* and *f-radius* for a neighborhood U_x of a point $x \in X$:

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}, \quad \operatorname{diam}_f(A) = \sup\{d_f(x, y) : x, y \in A\},$$
$$\operatorname{rad}(U_x) = \sup\{d(x, y) : y \in U_x\}, \qquad \operatorname{rad}_f(U_x) = \sup\{d_f(x, y) : y \in U_x\}.$$

The topology obtained from the metric d_f is usually strictly coarser than the original d topology. When we use a term like "open", we always refer to the original topology.

A point $x \in X$ is called *Lyapunov stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{rad}(U_x) < \delta$ implies $\operatorname{rad}_f(U_x) \leq \varepsilon$. This condition says exactly that the sequence $\{f^n : n \geq 0\}$ of iterates is equicontinuous at x. Hence, such a point is also called an *equicontinuity point*. We consider the associated point sets:

$$\operatorname{Eq}_\varepsilon(f) = \bigcup\{U_x \subset X : U_x \text{ is a neighborhood of a point } x \text{ with } \operatorname{rad}_f(U_x) \leq \varepsilon\},$$
$$\operatorname{Eq}(f) = \bigcap_{\varepsilon > 0} \operatorname{Eq}_\varepsilon(f).$$

As the notation suggests, $\operatorname{Eq}(f)$ is the set of equicontinuity points. If $\operatorname{Eq}(f) = X$, i.e. every point is equicontinuous, then the two metrics d and d_f are topologically equivalent and so, by compactness, they are uniformly equivalent. Such a system is called *equicontinuous*. Thus, (X, f) is equicontinuous exactly when the sequence $\{f^n : n \geq 0\}$ is uniformly equicontinuous.

If the G_δ set $\operatorname{Eq}(f)$ is dense in X then the system is called *almost equicontinuous*. On the other hand, $\operatorname{Eq}_\varepsilon(f) = \emptyset$ for some $\varepsilon > 0$ precisely when the system shows *sensitive dependence upon initial conditions*, or briefly (X, f) is *sensitive*. We define

$$\mathbb{L}_r := \sup\{\varepsilon : \text{for every } x \in X \text{ and every neighborhood } U_x \text{ of } x \text{ there exist } y \in U_x \text{ and a nonnegative integer } n \text{ with } d(f^n(x), f^n(y)) > \varepsilon\}$$

and call it the (*first*) *Lyapunov number*.

It can happen that $\operatorname{Eq}_\varepsilon(f) \neq \emptyset$ for all positive ε and yet the intersection $\operatorname{Eq}(f)$ is empty (see [3]). This cannot happen when the system is transitive ([2]) (Glasner and Weiss [9], Akin et al [2]).

([2]) We recall the definition in Section 3.

THEOREM 1.2. *Let (X, f) be a topologically transitive dynamical system. Exactly one of the following two cases holds:*

(i) $(\mathrm{Eq}(f) \neq \emptyset)$ *Assume there exists an equicontinuity point for the system. Then the equicontinuity points are exactly the transitive points, i.e.* $\mathrm{Eq}(f) = \mathrm{Trans}(f)$, *and the system is almost equicontinuous. The map f is a homeomorphism and the inverse system (X, f^{-1}) is almost equicontinuous. Furthermore, the system is* uniformly rigid, *meaning that some subsequence of $\{f^n : n = 0, 1, \ldots\}$ converges uniformly to the identity.*

(ii) $(\mathrm{Eq}(f) = \emptyset)$ *Assume the system has no equicontinuity points. Then the system is sensitive, i.e. there exists $\varepsilon > 0$ such that* $\mathrm{Eq}_\varepsilon(f) = \emptyset$.

COROLLARY 1.3. *If (X, f) is a minimal dynamical system then it is either sensitive or equicontinuous.*

Also let us define

$$\mathbb{L}_d := \sup\{\varepsilon : \text{in any opene } U \subset X \text{ there exist } x, y \in U \text{ and a positive integer } n \text{ with } d(f^n(x), f^n(y)) > \varepsilon\}$$

and call it the *second Lyapunov number.*

According to Proposition 1.1 we define

$$\overline{\mathbb{L}}_r := \sup\Big\{\varepsilon : \text{for every } x \in X \text{ and every open neighborhood } U_x \text{ of } x \text{ there exists } y \in U_x \text{ with } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\Big\},$$

$$\overline{\mathbb{L}}_d := \sup\Big\{\varepsilon : \text{in any opene } U \subset X \text{ there exist } x, y \in U \text{ with } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\Big\}.$$

Sometimes the following notations will also be useful:

$$\mathbb{L}_1 := \mathbb{L}_r, \quad \mathbb{L}_2 := \mathbb{L}_d, \quad \mathbb{L}_3 := \overline{\mathbb{L}}_r, \quad \mathbb{L}_4 := \overline{\mathbb{L}}_d.$$

So, various definitions of sensitivity formally give us different Lyapunov numbers—quantitative measures of these sensitivities.

In Section 2 we prove that for any topological dynamical system (X, f), we have $\mathbb{L}_d \leq 2\overline{\mathbb{L}}_r$. In Section 3 we examine the equalities between the Lyapunov numbers for topologically transitive systems, and in Section 4 for weakly mixing systems. In particular, we prove that for topologically weakly mixing minimal systems all Lyapunov numbers are the same. Finally, in Section 5 we give some examples and open problems for Lyapunov numbers.

2. A general inequality for Lyapunov numbers. Directly from the definitions,

$$\mathbb{L}_d \geq \overline{\mathbb{L}}_d \geq \overline{\mathbb{L}}_r \quad \text{and} \quad \mathbb{L}_d \geq \mathbb{L}_r \geq \overline{\mathbb{L}}_r.$$

PROPOSITION 2.1. *Let (X, f) be a sensitive topological dynamical system. Then $\mathbb{L}_d \leq 2\overline{\mathbb{L}}_r$.*

Proof. Fix a (small enough) $\delta > 0$, a point $x \in X$ and a neighborhood U_x of x. Let $U_0 = U_x$ and n_0 be the first positive integer for which $\operatorname{diam}(f^{n_0}(U_0)) > \mathbb{L}_d - \delta$. There exists $y_0 \in U_0$ such that $d(f^{n_0}(x), f^{n_0}(y_0)) > (\mathbb{L}_d - \delta)/2$. Choose an opene U_1 with its closure contained in U_0 such that $y_0 \in U_1$ and $\operatorname{diam}(f^m(U_1)) \leq \delta/2$ for every nonnegative integer $m \leq n_0$. Let n_1 be the first positive integer for which $\operatorname{diam}(f^{n_1}(U_1)) > \mathbb{L}_d - \delta$. By the definition of U_1, we clearly have $n_1 > n_0$.

We define recursively opene sets $U_2, U_3, \ldots$ and positive integers $n_2, n_3, \ldots$ as follows. If n_{k-1} is defined, there exists a point $y_{k-1} \in U_{k-1}$ such that $d(f^{n_{k-1}}(x), f^{n_{k-1}}(y_{k-1})) > (\mathbb{L}_d - \delta)/2$. So we can choose an opene U_k in U_{k-1} such that $y_{k-1} \in U_k$ and $\operatorname{diam}(f^m(U_{n_k})) \leq \delta/2$ for every nonnegative integer $m \leq n_{k-1}$. Let n_k be the first positive integer for which $\operatorname{diam}(f^{n_k}(U_k)) > \mathbb{L}_d - \delta$. As in the previous step, by the definition of U_k we clearly have $n_k > n_{k-1}$.

If y is a point of the nonempty intersection $\bigcap_k \overline{U_{n_k}}$, then obviously $y \in U$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \mathbb{L}_d/2 - \delta$. ∎

As a consequence of the inequalities at the beginning of Section 2 and Proposition 2.1 we conclude that $\mathbb{L}_i \leq 2\mathbb{L}_j$ for any $i, j \in \{1, 2, 3, 4\}$.

3. Lyapunov numbers for transitive maps. A topological dynamical system (X, f) is called *topologically transitive* if for any pair of opene subsets $U, V \in X$,

$$n_f(U, V) := \{n \in \mathbb{Z}_+ : U \cap f^{-n}(V) \neq \emptyset\}$$

is infinite. A point $x \in X$ is called a *transitive point* if its orbit $\{x, f(x), f^2(x), \ldots\}$ is dense in X. If (X, f) is topologically transitive and X is compact, then the set of transitive points is a G_δ dense subset of X.

If every point of a dynamical system (X, f) is transitive, then this system is called *minimal*. An f-invariant closed subset $M \subset X$ is called *minimal* if the orbit of any point of M is dense in M (in this case points of M are called *minimal*, too).

For a dynamical system (X, f), a point $x \in X$ and a set $U \subset X$ let

$$n_f(x, U) := \{n \in \mathbb{Z}_+ : f^n(x) \in U\}.$$

A point $x \in X$ is said to be *recurrent* if for every neighborhood U of x the set $n_f(x, U)$ is infinite.

A subset S of $\mathbb{Z}_+$ is *syndetic* if it has bounded gaps, i.e. there is $N \in \mathbb{N}$ such that $\{i, i+1, \ldots, i+N\} \cap S \neq \emptyset$ for every $i \in \mathbb{Z}_+$; and S is *thick* if it contains arbitrarily long runs of positive integers, i.e. there is a strictly increasing subsequence $\{n_i\}$ such that $S \supset \bigcup_{i=1}^{\infty} \{n_i, n_i + 1, \ldots, n_i + i\}$.

Some dynamical properties can be interpreted by using the notions of syndetic or thick subsets. For example, a classic result of Gottschalk states that $x \in X$ is a minimal point if and only if $n_f(x, U)$ is syndetic for any neighborhood U of x, and a topological dynamical system (X, T) is (topologically) weakly mixing (we recall the definition in Section 4) if and only if $n_f(U, V)$ is thick for any opene subsets U, V of X [7], [8].

THEOREM 3.1. *Let (X, f) be a sensitive topologically transitive dynamical system. Then $\mathbb{L}_d = \overline{\mathbb{L}}_d$.*

Proof. By the definition of $\mathbb{L}_d$, for any $\varepsilon < \mathbb{L}_d$ and for any opene $U \in X$ there are points $x, y \in U$ and a positive integer n_0 such that $d(f^{n_0}(x), f^{n_0}(y)) > \varepsilon$. Choose an arbitrary (small) $\delta > 0$. Let $U_x, U_y \subset U$ be neighborhoods of x and y such that $\operatorname{diam}(f^{n_0}(U_x)) < \delta$ and $\operatorname{diam}(f^{n_0}(U_y)) < \delta$. If $z \in U_x$ is a transitive point, there is a positive integer m for which $f^m(z) \in U_y$. By the triangle inequality we have $d(f^{n_0}(z), f^{n_0+m}(z)) > \varepsilon - 2\delta$.

Let U_z be a neighborhood of z such that $U_z \subset U_x$ and $f^m(U_z) \subset U_y$. Then obviously $\operatorname{diam}(f^{n_0}(U_z)) < \delta$ and $\operatorname{diam}(f^{n_0+m}(U_z)) < \delta$. Since a sensitive system has no isolated points, U_z is infinite. Therefore, the orbit of the point z visits U_z infinitely many times. If n_k is such that $f^{n_k}(z) \in U_z$, then $f^{n_0+n_k}(z) = f^{n_0}(f^{n_k}(z)) \subset f^{n_0}(U_z)$ and $f^{n_0+n_k+m}(z) = f^{n_0+m}(f^{n_k}(z)) \subset f^{n_0+m}(U_z) = f^{n_0}(f^m(U_z)) \subset f^{n_0}(U_y)$. Hence, by the triangle inequality, $d(f^{n_0+n_k}(z), f^{n_0+n_k+m}(z)) > \varepsilon - 2\delta$. From this we have $\overline{\mathbb{L}}_d > \limsup_{n\to\infty} d(f^n(z), f^n(f^m(z))) \geq \varepsilon - 2\delta$. Since $\delta > 0$ and $\varepsilon < \mathbb{L}_d$ were chosen arbitrarily, we conclude that $\mathbb{L}_d = \overline{\mathbb{L}}_d$. ∎

A topologically transitive dynamical system (X, f), where X has no isolated points, is called *ToM* if every point $x \in X$ is either (topologically) transitive or minimal. ToM systems were introduced by Downarowicz and Ye [6]. Since we do not require that both types are present (as in [6]), a minimal system is also ToM. If a ToM system is not minimal, then the set of minimal points is dense in X (because for a transitive, but nonminimal system, the set of nontransitive points is dense; see for instance [12]).

THEOREM 3.2. *Let (X, f) be a sensitive* ToM *system. Then $\mathbb{L}_r = \overline{\mathbb{L}}_r$.*

Proof. Fix a point $x \in X$. Let U_x be a neighborhood of x and let $\delta > 0$. By the definition of $\mathbb{L}_r$, there exist a point $y \in U_x$ and a positive integer m such that $d(f^m(x), f^m(y)) > \mathbb{L}_r - \delta$. Take a neighborhood $U_y \subset U_x$ of point y such that $\operatorname{diam}(f^m(U_y)) < \delta$.

Now, if x is a transitive point, one can just follow the idea of the proof of Theorem 3.1. If x is not transitive, then it is minimal. Since (X, f) is ToM, we can find a minimal point $z_1 \in U_y$ and therefore $d(f^m(x), f^m(z_1)) > \mathbb{L}_r - 2\delta$.

Consider the direct product system $(\overline{\mathrm{Orb}_f(x)} \times \overline{\mathrm{Orb}_f(z_1)}, f|_{\overline{\mathrm{Orb}_f(x)}} \times f|_{\overline{\mathrm{Orb}_f(z_1)}})$. Let M be a minimal subset of this system. Then obviously $M \cap M_x \neq \emptyset$, where $M_x := \{(x, z) : z \in \overline{\mathrm{Orb}_f(z_1)}\}$. Hence there is a point $(x, z_2) \in U_x \times \overline{\mathrm{Orb}_f(z_1)}$ which is minimal, and therefore (uniformly) *recurrent* for the map $f|_{\overline{\mathrm{Orb}_f(x)}} \times f|_{\overline{\mathrm{Orb}_f(z_1)}})$. Clearly, every point of the form $(x, f^k(z_2))$, $k = 0, 1, 2, \ldots$, will be uniformly recurrent too. Since z_1 is minimal, we can take a positive integer k such that $z_3 := f^k(z_2) \in U_y$. Therefore, $\limsup_{n\to\infty} d(f^n(x), f^n(z_3)) \geq \mathbb{L}_r - 2\delta$. Since x and $\delta > 0$ were chosen arbitrarily, we get $\mathbb{L}_r = \overline{\mathbb{L}}_r$. ∎

As a corollary of the last two theorems we conclude that the equalities $\mathbb{L}_r = \overline{\mathbb{L}}_r$ and $\mathbb{L}_d = \overline{\mathbb{L}}_d$ hold for minimal dynamical systems. And what about dynamical systems for which $\mathbb{L}_r = \mathbb{L}_d$?

4. Lyapunov numbers for weakly mixing maps. Recall that a topological dynamical system (X, f) is called (*topologically*) *weakly mixing* if for any opene $U_1, U_2, V_1, V_2 \in X$ there is a nonnegative integer n such that $U_1 \cap f^{-n}(V_1) \neq \emptyset$ and $U_2 \cap f^{-n}(V_2) \neq \emptyset$; in other words, if its direct product $(X \times X, f \times f)$ is topologically transitive.

Theorem 4.1. *Let (X, f) be a topologically weakly mixing dynamical system. Then:*

(1) $\mathbb{L}_d = \overline{\mathbb{L}}_d = \mathrm{diam}(X)$.
(2) $\mathbb{L}_r = \overline{\mathbb{L}}_r$.
(3) *If, in addition, (X, f) is minimal, then* $\mathbb{L}_r = \overline{\mathbb{L}}_r = \mathbb{L}_d = \overline{\mathbb{L}}_d = \mathrm{diam}(X)$.

Proof. (1) Since a weakly mixing system is topologically transitive, from Theorem 3.1 we have $\mathbb{L}_d = \overline{\mathbb{L}}_d$. Since (X, f) is topologically weakly mixing, also the direct product $(X \times X, f \times f)$ is topologically transitive. So, in every open set in the product, in particular, in the Cartesian square of every ball U in X, there is a transitive point of $(X \times X, f \times f)$, i.e., a pair of points $x, y \in U$. Such a pair visits all places in $X \times X$ infinitely many times. This means that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) = \mathrm{diam}(X) = \overline{\mathbb{L}}_d$.

(2) Let $x \in X$. Since (X, f) is weakly mixing, there is a point $z \in X$ such that for any neighborhood G of z and any opene U, V in X there exist infinitely many positive integers n for which $f^n(x) \in G$ and $f^n(U) \cap V \neq \emptyset$ ([3]).

By the definition of $\mathbb{L}_r$, for the point z and any (small enough) positive δ there is a point $y \in X$ and a positive integer k such that $d(f^k(y), f^k(z)) > \mathbb{L}_r - \delta$.

Now, let U_x be a neighborhood of x, and let G_z and V_y be open balls of radius δ centered at $f^k(z)$ and $f^k(y)$, respectively. Suppose also $G_z \cap V_y = \emptyset$. We will find a point in U_x by using the above property from [3]. Let n_0 be a positive integer such that $f^{n_0}(x) \in G_z$ and $f^{n_0}(U_x) \cap V_y \neq \emptyset$. Put $U_0 := U_x \cap f^{-n_0}(V_y)$. Obviously, U_0 is an opene subset of U_x, $\overline{U_0} \subset \overline{U_x}$ and $x \notin U_0$. Define inductively opene sets $U_1, U_2, \ldots$ and positive integers n_i as follows. Let n_k be a (large enough) positive integer, say $n_k \geq k$, such that $f^{n_k}(x) \in G_z$ and $f^{n_k}(U_{k-1}) \cap V_y \neq \emptyset$. Define $U_k := U_{k-1} \cap f^{-n_k}(V_y)$. It is clear that U_k is an opene subset in X and $U_i \subset U_{i-1}$ for any $i \geq 1$. Hence $\overline{U_0} \supset \overline{U_1} \supset \cdots$. If u is a point of the nonempty intersection $\bigcap_i \overline{U_i}$, then for any natural i we have $f^{n_i}(u) \in \overline{V}_y$ and $f^{n_i}(x) \in G_z$. Therefore, $\limsup_{n\to\infty} d(f^n(x), f^n(u)) \geq \mathbb{L}_r - 3\delta$. Since $\delta > 0$ is arbitrary, we have $\mathbb{L}_r = \overline{\mathbb{L}}_r$.

(3) Statements (1) and (2) imply that it is sufficient to prove $\mathbb{L}_r = \operatorname{diam}(X)$. Let $x \in X$ and let U_x be a neighborhood of x. There are two opene (infinite) sets V_x and V_y in X and a positive (small enough) number δ such that the distance between V_x and V_y is larger than or equal to $\operatorname{diam}(X) - \delta$.

As mentioned before, since (X, f) is minimal, any point of X is uniformly recurrent. In particular, $n_f(x, V_x)$ is a syndetic subset of $\mathbb{Z}_+$. On the other hand, (X, f) is also a topologically weakly mixing dynamical system, which in turn means that $n_f(U, V_y)$ is a thick subset of $\mathbb{Z}_+$. Hence $n_f(x, V_x) \cap n_f(U, V_y) \neq \emptyset$ and therefore there exist a point $y \in U$ and a positive integer $k \in n_f(x, V_x) \cap n_f(U, V_y)$ such that $f^k(x) \in V_x$ and $f^k(y) \in V_y$. So, $d(f^k(x), f^k(y)) \geq \operatorname{diam}(X) - \delta$. Since $\delta > 0$ was arbitrary, we get $\mathbb{L}_r = \operatorname{diam}(X)$. ∎

5. Concluding remarks. Firstly, let us remark that there are topologically weakly mixing (even topologically mixing) systems for which $\mathbb{L}_r = \overline{\mathbb{L}}_r = \operatorname{diam}(X)/2$. For instance, the continuous interval map $g : [0, 1] \to [0, 1]$, where $g(x) = 3((x - 1/3) - |x - 1/3| + |x - 2/3|)$, is topologically mixing and one of its fixed points is $1/2$, therefore clearly $\mathbb{L}_r = 1/2$.

Also (as we will see in Proposition 5.1 below) there are dynamical systems for which $\mathbb{L}_r = 2\overline{\mathbb{L}}_r$, but this equality is still an open question for topologically transitive maps (nonminimal by Theorem 3.2).

Two more open questions:

1. Does there exist a nontransitive dynamical system (X, f) for which $\mathbb{L}_d > \overline{\mathbb{L}}_d$ and/or $\mathbb{L}_r > \overline{\mathbb{L}}_d$?
2. Does there exist a minimal dynamical system (X, f) for which $\mathbb{L}_d > \mathbb{L}_r$? (3)

(3) According to a remark by T. Downarowicz, such minimal dynamical systems do exist (see [15] for details). We received this remark after the article was submitted.

Proposition 5.1. *There exists a topological dynamical system (X, f) for which $\mathbb{L}_r = 2\overline{\mathbb{L}}_r$.*

Proof. We define X to be a compact surface in $\mathbb{R}^3$ which is homeomorphic to a two-dimensional disk in $\mathbb{R}^2$. More precisely, the cylindrical coordinates of a point $(x, y, z) \in X$ have the form (r, φ, z), where $r = \sqrt{x^2 + y^2}$ and φ is an angle for which $x = r\cos\varphi$ and $y = r\sin\varphi$. In other words, (r, φ) are the polar coordinates of (x, y), and z remains unchanged. Let $h(r) = 8r(1 - r)$. Now, define X as the set of points with cylindrical coordinates $(r, \varphi, h(r))$, where $0 \le r \le 1$, $\varphi \in \mathbb{R}$, and let the Euclidean metric d (in $\mathbb{R}^3$) be the metric on X.

Now we define a continuous map f from X to itself as follows: $(r, \varphi, h(r)) \mapsto (g(r), 2\varphi, h(g(r)))$, where g is a continuous map $[0, 1] \to [0, 1]$ with $g(0) = 0$, $g(1) = 1$ and $g(x) > x$ for all $x \in (0, 1)$. From these properties one can easily deduce that $\lim_{n\to\infty} g^n(x) = 1$ for any $x \in (0, 1]$. For example, let $g(x) = 2x - x^2$.

Let $p \in X$ and U be a neighborhood of p. If $p \neq (0, 0, 0)$, then for any $\delta > 0$ there are $n \in \mathbb{N}$ and $q \in U$ such that $d(f^n(p), f^n(q)) > 2 - \delta$. If $p = (0, 0, 0)$, then there are $n \in \mathbb{N}$ and $q \in U$ for which $f^n(q)$ lies on a circle in X with center $(0, 0, 2)$ (in $\mathbb{R}^3$) and radius $1/2$. For those n and q we have $d(f^n(p), f^n(q)) > 2$ and so $\mathbb{L}_r \ge 2$.

Now, let $p = (0, 0, 0)$. Then $\lim_{n\to\infty} d(f^n(p), f^n(q)) = 1$ for any $q \neq p$. So $\overline{\mathbb{L}}_r \le 1$. Since $\mathbb{L}_r \le 2\overline{\mathbb{L}}_r$ (by Proposition 2.1), this gives $\mathbb{L}_r = 2\overline{\mathbb{L}}_r$. ∎

The idea of introducing and studying the Lyapunov numbers comes from the following:

1. If some practical assumption holds about the behavior of a particular system, for example, a physical object, we need to know how far we can go wrong in calculations, if we want to predict the evolution of the system over a long term. Merely knowing that there could exist errors in the calculations of the future behavior of the system is not that useful, since from the practical point of view, the existence of errors in calculations of almost all natural systems (as a result of inaccurate initial data) is a well-known fact. So, a quantitative analysis of sensitivity that determines to what extent one's calculations are accurate is of great interest. Comparison of different Lyapunov numbers (the ones which are determined by the upper limit and the ones without limit) demonstrates that errors in calculations cannot disappear (decrease) in time. That is, we cannot expect that, for example, after 10000 or 1000000 steps the accuracy of our prediction increases significantly (which seems commonsensical).

2. According to the Auslander theorem, one of the most important theorems in topological dynamics, any proximal cell (i.e., $\mathrm{Prox}_f(x) := \{y \in X : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$) contains a minimal point [4]. This im-

plies, in particular, that a distal point is always minimal. It should be noted that if (X, f) is a weakly mixing dynamical system then for every $x \in X$, the proximal cell $\mathrm{Prox}_f(x)$ is dense in X [3]. What about this property for sensitive topologically transitive systems, in particular, for Devaney systems (i.e., topologically transitive systems with a dense set of periodic points)? There is a direct connection between this question and the following one: When does $\mathbb{L}_r = \overline{\mathbb{L}}_r$ hold for a sensitive topologically transitive system?

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REFERENCES

[1] Ch. Abraham, G. Biau and B. Cadre, *On Lyapunov exponent and sensitivity*, J. Math. Anal. Appl. 290 (2004), 395–404.

[2] E. Akin, J. Auslander and K. Berg, *When is a transitive map chaotic?*, in: Convergence in Ergodic Theory and Probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ. 5, de Gruyter, Berlin, 1996, 25–40.

[3] E. Akin and S. Kolyada, *Li–Yorke sensitivity*, Nonlinearity 16 (2003), 1421–1433.

[4] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Math. Stud. 153, North-Holland, Amsterdam, 1988.

[5] J. Auslander and J. Yorke, *Interval maps, factors of maps and chaos*, Tohoku Math. J. 32 (1980), 177–188.

[6] T. Downarowicz and X. Ye, *When every point is either transitive or periodic*, Colloq. Math. 93 (2002), 137–150.

[7] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.

[8] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, M. B. Porter Lectures, Princeton Univ. Press, Princeton, NJ, 1981.

[9] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity 6 (1993), 1067–1075.

[10] J. Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps*, Comm. Math. Phys. 70 (1979), 133–160.

[11] W. Huang, P. Lu and X. Ye, *Measure-theoretical sensitivity and equicontinuity*, Israel J. Math. 183 (2011), 233–283.

[12] S. Kolyada and L. Snoha, *Some aspects of topological transitivity—a survey*, in: Iteration Theory (ECIT 94, Opava), Grazer Math. Ber. 334, Karl-Franzens-Univ. Graz, Graz, 1997, 3–35.

[13] T. K. S. Moothathu, *Stronger forms of sensitivity for dynamical systems*, Nonlinearity 20 (2007), 2115–2126.

[14] D. Ruelle, *Dynamical systems with turbulent behavior*, in: Mathematical Problems in Theoretical Physics (Rome, 1977), Lecture Notes in Phys. 80, Springer, Berlin, 1978, 341–360.

[15] M. Štefánková, *Strange chaotic triangular maps*, Chaos Solitons Fractals 45 (2012), 1188–1191.

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