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ON THE HAUSDORFF–YOUNG THEOREM FOR COMMUTATIVE HYPERGROUPS

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Abstract. We study the Hausdorff–Young transform for a commutative hypergroup K and its dual space $\hat{K}$ by extending the domain of the Fourier transform so as to encompass all functions in $L^p(K,m)$ and $L^p(\hat{K},\pi)$ respectively, where $1 \le p \le 2$. Our main theorem is that those extended transforms are inverse to each other. In contrast to the group case, this is not obvious, since the dual space $\hat{K}$ is in general not a hypergroup itself.

Introduction. There exist a lot of results on the Hausdorff–Young transform on groups and its applications (see for instance [HR]). Estimates for the norm of the L^p-Fourier transform on locally compact groups are established by Russo [Ru1–Ru3] and Fournier [F1–F2]. For groups which are neither compact nor Abelian but which are unimodular, the Hausdorff–Young transform has been defined and a Hausdorff–Young theorem has been proved by Kunze [K]. Moreover, Führ [Fü] studied Hausdorff–Young inequalities for certain group extensions, by using Mackey's theory of induced representations.

In this paper we introduce a Hausdorff–Young transform for commutative hypergroups. Our paper is structured in the following way. First we give some definitions, general notations and basic facts. In Section 2 we prove our main result, which states that the Hausdorff–Young transform and the inverse Hausdorff–Young transform are indeed inverse to each other. In contrast to the Abelian group case, this is not obvious, since the dual space $\hat{K}$ of an arbitrary hypergroup K is in general not a hypergroup. To overcome these difficulties we need some results on the relations between convolution and Fourier transform, which are of general interest. Section 3 deals with a few dual results. In Section 4 we give some useful applications in harmonic analysis.

1. Preliminaries. Hypergroups generalize locally compact groups. For the theory of hypergroups we refer to [BH] and [J]. A hypergroup K is a

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locally compact Hausdorff space with a convolution, i.e. a map $K \times K \to M^1(K)$, $(x,y) \mapsto \omega(x,y)$ ($M^1(K)$ is the space of probability measures on K) and an involution, i.e. $K \to K$, $x \mapsto \tilde{x}$, satisfying certain axioms (see [BH]).

We assume throughout that K is a commutative hypergroup.

For a locally compact Hausdorff space X let $C(X)$, $C^b(X)$, $C_0(X)$, $C_c(X)$ denote the spaces of all continuous functions on X, those which are bounded, which vanish at infinity, and which have compact support, respectively.

For $x \in K$ we define the translation operator on $C(K)$ by setting

$$T_x f(y) = \int_K f(z)\, d(\omega(x,y))(z).$$

Spector [S] has proved that each commutative hypergroup possesses a Haar measure m, which is translation invariant. We denote by $\mathbb{S}(K)$ the m-integrable simple functions on K. The Banach spaces $L^p(K,m)$, $1 \le p \le \infty$, are invariant under the translation actions T_x, $x \in K$, and $L^1(K,m)$ acts on $L^p(K,m)$ via

$$g * f(x) = \int_K g(y) T_{\tilde{y}} f(x)\, dm(y)$$

for $g \in L^1(K,m)$, $f \in L^p(K,m)$, $1 \le p \le \infty$.

There exists a net $(g_i)_{i\in I}$ of functions $g_i \in C_c(K)$, $g_i \ge 0$, $\int_K g_i(x)\, dm(x) = 1$ such that $\lim_i \|g_i * f - f\|_p = 0$ for all $f \in L^p(K,m)$, $1 \le p < \infty$ (see [**?**, Lemma 1]).

The symmetric structure space of the commutative Banach $*$-algebra $L^1(K,m)$ is a locally compact Hausdorff space, which is defined by

$$\hat{K} = \{\alpha \in C^b(K) : \alpha(e) = 1,\, T_x\alpha(y) = \alpha(x)\alpha(y),\, \alpha(\tilde{x}) = \overline{\alpha(x)} \quad \text{for all } x,y \in K\},$$

where $\hat{K}$ is equipped with the compact-open topology which is equal to the Gelfand topology. The Fourier transform of $f \in L^1(K,m)$ (resp. the Fourier–Stieltjes transform of $\mu \in M(K)$) is defined by

$$\hat{f}(\alpha) = \int_K f(x)\overline{\alpha(x)}\, dm(x) \qquad (\text{resp. } \hat{\mu}(\alpha) = \int_K \overline{\alpha(x)}\, d\mu(x)).$$

for $\alpha \in \hat{K}$. We have $\hat{f} \in C_0(\hat{K})$ with $\|\hat{f}\|_\infty \le \|f\|_1$ and $\hat{\mu} \in C^b(\hat{K})$. There exists a unique regular positive Borel measure π on $\hat{K}$ (called the *Plancherel measure*) such that

$$\int_K |f(x)|^2\, dm(x) = \int_{\hat{K}} |\hat{f}(\alpha)|^2\, d\pi(\alpha)$$

for all $f \in L^1(K,m) \cap L^2(K,m)$ (see [BH, pp. 84 ff.]). Since $\hat{K}$ is a locally compact Hausdorff space, we note that every compact set $C \in \hat{K}$ has finite measure $\pi(C)$. We emphasize that the support of π, $\operatorname{supp}\pi =: \mathcal{S}$, defined as in [BH, p. 8], can be a proper subset of $\hat{K}$. The extension of the Fourier transform from $L^1(K,m) \cap L^2(K,m)$ to $L^2(K,m)$ is called the *Plancherel transform*, and denoted by $\wp(f)$ for $f \in L^2(K,m)$. The Plancherel transform is an isometric isomorphism from $L^2(K,m)$ onto $L^2(\mathcal{S},\pi)$, and Parseval's formula

$$\int_K f(x)g(x)\,dm(x) = \int_{\mathcal{S}} \wp f(\alpha)\wp g(\bar{\alpha})\,d\pi(\alpha)$$

holds for $f, g \in L^2(K,m)$.

The inverse Fourier transform of $f \in L^1(\mathcal{S},\pi)$ (resp. the inverse Fourier–Stieltjes transform of $\mu \in M(\hat{K})$) is defined by

$$\check{f}(x) = \int_{\mathcal{S}} f(\alpha)\alpha(x)\,d\pi(\alpha) \qquad \Big(\text{resp. } \check{\mu}(x) = \int_{\hat{K}} \alpha(x)\,d\mu(\alpha)\Big)$$

for $x \in K$. We have $\check{f} \in C_0(K)$, $\check{\mu} \in C^b(K)$ and $\|\check{f}\|_\infty \le \|f\|_1$. An inversion theorem holds: if $f \in L^1(K,m)$ and $\hat{f} \in L^1(\mathcal{S},\pi)$ then $f = (\hat{f})^\vee$ with equality in $L^1(K,m)$. Furthermore, an inverse uniqueness theorem is valid: If $f \in L^1(K,m)$ and $\check{f} = 0$, then $f = 0$.

2. Main results. We want to extend the domain of the Fourier transform so as to encompass all functions in $L^p(K,m)$ where $1 \le p \le 2$. This follows as indicated below from the Riesz–Thorin convexity theorem [DS, VI.10.11].

The Fourier transform coincides on $L^1(K,m) \cap L^2(K,m)$ with the Plancherel transform. Therefore the Riesz–Thorin convexity theorem yields the inequality

$$\|\hat{f}\|_q \le \|f\|_p$$

for $1 \le p \le 2$, $1/p + 1/q = 1$, and for all $f \in \mathbb{S}(K)$. Since we can approximate each function $f \in C_c(K)$ uniformly by functions in $\mathbb{S}(K)$, this inequality holds for all $f \in C_c(K)$ and the mapping

$$f \mapsto \hat{f}, \qquad C_c(K) \to L^q(\mathcal{S},\pi),$$

can be extended uniquely by continuity to the whole of $L^p(K,m)$. This extended map is called the *Hausdorff–Young transform*. To sum up, we have the following important result.

Proposition 2.1 (Hausdorff–Young). *Let $1 \le p \le 2$, $1/p + 1/q = 1$, and $f \in L^p(K,m)$. The Hausdorff–Young transform $f \mapsto \hat{f}$ is a linear mapping from $L^p(K,m)$ into $L^q(\mathcal{S},\pi)$ such that $\|\hat{f}\|_q \le \|f\|_p$.*

In the same way we can extend the inverse Fourier transform $f \mapsto \check{f}$ from $C_c(\mathcal{S})$ into $C_0(K) \subseteq L^\infty(K,m)$ by using the Riesz–Thorin convexity theorem again. Its extension maps from $L^p(\mathcal{S},\pi)$, $1 \le p \le 2$, into $L^q(K,m)$, $1/p+1/q=1$.

PROPOSITION 2.2 (Inverse Hausdorff–Young). *Let* $1 \le p \le 2$, $1/p + 1/q = 1$, *and* $f \in L^p(\mathcal{S},\pi)$. *The inverse Hausdorff–Young transform* $f \mapsto \check{f}$ *is a linear mapping from* $L^p(\mathcal{S},\pi)$ *into* $L^q(K,m)$ *such that* $\|\check{f}\|_q \le \|f\|_p$.

For later results it is important to known whether the L^{p_1}-transform and the L^{p_2}-transform of a function f, which is contained in two different spaces $L^{p_1}(K,m)$ and $L^{p_2}(K,m)$, agree π-almost everywhere on $\hat{K}$. The same question arises for the dual versions.

PROPOSITION 2.3. *Let* $1 \le p_1, p_2 \le 2$.

(i) *For* $f \in L^{p_1}(K,m) \cap L^{p_2}(K,m)$ *the* L^{p_1}*-transform and the* L^{p_2}*-transform of* f *agree* π*-almost everywhere on* $\hat{K}$.
(ii) *For* $f \in L^{p_1}(\mathcal{S},\pi) \cap L^{p_2}(\mathcal{S},\pi)$ *the inverse* L^{p_1}*-transform and the inverse* L^{p_2}*-transform of* f *agree* m*-almost everywhere on* K.

Proof. The proof is similar to that of [HR, (31.26)]. ∎

Now it is natural to ask whether the inverse Hausdorff–Young transform is really the inverse mapping of the Hausdorff–Young transform. This turns out to be true, but in order to prove this we have to take into account that $\hat{K}$ is not a hypergroup in general. Thus we need a few preparatory results.

LEMMA 2.4. *For* $f, g \in L^2(K,m)$ *and* $h \in L^1(K,m)$ *we have* $f * g \in C_0(K)$ *and*

$$\int_K f * g(y) h(\tilde{y})\, dm(y) = \int_K f(\tilde{y}) g * h(y)\, dm(y).$$

Proof. It is well-known that $f * g \in C_0(K)$ for $f, g \in L^2(K,m)$ and $g * h \in L^2(K,m)$. Furthermore, $f * g(x) = g * f(x)$ by the commutativity of K. Hence we conclude, applying Fubini's theorem,

$$\begin{aligned}\int_K f * g(y) h(\tilde{y})\, dm(y) &= \int_K \int_K f(\tilde{x}) T_y g(x)\, dm(x)\, h(\tilde{y})\, dm(y) \\ &= \int_K f(\tilde{x}) \int_K T_y g(x) h(\tilde{y})\, dm(y)\, dm(x) = \int_K f(\tilde{x}) g * h(x)\, dm(x). \ \blacksquare\end{aligned}$$

PROPOSITION 2.5. *Let* K *be a commutative hypergroup and* $1 \le p \le 2$.

(i) *For* $f \in L^p(K,m)$ *and* $\varphi \in L^p(\mathcal{S},\pi)$ *we have* $(\hat{f}\varphi)^\vee = f * \check{\varphi}$.
(ii) *For* $\mu \in M(K)$ *and* $\varphi \in L^p(\mathcal{S},\pi)$ *we have* $(\hat{\mu}\varphi)^\vee = \mu * \check{\varphi}$ m*-almost everywhere. In particular, for* $f \in L^1(K,m)$ *and* $\varphi \in L^p(\mathcal{S},\pi)$ *we have* $(\hat{f}\varphi)^\vee = f * \check{\varphi}$ m*-almost everywhere.*

Proof. (i) $\hat{f}\varphi$ is an element in $L^1(\mathcal{S},\pi)$. Hence the inverse Fourier transform is well-defined. Choosing $f \in C_c(K)$, $\varphi \in C_c(\mathcal{S})$ we obtain, with Fubini's theorem,

$$\begin{aligned}(\hat{f}\varphi)^\vee(x) &= \int_{\mathcal{S}} \hat{f}(\alpha)\varphi(\alpha)\alpha(x)\,d\pi(\alpha) = \int_{\mathcal{S}}\int_K f(y)\bar{\alpha}(y)\,dm(y)\,\varphi(\alpha)\alpha(x)\,d\pi(\alpha)\\ &= \int_{\mathcal{S}}\int_K \alpha(x)\bar{\alpha}(y)f(y)\varphi(\alpha)\,dm(y)d\pi(\alpha) = \int_K T_x\check{\varphi}(\tilde{y})f(y)\,dm(y)\\ &= f * \check{\varphi}(x).\end{aligned}$$

Using the continuity of the transformation and the convolution the statement follows from the denseness of $C_c(K)$ in $L^p(K,m)$ and the denseness of $C_c(\mathcal{S})$ in $L^p(\mathcal{S},\pi)$. Indeed, choosing a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$ such that $\lim_{n\to\infty}\|\varphi_n-\varphi\|_p = 0$ and using Hölder's inequality we obtain, for each $f \in C_c(K)$,

$$\begin{aligned}\|(\hat{f}\varphi)^\vee - f*\check{\varphi}\|_\infty &\le \|\hat{f}(\varphi-\varphi_n)\|_1 + \|f*\check{\varphi}_n - f*\check{\varphi}\|_\infty\\ &\le 2\|f\|_p\|\varphi-\varphi_n\|_p \to 0 \qquad \text{as } n\to\infty.\end{aligned}$$

(ii) By (2.2.15) in [BH] we know already that $(\hat{\mu}\varphi)^\vee = \mu * \check{\varphi}$ for all $\varphi \in C_c(\mathcal{S})$. For each $\varphi \in L^p(\mathcal{S},\pi)$, the denseness of $C_c(\mathcal{S})$ in $L^p(\mathcal{S},\pi)$ yields the existence of a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$ which converges to φ in $L^p(\mathcal{S},\pi)$. Hence,

$$\begin{aligned}\|(\hat{\mu}\varphi)^\vee - \mu*\check{\varphi}\|_q &\le \|\hat{\mu}(\varphi-\varphi_n)\|_p + \|\mu*(\varphi-\varphi_n)^\vee\|_q\\ &\le 2\|\mu\|\|\varphi-\varphi_n\|_p \to 0\end{aligned}$$

as $n\to\infty$. The second statement follows by embedding $L^1(K,m)$ into $M(K)$ via the mapping $f\mapsto fm$, $L^1(K,m)\to M(K)$. ∎

We also need the following lemma proved in [FL, Theorem 3.1].

LEMMA 2.6. *Given $\alpha\in\mathcal{S}$ and a compact neighborhood C of α there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c(K)$ such that $\|(f_n*f_n^*)^\wedge - \chi_C\|_1 \to 0$ as $n\to\infty$.*

The following proposition is essential for our main theorem.

PROPOSITION 2.7.

(i) *Let $1\le p\le 2$ and $f\in L^p(K,m)$ be such that the Hausdorff–Young transform $\hat{f}$ belongs to $L^2(\mathcal{S},\pi)$. Then $f\in L^2(K,m)$ and $f = \wp^{-1}(\hat{f})$ m-almost everywhere.*
(ii) *The same holds true for the dual $\mathcal{S}$: If $1\le p\le 2$ and $\varphi\in L^p(\mathcal{S},\pi)$ such that the inverse Hausdorff–Young transform $\check{\varphi}$ belongs to $L^2(K,m)$, then $\varphi\in L^2(\mathcal{S},\pi)$ and $\varphi = \wp(\check{\varphi})$ π-almost everywhere.*

Proof. (i) There exists a net $(k_i)_{i\in I}$ in $C_c(K)$ such that $k_i * f \to f$ in $L^p(K,m)$. We can choose $(k_i)_{i\in I}$ such that $(\hat{k}_i)_{i\in I}$ converges uniformly to 1 on compact subsets of $\mathcal{S}$ (see [?]). Further $k_i * f \in L^p(K,m) \cap C_0(K) \subseteq L^p(K,m) \cap L^\infty(K,m) \subseteq L^2(K,m)$. Thus, $(k_i * f)^\wedge = \wp(k_i * f) \in L^2(\mathcal{S},\pi) \cap L^q(\mathcal{S},\pi)$, $1/p + 1/q = 1$. Furthermore, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c(K)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Since

$$\|)(k_i * f)^\wedge - \hat{k}_i \hat{f}\|_q \leq \|(k_i * f)^\wedge - (k_i * f_n)^\wedge\|_q + \|\hat{k}_i \hat{f}_n - \hat{k}_i \hat{f}\|_q \to 0$$

as $n \to \infty$, we conclude $(k_i * f)^\wedge = \hat{k}_i \hat{f}$ π-almost everywhere. Hence, by Plancherel's theorem we have

$$\|k_i * f - \wp^{-1}(\hat{f})\|_2 = \|(k_i * f)^\wedge - \hat{f}\|_2 = \|\hat{k}_i \hat{f} - \hat{f}\|_2.$$

We can choose for each $\varepsilon > 0$ a compact set $C \subseteq \mathcal{S}$ such that

$$\int_C |(\hat{k}_i - 1)\hat{f}(\alpha)|^2 \, d\pi(\alpha) + \int_{\mathcal{S}\setminus C} |(\hat{k}_i - 1)\hat{f}(\alpha)|^2 \, d\pi(\alpha)$$
$$< \int_C |(\hat{k}_i - 1)\hat{f}(\alpha)|^2 \, d\pi(\alpha) + \varepsilon/2 \to \varepsilon/2.$$

Thus $\|k_i * f - \wp^{-1}(\hat{f})\|_2 \to 0$ and we obtain $f = \wp^{-1}(\hat{f})$ m-almost everywhere.

(ii) Let $\alpha \in \mathcal{S}$, C be a compact neighborhood of α and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C_c(K)$ such that $\|(f_n * f_n^*)^\wedge - \chi_C\|_1 \to 0$ as $n \to \infty$. Put $\psi = (f_n * f_n^*)^\wedge \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi) \cap C_0(\mathcal{S})$. For any $h \in L^1(K,m)$ we have, applying Parseval's formula and Proposition 2.5(ii),

$$\int_{\mathcal{S}} \wp(\check{\varphi})(\bar{\alpha})\hat{h}(\alpha)\psi(\alpha) \, d\pi(\alpha) = \int_K \check{\varphi}(x)\wp^{-1}(\hat{h}\psi)(x) \, dm(x)$$
$$= \int_K \check{\varphi}(x) h * \check{\psi}(x) \, dm(x).$$

Defining $\tilde{\psi}$ and $\tilde{h}$ by $\tilde{\psi}(\alpha) := \psi(\bar{\alpha})$ and $\tilde{h}(x) = h(\tilde{x})$ we easily see that $\tilde{h} * (\tilde{\psi})^\vee(\tilde{x}) = h * \check{\psi}(x)$. Applying successively Fubini's theorem, Proposition 2.5(i) and Lemma 2.4, we obtain

$$\int_{\mathcal{S}} \varphi(\bar{\alpha})\hat{h}(\alpha)\psi(\alpha) \, d\pi(\alpha) = \int_K h(\tilde{x})(\varphi\tilde{\psi})^\vee(x) \, dm(x) = \int_K h(\tilde{x})\check{\varphi} * (\tilde{\psi})^\vee(x) \, dm(x)$$
$$= \int_K \check{\varphi}(\tilde{x})\tilde{h} * (\tilde{\psi})^\vee(x) \, dm(x) = \int_K \check{\varphi}(x) h * (f_n * f_n^*)(x) \, dm(x).$$

Hence, we have

$$\int_{\mathcal{S}} (\wp(\check{\varphi})(\bar{\alpha}) - \varphi(\bar{\alpha}))\hat{h}(\alpha)(f_n * f_n^*)^\wedge(\alpha) \, d\pi(\alpha) = 0$$

for all $h \in L^1(K,m)$. Since $\{\hat{h} : h \in L^1(K,m)\}$ is uniformly dense in $C_0(\hat{K})$ (see Theorem 2.2.4(ix) in [BH]), we conclude that $(\wp(\check{\varphi}) - \varphi)(\bar{\alpha})(f_n * f_n^*)^\wedge(\alpha) = 0$ for each $n \in \mathbb{N}$ and for almost all $\alpha \in \mathcal{S}$. So $(\wp(\check{\varphi}) - \varphi)(\bar{\alpha})(f_n * f_n^*)^\wedge(\alpha) = 0$ for all $\alpha \in \mathcal{S} \setminus N$, where N is a π-zero set, for all $n \in \mathbb{N}$.

Further, as $\|(f_n * f_n^*)^\wedge - \chi_C\|_1 \to 0$ we can find a subsequence $(f_{n_k} * f_{n_k}^*)_{k\in\mathbb{N}}$ of $(f_n * f_n^*)_{n\in\mathbb{N}}$ such that $(f_{n_k} * f_{n_k}^*)^\wedge(\alpha) - \chi_C(\alpha) \to 0$ for almost all $\alpha \in \mathcal{S}$. Thus $\wp(\check{\varphi}) = \varphi$ π-almost everywhere on C. Therefore $\wp(\check{\varphi}) = \varphi$ π-almost everywhere, and in particular $\varphi \in L^2(\mathcal{S}, \pi)$. ∎

The last proposition leads to our main theorem, which implies that the inverse Hausdorff–Young transformation is indeed the inverse mapping to the Hausdorff–Young transformation.

THEOREM 2.8. *Let K be a commutative hypergroup, $1 \le p \le 2$ and $1 \le r \le 2$.*

(i) *For $f \in L^p(K, m)$ such that $\hat{f} \in L^r(\mathcal{S}, \pi)$ we have $(\hat{f})^\vee = f$ in $L^p(K, m)$.*
(ii) *For $g \in L^p(\mathcal{S}, \pi)$ such that $\check{g} \in L^r(K, m)$ we have $(\check{g})^\wedge = g$ in $L^p(\mathcal{S}, \pi)$.*

Proof. First let $f \in L^p(K, m)$ be such that $\hat{f} \in L^r(\mathcal{S}, \pi)$. Then $\hat{f} \in L^q(\mathcal{S}, \pi) \cap L^r(\mathcal{S}, \pi) \subseteq L^2(\mathcal{S}, \pi)$, $1/p + 1/q = 1$, and by Proposition 2.7 $f = \wp^{-1}(\hat{f}) = (\hat{f})^\vee$, since the inverse Hausdorff–Young transform and the inverse Plancherel transform coincide on $L^2(\mathcal{S}, \pi) \cap L^r(\mathcal{S}, \pi)$. The second statement follows in a similar manner by Proposition 2.7. ∎

REMARK. The special case $r = 1$ in Theorem 2.8 is of particular interest. If $f \in L^p(K, m)$ and $\hat{f} \in L^1(\mathcal{S}, \pi)$, then the integral $\int_{\mathcal{S}} \hat{f}(\alpha)\alpha(x)\, d\pi(\alpha)$ equals $f(x)$ m-almost everywhere.

COROLLARY 2.9 (Uniqueness theorem). *Let K be a commutative hypergroup and $1 \le p \le 2$.*

(i) *If $f \in L^p(K, m)$ is such that $\hat{f} = 0$ almost everywhere on $\mathcal{S}$, then $f = 0$ almost everywhere.*
(ii) *If $g \in L^p(\mathcal{S}, \pi)$ is such that $\check{g} = 0$ almost everywhere, then $g = 0$ almost everywhere.*

Another consequence of Proposition 2.5 is the following corollary.

COROLLARY 2.10. *Let $1 \le p \le 2$ and $1/p + 1/q = 1$. Suppose that $f \in L^p(K, m)$, $g \in L^p(\mathcal{S}, \pi)$ and $x \in K$. Further, let $\varphi \in L^2(\mathcal{S}, \pi)$ and $\beta \in \hat{K}$. Then:*

(i) *$(T_x f)^\wedge(\alpha) = \alpha(x)\hat{f}(\alpha)$ for π-almost all $\alpha \in \hat{K}$ and $(\hat{\varepsilon}_x g)^\vee = T_{\tilde{x}}\check{g}$ m-almost everywhere.*
(ii) *Define f^* by $f^*(x) = \overline{f(\tilde{x})}$. Then $(f^*)^\wedge = \overline{\hat{f}}$ π-almost everywhere and $(\bar{g})^\vee = (\check{g})^*$ m-almost everywhere.*

Proof. (i) $(T_x f)^\wedge(\alpha) = (\varepsilon_{\tilde{x}} * f)^\wedge(\alpha) = \hat{\varepsilon}_{\tilde{x}}(\alpha)\hat{f}(\alpha) = \alpha(x)\hat{f}(\alpha)$ for π-almost all $\alpha \in \hat{K}$. The second statement follows from Proposition 2.5(ii).

(ii) See [BH, (2.2.32), (2.2.15)] and argue as in the proof of Proposition 3.1 below. ∎

3. Further convolution results. In this section we give some further convolution results concerning the Hausdorff–Young transformation.

The following result is proved by standard arguments. We include the proof for completeness.

PROPOSITION 3.1. *Let $1 \le p \le 2$, $1/p + 1/q = 1$. For $f \in L^p(K,m)$ and each measure $\mu \in M(K)$ we have $(\mu * f)^\wedge = \hat{\mu}\hat{f}$ π-almost everywhere. In particular, for each function $g \in L^1(K,m)$, $(g * f)^\wedge = \hat{g}\hat{f}$ π-almost everywhere.*

Proof. By [BH, Lemma 1.4.6], $\mu * f \in L^p(K,m)$. Thus $(\mu * f)^\wedge$ is defined thanks to Proposition 2.1. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(K,m) \cap L^p(K,m)$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Then $(\mu * f_n)^\wedge = \hat{\mu}\hat{f_n}$ for each $n \in \mathbb{N}$ (see [BH, (2.2.15)]). Since the Hausdorff–Young transform is norm decreasing, we obtain $(\mu * f)^\wedge = \hat{\mu}\hat{f}$ π-almost everywhere. The second statement follows by embedding $L^1(K,m)$ into $M(K)$ via the mapping $f \mapsto fm$, $L^1(K,m) \to M(K)$. ∎

Taking $p = 2$ we can conclude immediately

PROPOSITION 3.2. *Let $f, g \in L^2(\mathcal{S},\pi)$. Then $(fg)^\vee = \wp^{-1}(f) * \wp^{-1}(g)$ m-almost everywhere.*

Proof. Use [BH, (2.2.15)] and Parseval's identity. ∎

COROLLARY 3.3. *We have the equality $L^2(K,m) * L^2(K,m) = L^1(\mathcal{S},\pi)^\vee$. In particular, $L^2(K,m) * L^2(K,m)$ is a linear space.*

In order to obtain a dual version of the last corollary we need some preliminary results.

Applying the Plancherel transform we can define a (rather weak) translation operator for $L^2(\mathcal{S},\pi)$ (see [DSL]). For every $f \in L^\infty(K,m)$ define $M_f \in B(L^2(\mathcal{S},\pi))$ by

$$M_f(\varphi) := \wp(\bar{f}\wp^{-1}(\varphi)), \qquad \varphi \in L^2(\mathcal{S},\pi).$$

Then M_f is a bounded linear operator satisfying $\|M_f(\varphi)\|_2 \le \|f\|_\infty \|\varphi\|_2$. We call M_α, $\alpha \in \hat{K}$, the *translation operator on* $L^2(\mathcal{S},\pi)$.

Furthermore, we can introduce an action of $L^1(\mathcal{S},\pi)$ on $L^2(\mathcal{S},\pi)$. Given $\psi \in C_c(\mathcal{S})$ and $\varphi \in L^2(\mathcal{S},\pi)$ we define the $L^2(\mathcal{S},\pi)$-valued integral

$$\psi * \varphi := \int_{\mathcal{S}} \psi(\alpha) M_{\bar{\alpha}}(\varphi)\, d\pi(\alpha) \in L^2(\mathcal{S},\pi).$$

For any $\psi \in L^1(\mathcal{S},\pi)$ we can define $\psi * \varphi$ by setting $\psi * \varphi := \lim_{n\to\infty} \psi_n * \varphi \in L^2(\mathcal{S},\pi)$, where any sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$ converges to ψ in $L^1(\mathcal{S},\pi)$. Then $\|\psi * \varphi\|_2 \le \|\psi\|_1 \|\varphi\|_2$.

Proposition 3.4. *Let $\mathcal{S}$ be compact and let $\psi \in L^1(\mathcal{S},\pi)$ and $\varphi \in L^2(\mathcal{S},\pi)$. Then $(\psi * \varphi)^\vee = \check{\psi}\check{\varphi}$.*

Proof. Choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C_c(\mathcal{S})$ such that $\|\psi_n - \psi\|_1 \to 0$ as $n \to \infty$. Since $\mathcal{S}$ is compact, $\psi_n * \varphi \in L^1(\mathcal{S},\pi)$ for all $n \in \mathbb{N}$ and by Fubini's theorem we deduce

$$(\psi_n * \varphi)^\vee(x) = \int_{\mathcal{S}}\int_{\mathcal{S}} \psi_n(\beta) M_{\bar{\beta}}\varphi(\alpha)\alpha(x)\, d\pi(\beta)\, d\pi(\alpha) = \check{\psi}_n(x)\check{\varphi}(x)$$

for all $x \in K$. The statement follows from

$$\|(\psi * \varphi)^\vee - \check{\psi}\check{\varphi}\|_\infty \le \|(\psi * \varphi) - (\psi_n * \varphi)\|_2 + \|\check{\psi}_n - \check{\psi}\|_\infty \|\check{\varphi}\|_\infty \to 0$$

as $n \to \infty$. ∎

Corollary 3.5. *Let $\mathcal{S}$ be compact. Then $(L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi))^\vee = L^1(K,m)$ as linear spaces.*

Proof. Let $\psi,\varphi \in L^2(\mathcal{S},\pi)$. By Proposition 3.4, $(\psi*\varphi)^\vee = \check{\psi}\check{\varphi} \in L^1(K,m)$. Conversely, for each $h \in L^1(K,m)$ there exist $\psi,\varphi \in L^2(\mathcal{S},\pi)$ such that $h = \check{\psi}\check{\varphi}$ in $L^1(K,m)$. Hence $h \in (L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi))^\vee$. ∎

Corollary 3.6. *Let $\mathcal{S}$ be compact. Then $L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi) = L^1(K,m)^\wedge$ as linear spaces. In particular $L^2(\mathcal{S},\pi) * L^2(\mathcal{S},\pi)$ is a linear space.*

Concerning the translation on the dual, the following result holds and can be proved using Proposition 3.4.

Corollary 3.7. *Let $\mathcal{S}$ be compact and $\varphi \in L^2(\mathcal{S},\pi)$. Then $(M_\alpha\varphi)(\beta) = (M_\beta\varphi)(\alpha)$ for π-almost all $\alpha,\beta \in \hat{K}$.*

Proof. There exists a unique $g \in L^2(K,m)$ such that $\varphi = \wp(g)$ in $L^2(\mathcal{S},\pi)$. Let $f \in L^2(K,m)$ be arbitrary. By Proposition 3.4 and Parseval's identity we have

$$\begin{aligned}\wp(f) * \wp(g)(\alpha) = (fg)^\wedge(\alpha) &= \int_K f(x)g(x)\overline{\alpha(x)}\, dm(x)\\ &= \int_{\mathcal{S}} \wp(f)(\beta)\wp(\bar{\alpha}g)(\bar{\beta})\, d\pi(\beta).\end{aligned}$$

By definition, $\wp(f) * \wp(g)(\alpha) = \int_{\mathcal{S}} \wp(f)(\beta) M_{\bar{\beta}}(\wp(g))(\alpha)\, d\pi(\beta)$. Hence,

$$\int_{\mathcal{S}} \wp(f)(\beta)[M_{\bar{\beta}}(\wp(g))(\alpha) - \wp(\bar{\alpha}g)(\bar{\beta})]\, d\pi(\beta) = 0.$$

Since f was arbitrary, we conclude that $M_{\bar{\beta}}(\varphi)(\alpha) = \wp(\bar{\alpha}g)(\bar{\beta}) = M_\alpha(\varphi)(\bar{\beta})$ for π-almost all $\alpha,\beta \in \hat{K}$. ∎

4. Further consequences of the main theorem. In this section we mention some further consequences of our main theorem of Section 2.

Proposition 4.1 (Generalization of Parseval's identity). *For* $1 \leq p \leq 2$ *and* $1/p + 1/q = 1$ *we have the following:*

(i) *For* $f \in L^p(K, m)$, $g \in L^p(\mathcal{S}, \pi)$ *we have*

$$\int_K f(x)\overline{\check{g}(x)}\, dm(x) = \int_{\mathcal{S}} \hat{f}(\alpha)\overline{g(\alpha)}\, d\pi(\alpha).$$

(ii) *If* K *is compact,* $f \in L^p(K, m)$ *and* $g \in L^q(K, m)$ *is such that* $\hat{g} \in L^p(\mathcal{S}, \pi)$*, then*

$$\int_K f(x)\overline{g(x)}\, dm(x) = \int_{\mathcal{S}} \hat{f}(\alpha)\overline{\hat{g}(\alpha)}\, d\pi(\alpha).$$

(iii) *If* $\mathcal{S}$ *compact,* $\varphi \in L^p(\mathcal{S}, \pi)$ *and* $\psi \in L^q(\mathcal{S}, \pi)$ *is such that* $\check{\psi} \in L^p(K, m)$*, then*

$$\int_{\mathcal{S}} \varphi(\alpha)\overline{\psi(\alpha)}\, d\pi(\alpha) = \int_K \check{\varphi}(x)\overline{\check{\psi}(x)}\, dm(x).$$

Proof. The proof is similar to the proof of [HR, 31.48]. ∎

Proposition 4.2. *Let* $1 < p < 2$ *and* $1/p + 1/q = 1$*. The mapping* $f \mapsto \hat{f}$*,* $L^p(K, m) \to L^q(\mathcal{S}, \pi)$*, is onto if and only if* K *is finite.*

Proof. If K is finite the mapping is obviously onto. Conversely, let K be infinite and suppose that every function in $L^q(\mathcal{S}, \pi)$ is the Hausdorff–Young transform of a function in $L^p(K, m)$. Thus the mapping $f \mapsto \hat{f}$, $L^p(K, m) \to L^q(\mathcal{S}, \pi)$, is linear, bijective and continuous. Hence by the open mapping theorem the inverse mapping is also continuous, and hence there exists a constant $C > 0$ such that $\|\hat{f}\|_q \leq \|f\|_p \leq C\|\hat{f}\|_q$.

Now consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(K, m)$ which converges weakly to zero in $L^p(K, m)$ and satisfies $\|f_{n_1} + \cdots + f_{n_m}\|_p = m^{1/p}$ for all subsets $\{f_{n_1}, \ldots, f_{n_m}\}$ of $(f_n)_{n\in\mathbb{N}}$, $m \in \mathbb{N}$. Such a sequence exists by [H, Lemma A]. The sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ converges weakly to zero in $L^q(\mathcal{S}, \pi)$ by Proposition 4.1. By Lemma B in [H] there exists a subsequence $(\hat{f}_{n_k})_{k\in\mathbb{N}}$ of $(\hat{f}_n)_{n\in\mathbb{N}}$ and a constant $A > 0$ such that $\|\sum_{k=1}^m \hat{f}_{n_k}\|_q \leq Am^{1/2}$. It follows that

$$m^{1/p} = \|f_{n_1} + \cdots + f_{n_m}\|_p \leq C\Big\|\sum_{k=1}^{m} \hat{f}_{n_k}\Big\|_q \leq ACm^{1/2}$$

for all $m \in \mathbb{N}$. We see at once that $1/p \leq 1/2$, which contradicts our hypothesis. Hence the mapping $f \mapsto \hat{f}$, $L^p(K, m) \to L^q(\mathcal{S}, \pi)$, cannot be onto. ∎

Considering the dual case, we can still show that the range of the mapping $f \mapsto \check{f}$, $L^p(\mathcal{S}, \pi) \to L^q(K, m)$, is always dense in $L^q(K, m)$. To verify this we need the following lemma. The dual spaces $\hat{K}$ or $\mathcal{S}$ do not, in general, carry a dual hypergroup structure. Therefore, there does not exist a dual version of the following lemma and we cannot say anything about the range of the mapping $f \mapsto \hat{f}$, $L^p(K,m) \to L^q(\mathcal{S},\pi)$.

LEMMA 4.3. *Let A be a compact subset of K, and H an open subset of K such that $A \subseteq H$. Then there is a function $\psi \in L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi)$ such that $\check{\psi} \in C_c(K)$ and $\chi_A \le \check{\psi} \le \chi_H$.*

Proof. We may suppose that H has compact closure in K. Let P be an m-measurable symmetric neighborhood of $e \in K$ such that $P * P * A \subseteq H$. Let

$$f = \frac{1}{m(P)} \chi_{P*A} * \chi_P.$$

By Corollary 3.3 there exists $\psi \in L^1(\mathcal{S},\pi)$ such that $f = \check{\psi}$. Since

$$\check{\psi}(x) = f(x) = \frac{1}{m(P)} \Big(\int_P \omega(x,y)(P*A)\, dm(y) \Big)$$

and $(P*\{x\}) \cap (P*A) = \emptyset$ if and only if $\tilde{P} * P * A \cap \{x\} = \emptyset$, it is immediate that $\chi_A \le \check{\psi} \le \chi_{P*P*A} \le \chi_H$. Since $\check{\psi} \in L^2(K,m)$, Proposition 2.7 implies $\psi \in L^2(\mathcal{S},\pi)$. ∎

PROPOSITION 4.4. *Let $1 < p \le 2$ and $1/p + 1/q = 1$. Then $L^p(\mathcal{S},\pi)^\vee$ is a dense linear subspace of $L^q(K,m)$.*

Proof. For $p = 2$ the statement is obviously true. Therefore suppose $1 < p < 2$. Consider an m-measurable subset B of K such that $m(B) < \infty$. Given $\varepsilon > 0$, let A be a compact subset of B, and H an open subset of K, such that $B \subseteq H$ and $m(H \setminus A) < \varepsilon^q$. By Lemma 4.3 there exists a function $f \in (L^1(\mathcal{S},\pi) \cap L^2(\mathcal{S},\pi))^\vee \subseteq L^p(\mathcal{S},\pi)^\vee$ such that $\chi_A \le f \le \chi_H$. Then $\|f - \chi_B\|_q < \|\chi_H - \chi_A\|_q < \varepsilon$. Now linear combinations of functions χ_B are dense in $L^q(K,m)$, and so $L^p(\mathcal{S},\pi)^\vee$ is dense in $L^q(K,m)$. ∎

Before we proceed, we mention another consequence of Lemma 4.3.

PROPOSITION 4.5. *$L^1(\mathcal{S},\pi)^\vee * C_c(K) = C_c(K)$.*

Concluding, we give two further results which are very interesting in the context of harmonic analysis.

PROPOSITION 4.6. *Let $1 \le p \le 2$, $\mu \in M(K)$, $f \in L^p(K,m)$ and suppose that $\hat{\mu} = \hat{f}$ π-almost everywhere on $\hat{K}$. Then $f \in L^1(K,m)$, μ is absolutely continuous and $\mu = f dm$.*

Proof. The proof is similar to the proof of [HR, 31.33]. ∎

PROPOSITION 4.7. *Suppose that $f \in L^1(K,m) \cap L^\infty(K,m)$ and $\hat{f}$ is nonnegative. Then $\hat{f} \in L^1(\mathcal{S},\pi)$ and $\|\hat{f}\|_1 \le \|f\|_\infty$.*

Proof. Since $f \in L^1(K,m) \cap L^\infty(K,m)$, by Hölder's interpolation theorem $f \in L^2(K,m)$ and hence $\hat{f} = \wp(f) \in L^2(\mathcal{S},\pi)$. Let $(k_i)_{i\in I} \in C_c(K)$ be an approximate identity in $L^1(K,m)$. Then by Parseval's theorem, for all $i \in I$,

$$\int_{\mathcal{S}} \hat{f}(k_i * k_i^*)^\wedge \, d\pi = \int_K f(k_i * k_i^*) \, dm \le \|f\|_\infty \|k_i * k_i^*\|_1 \le \|f\|_\infty.$$

We observe that $\hat{f}(k_i*k_i^*)^\wedge$ converges pointwise to $\hat{f}$ and $\hat{f}|\hat{k}_i|^2$ is nonnegative by assumption. Hence applying Fatou's lemma, we obtain

$$\int_{\mathcal{S}} \hat{f} \, d\pi = \int_{\mathcal{S}} \lim_i \hat{f}(k_i * k_i^*)^\wedge \, d\pi \le \lim_i \sup \int_{\mathcal{S}} \hat{f}(k_i * k_i^*)^\wedge \, d\pi \le \|f\|_\infty. \ \blacksquare$$

Remark. We do not know whether Propositions 4.6 and 4.7 admit dual versions.

Remark. We remark that Rodionov established expansions of functions in L^p with respect to systems similar to orthogonal ones. His results are analogues of the Hausdorff–Young theorems in the theory of trigonometric series (see [R]). However, Rodionov's results apply only to a few polynomial hypergroups, since orthonormal polynomials which are also bounded are very rare.

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